The Riemann-Stieltjes integral

and some applications in complex analysis and probability theory

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Abstract

The purpose of this essay is to prove the existence of the Riemann-Stieltjes integral. After doing so, we present some applications in complex analysis, where we define the complex curve integral as a special case of the Riemann-Stieltjes integral, and then focus on Cauchy's celebrated integral theorem. To show the versatility of the Riemann-Stieltjes integral, we also present some applications in probability theory, where the integral generates a general formula for the expectation, regardless of its underlying distribution.

Sammanfattning

Syftet med denna uppsats är att bevisa existensen av Riemann-Stieltjes integralen. Därefter ges tillämpningar inom komplex analys, där vi definierar den komplexa kurvintegralen som ett specialfall av Riemann-Stieltjes integralen och sedan koncentrerar oss på Cauchys hyllade integralsats. För att visa på mångsidigheten hos Riemann-Stieltjes integralen beskrivs även tillämpningar inom sannolikhetsteori, där integralen ger en generell formel för väntevärdet, oberoende av dess underliggande fördelning.

Contents

1. Introduction

A basic tool in natural science is the integral. It has countless applications ranging from the most simple in calculus such as computing area and volume to more advanced topics such as electrodynamics, where integrals are used to describe interactions between electric charges. There are several different types of integrals (see e.g. [11]). In this essay we have chosen to focus on the so called Riemann-Stieltjes integral. After proving its existence, we give some applications within complex analysis and probability theory. However, let us first introduce some basic concepts and notations.

Let A be a non-empty open subset of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $[a, b]$ be an interval. Furthermore, let $f : A \to \mathbb{K}$ be a continuous function and $g : [a, b] \to \mathbb{K}$ a function of bounded variation. A partition $\pi = \{t_k\}$ of the interval $[a, b]$ is an ordered set of points satisfying

$$a = t_0 < t_1 < \ldots < t_n = b \ .$$

Furthermore, we introduce the following notation for the norm of a partition

$$\|\pi\| = \max_k (t_k - t_{k-1}) \ .$$

Now for each subinterval $[t_{k-1}, t_k] \subset [a, b]$, $k = 1, 2, \ldots, n$, let $\sigma = \{s_k\}$ be an ordered set of points $s_k \in [t_{k-1}, t_k]$. We then define the *Riemann-Stieltjes sum* corresponding to π, σ, f, and g as

$$S_{\pi,\sigma}(f, g) = \sum_{k=1}^{n} f(s_k)\big(g(t_k) - g(t_{k-1})\big) \ .$$

Furthermore, for the continuous function $f : A \to \mathbb{K}$ we introduce the modulus of continuity $\mu(\delta, f)$ defined for $\delta > 0$ as

$$\mu(\delta, f) = \max\left\{|f(t_1) - f(t_2)| : t_1, t_2 \in [a, b], |t_1 - t_2| \le \delta\right\} \ ,$$

and we denote the total variation of a function $g : [a, b] \to \mathbb{K}$ of bounded variation as $V_a^b(g)$.

The main aim of this essay is to prove the following theorem.

Theorem 3.2. *Let $[a, b]$ be an interval and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $f : [a, b] \to \mathbb{K}$ be a continuous function and $g : [a, b] \to \mathbb{K}$ a function of bounded variation. There exists exactly one element $J \in \mathbb{R}$ such that*

$$|J - S_{\pi,\sigma}(f, g)| \le 2\mu(\delta, f)V_a^b(g) \ ,$$

for any $\delta > 0$ such that $\|\pi\| \le \delta$.

From Theorem 3.2 we can make a well-posed definition of the *Riemann-Stieltjes integral* as

$$\int_a^b f(t)dg(t) = J \ .$$

The integral bears a resemblance to the Riemann integral encountered in elementary calculus (see e.g. [1]). If we let $g(t) = t$ we get exactly the Riemann integral. Hence the name ”Riemann-Stieltjes integral”, since it is a generalization of Riemann's integral made by the Dutch astronomer Thomas Jan Stieltjes. The proof of Theorem 3.2 follows the outline in [8].

A useful consequence of the Riemann-Stieltjes integral is that, under the assumption that g has a continuous derivative in $[a, b]$, we have that

$$\int_a^b f(t)dg(t) = \int_a^b f(t)g'(t)dt \ .$$

The left hand side above is our Riemann-Stieltjes integral, whilst the right hand side is a Riemann integral. In section 4 we make use of this property. We there change our notation from g to z, which is more commonly used for complex valued functions.

Let $A \subset \mathbb{C}$ be a non-empty open set, and let $f : A \to \mathbb{C}$ be a continuous function. Furthermore, let $z : [a, b] \to A$ be a continuous function of bounded variation. We consider an oriented curve γ in the complex plane, which happens to be generated by the parametrization z. Let γ be given by

$$\gamma = \{z = z(t) : t \in [a, b]\} \ .$$

Now, we can use Theorem 3.2 to define the complex curve integral along γ as

$$\int_\gamma f(z)dz = \int_a^b (f \circ z)(t)dz(t) \ . \tag{1.1}$$

Before discussing any assumptions of f and z we note that a continuous function need not be of bounded variation, nor does a function of bounded variation need to be continuous. However, we require that z is both continuous and of bounded variation in order for Theorem 3.2 to apply here. We also point out that we have chosen a parametrization and that the curve of interest happens to be generated by this parametrization. This since we then can avoid the requirement that the parametrization function has a non zero first order derivative. Throughout section 4 we therefore work with the parametrizations themselves and not with the equivalent classes of parametrized curves. For simplicity, we shall also assume that the parametrization z that generates the curve γ is continuously differentiable so that equation (1.1) holds.

Our focus for the complex curve integral is Cauchy's greatly celebrated integral theorem.

Theorem 4.3 (Cauchy's integral theorem)**.** *Let D be a simply-connected domain in the complex plane, and let $f : D \to \mathbb{C}$ be a holomorphic function. Then for any closed, simple curve $\gamma \subset D$ generated by a continuously differentiable parametrization*

$$\int_\gamma f(z)dz = 0 \ .$$

In Theorem 4.3, the assumption that the curve $\gamma \subset D$ is generated by a continuously differentiable parametrization could be weakened. However, more technical details and approximation arguments would be required. Therefore, we shall stick to the stronger assumption, enabling a clearer outline and greater emphasis to main methods.

Due to the importance of Theorem 4.3, three different proofs are presented in section 4. First, we consider Cauchy's original complete works from 1825 and 1847 ([3] and [4] respectively). His results are presented in Theorem 4.2. Secondly, we present a proof of Theorem 4.3 using real analytic methods in which the key ingredient is Green's theorem. The main outline here follows from Greene and Krantz in [7]. We are well aware that Green's theorem requires that the integrand f has a continuous first order derivative, and that Theorem 4.3 is true even without this requirement. A more general real analysis proof was presented by Goursat in 1884 (see e.g. [6] and [18]). However, Goursat's proof is more technically complicated which is why we have chosen the more straightforward version with Green's theorem in this essay. We also present third and final proof without the additional requirement on f. This proof is inspired by Dixon's proof in [5], which is built around the idea of homotopic curves.

To demonstrate the versatility of the Riemann-Stieltjes integral, we end this essay by presenting an application of it in probability theory. In section 5 we therefore consider random processes and particularly the expected value of random variables. By following the ideas of Anevski in [2], we reach the conclusion that the expected value of random variables can be interpreted as the Riemann-Stieltjes integral of the random variable with respect to its distribution function. The results are presented in Theorem 5.4 and Theorem 5.7 for the one- and n-dimensional cases respectively.

The main purpose of defining expected values with the Riemann-Stieltjes integral is that it provides a definition of the expectation of a sum of one continuous and one discrete random variable as the sum of the two separate expectations of the continuous and discrete variables. We present this result in Theorem 5.8, which is valid regardless of the type of the random variables.

Theorem 5.8. *If X and Y are two random variables and a and b are two real valued constants, then the expectation of X and Y satisfies*

$$E(aX + bY) = aE(X) + bE(Y) \ .$$

This essay is structured as follows. In section 2, we begin by introducing necessary background theory and notations. For further details related to real analysis see e.g. [15], for more on complex analysis see e.g. [7] or [14], and for topology see e.g. [17]. In section 3, we then present the main proof of this essay, namely for the existence of the Riemann-Stieltjes integral (Theorem 3.2). Three different proofs of Cauchy's integral theorem (Theorem 4.3) then follow in section 4. Finally, in section 5 we present a connection between the Riemann-Stieltjes integral and the expectation of random variables. For further reading on probability theory and on the Riemann-Stieltjes integral in probability theory see e.g. [9] or [13].

2. Preliminaries

In this section we form a foundation for the key concepts of the essay by introducing some basic theory. Since our great Riemann-Stieltjes integral requires functions of bounded variation, we begin by discussing what they are. To prove the existence of the Riemann-Stieltjes integral we also need to take a tour into the world of topology, and therefore some basic topology is included. To finish up, we present some basic concepts of complex analysis which will lay ground for the applications of our Riemann-Stieltjes integral in complex analysis.

2.1. **Functions of bounded variation.**

The concept of functions of bounded variation first arose in the late 19^{th} century in connection with the alignment of curves when studying convergence of Fourier series. It is precisely with respect to these functions that we may define the Riemann-Stieltjes integral of continuous functions.

Definition 2.1. We define a *partition* $\pi = \{t_k\}$ of an interval $[a, b]$ as a subdivision of $[a, b]$ such that

$$a = t_0 < t_1 < ... < t_n = b \ ,$$

with $n \in \mathbb{N}$ and $t_k \in [a, b]$, $k = 0, 1, ..., n$.

A *refinement* π' of a partition π is a further subdivision of $[a, b]$ such that every element of π is a subset of some element of π'. Furthermore, the *total refinement* π_T of π is the partition that corresponds to adding up all subintervals of $[a, b]$.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $g : [a, b] \to \mathbb{K}$, and let π be a partition of $[a, b]$.

Definition 2.2. A function $g : [a, b] \to \mathbb{K}$ is said to be of *bounded variation* on $[a, b]$ if there exists an $L \in \mathbb{R}$ such that

$$L_\pi(g) = \sum_{k=1}^{n} |g(t_k) - g(t_{k-1})| \leq L \ ,$$

where $L_\pi(g)$ is a sum corresponding to a partition π. We define the *total variation* of a function g on the interval $[a, b]$ as

$$V_a^b(g) = \sup_\pi L_\pi(g) \ .$$

Since every sum $L_\pi(g)$ is bounded, so will the set of all sums $L_\pi(g)$ be. By our definition of $\mathbb{K}$, Definition 2.2 is valid for both real and complex valued functions g. In fact, every real valued function that is monotone on a given interval satisfy the conditions of being of bounded variation. For such a function g, the total variation of g on $[a, b]$ will be $V_a^b(g) = |g(a) - g(b)|$.

The following theorem sums up important properties for the total variation.

Theorem 2.3. *Let* $g : [a, b] \to \mathbb{K}$ *be of bounded variation.*

I. *If* $[x_1, x_2] \subset [a, b]$, g *will also be of bounded variation on* $[x_1, x_2]$ *and*

$$V_{x_1}^{x_2}(g) \leq V_a^b(g) \ .$$

That is, the total variation increases with a larger interval.

II. *If $a < c < b$, we can write the total variation as*

$$V_a^b(g) = V_a^c(g) + V_c^b(g) \ .$$

That is, the total variation is additive on consecutive subintervals.

Proof. To prove part (I) we construct two intervals $I = [a, b]$ and $I_1 = [x_1, x_2]$ with total variations $V = V_a^b(g)$ and $V_1 = V_{x_1}^{x_2}(g)$. Let π_1 be a partition of I_1, and let π be π_1 with the points a and b adjoined. That is, $\pi = \pi_1 \cup \{a, b\}$, setting π as a partition of the interval I. Let $L_\pi(g)$ and $L_{\pi_1}(g)$ be sums constructed as in Definition 2.2. We have

$$L_{\pi_1}(g) \leq L_\pi(g) \ .$$

and it follows that

$$V_1 \leq V \ .$$

For part (II) we construct three intervals $I = [a, b]$, $I_1 = [a, c]$ and $I_2 = [c, b]$. Also, set $V = V_a^b(g)$, $V_1 = V_a^c(g)$ and $V_2 = V_c^b(g)$ as the total variations on each interval. Let π_1 and π_2 be partitions of I_1 and I_2 respectively. Then $\pi = \pi_1 \cup \pi_2$ will be a partition of I, and we have

$$L_\pi(g) = L_{\pi_1}(g) + L_{\pi_2}(g) \ .$$

Since $V = \sup_\pi L_\pi(g)$ by definition

$$L_{\pi_1}(g) + L_{\pi_2}(g) \leq V \ .$$

Furthermore, $V_1 = \sup_{\pi_1} L_{\pi_1}(g)$ and $V_2 = \sup_{\pi_2} L_{\pi_2}(g)$, so

$$L_{\pi_1}(g) + L_{\pi_2}(g) \leq V_1 + V_2 \leq V \ . \tag{2.1}$$

Conversely, if π is a partition of I, let π' be π with the point c adjoined. By (I) we then have $L_\pi(g) \leq L_{\pi'}(g)$ and furthermore, if π' splits into partitions π_1 of I_1 and π_2 of I_2

$$L_\pi(g) \leq L_{\pi'}(g) = L_{\pi_1}(g) + L_{\pi_2}(g) \leq V_1 + V_2 \ .$$

Since $V = \sup_\pi L_\pi(g)$ we get that

$$V \leq V_1 + V_2 \ . \tag{2.2}$$

Equations (2.1) and (2.2) will leave us with $V = V_1 + V_2$ which completes the proof. □

2.2. **Basic topology.**

This section is dedicated to an introduction of topological spaces and is based on Simmons's introductory book on topology and modern analysis [17]. A topological space can be thought of as a set which only contains structure that is relevant to the continuity of functions defined on the set. A part of the general theory behind topological spaces concerns metric spaces in which we deal with two elementary concepts: convergent series of real or complex numbers and continuous functions of real or complex variables. In this essay we will focus on the concept of the difference between two real or complex numbers or functions, and particularly that the absolute value of such a difference represents the distance between the two.

We will begin by defining a metric space and present its elementary properties, leading up to Cantor's intersection theorem which represents the final argument in proving the existence of our Riemann-Stieltjes integral.

Definition 2.4. Let X be a non-empty set. We define a *metric* on X as a real function $d : X \times X \to \mathbb{R}$ such that

I. $d(x, y) \geq 0$, and $d(x, y) = 0 \;\Leftrightarrow\; x = y$
II. $d(x, y) = d(y, x)$
III. $d(x, y) \leq d(x, w) + d(w, y)$

for all elements x, y and w in X. We refer to the metric $d(x, y)$ as the *distance* between x and y.

Definition 2.5. We define a *metric space* as a non-empty set X with a metric d on X.

Let $z = x + iy$, $x, y \in \mathbb{R}$, be a complex number. The distance from z to the origin is given by the absolute value $|z| = \sqrt{x^2 + y^2}$. If $z_1 = x_1 + iy_1$, $x_1, y_1 \in \mathbb{R}$, is another complex number, the distance between z and z_1 is given by $d(z, z_1) = |z - z_1|$. Thus, the complex plane is a metric space with a metric d representing distance and satisfying

I. $|z| \geq 0$, and $|z| = 0 \;\Leftrightarrow\; z = 0$
II. $|-z| = |z|$
III. $|z + z_1| \leq |z| + |z_1|$.

As mentioned, one of the elementary concepts for metric spaces is convergent sequences of real or complex numbers. Therefore, we shall consider the circumstances under which a series converges in a metric space.

Definition 2.6. Let X be a metric space with metric d, and let

$$\{x_n\} = \{x_1, x_2, ..., x_n\}$$

be a sequence of points in X. If there, for each $\varepsilon > 0$, exists a point $x \in X$ and a positive integer N such that

$$n \geq N \Rightarrow d(x_n, x) < \frac{\varepsilon}{2} \ ,$$

then the sequence $\{x_n\}$ is said to be *convergent* and converges to the point x, called its limit.

Equivalently, the sequence $\{x_n\}$ is convergent if there exist a point $x \in X$ such that $d(x_n, x)$ tends to zero with increasing n.

Definition 2.7. Let X be a metric space with metric d, and let $\{x_n\}$ be a convergent sequence in X. If, for each $\varepsilon > 0$, there exists a positive integer N such that $m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon$, then $\{x_n\}$ is called a *Cauchy sequence.*

It follows directly from Definition 2.7 that every convergent sequence is a Cauchy sequence since, for some integers $m, n \geq N$, the convergence of a sequence $\{x_n\}$ implies that

$$m, n \geq N \Rightarrow d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$

However, the converse, that every Cauchy sequence is convergent, is not necessarily true. For metric spaces in which every Cauchy sequence is convergent we reach the following definition.

Definition 2.8. We define a *complete metric space* as a metric space in which every Cauchy sequence is convergent and converges to a point in the metric space.

Let X be a metric space with metric d and let A be a subset of X. Then a point x is called a *limit point* of A if each open sphere centred around x contains at least one point $x_0 \in A$ different form x. Furthermore, we say that a subset A of a metric space X is closed if it contains all of its limit points. The union of A and the set of all its limit points is called the closure of A, and is denoted by $\overline{A}$.

Now, let X be a metric space with metric d, and let F be a non-empty subset of X in which the function d is defined only for points in F. Then F is itself a metric space with the particular metric d and we call it a subspace of X.

Theorem 2.9. *Let X be a complete metric space, and let F be a subspace of X. F being complete is equivalent to F being closed.*

Proof. First, we assume that the subspace F is complete and show that it must be closed. For positive integers n, let $B(x, \frac{1}{n}) \subset F$ be a sphere of radius $\frac{1}{n}$ centred around x containing a different point x_n. Since $d(x_n, x) < \frac{1}{n}$ tends towards zero with increasing n, x is a limit point of F. The sequence $\{x_n\}$ thereby converges to x in X, and is therefore a Cauchy sequence in X by the argument following Definition 2.7. Since F is complete, x must be a point of F, which implies that F must be closed.

Conversely, we assume that F is closed and show that F is complete. Let $\{x_n\}$ be a Cauchy sequence in F and thereby also a Cauchy sequence in X. The completeness of X implies that $\{x_n\}$ converges to a point x in X. Since $\{x_n\} \subset F$, x will be a limit point in F. A closed set contains all its limit points, which implies that $x \in F$. Thus each Cauchy sequence $\{x_n\}$ in F is convergent and converges to a point in F, and F is thereby complete by Definition 2.8. □

Definition 2.10. Let A be a non-empty set of a metric space X with metric d. We define the *diameter of the set A* as

$$d(A) = \sup \left\{ d(x_1, x_2) : x_1, x_2 \in A \right\} .$$

Now, let $\{F_n\}_{n=1}^{\infty}$ be a a decreasing sequence of X, that is

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq ...$$

Then the sequence $\{d(F_n)\}_{n=1}^{\infty}$ will also be decreasing. Conditions under which an intersection of such a sequence will be non-empty is provided in the following theorem by Cantor.

Theorem 2.11 (Cantor's intersection theorem)**.** *Let X be a complete metric space, and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$ as $n \to \infty$. Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.*

Proof. The assumption that $d(F_n) \to 0$ implies that F cannot contain more than one point. Therefore, in order to prove the theorem it is sufficient to prove that F is non-empty.

Each subset F_n is closed and thereby also complete by Theorem 2.9. Let x_n be a point in F_n, thus forming a sequence $\{x_n\}$ in X. We have assumed that $d(F_n) \to 0$ as $n \to \infty$. Therefore, for $\varepsilon > 0$, there exists a positive integer N

such that $n \geq N \Rightarrow d(F_n) < \varepsilon$. $\{F_n\}$ is furthermore a decreasing sequence. Therefore, given any $n, m \geq N$, we have that $x_n, x_m \in F_n$ and $d(x_n, x_m) < \varepsilon$. By Definition 2.7 $\{x_n\}$ is thereby a Cauchy sequence. Since every F_n is closed, $F = \bigcap_{n=1}^{\infty} F_n$ will be closed and thereby also complete. At length the Cauchy sequence will be in F, implying that $\lim_{n\to\infty} x_n = x \in F$. □

2.3. **Basic complex analysis.**

Holomorphic functions and their properties are of great importance for the theory behind complex integration, and in this essay they are the foundation on which the entire section on applications of the Riemann-Stieltjes integral in complex analysis is built. Another important aspect concerns the curves along which integration is performed. In this section we therefore also consider curves that can be continuously "deformed" into other more conveniently shaped curves. This opens up to a new approach to treating complex integrals.

Definition 2.12. Let f be a complex valued function defined in a neighbourhood of a point $z_0 \in \mathbb{C}$. Then f is *complex differentiable* at z_0 if the limit

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, and we define the complex derivative as this limit, denoted $f'(z_0)$.

Since $h \in \mathbb{C}$, it can approach zero in numerous ways. However, the limit must tend towards a unique value $f'(z_0)$ independent of how h approaches zero.

Definition 2.13. Let $A \subset \mathbb{C}$ be a non-empty open set. A function $f : A \to \mathbb{C}$ is *holomorphic* on A if f is complex differentiable at every point of A.

A necessary condition for a function to be complex differentiable at a point z_0 is that the Cauchy-Riemann equations hold at z_0. That is, if the function $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$ then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.3}$$

must hold at $z_0 = x_0 + iy_0$. Equations (2.3) are called the Cauchy-Riemann equations. Consequently, for a function to be holomorphic in an open set $A \subset \mathbb{C}$, Cauchy-Riemanns' equations must hold at every point of A.

Theorem 2.14. *If f is a holomorphic function in a non-empty open set $A \subset \mathbb{C}$, then the Cauchy-Riemann equations hold at every point in A.*

For further details and a proof of Theorem 2.14 we refer to e.g. pages 73-76 in [16]. We point out that the reverse implication does not apply without some additional requirements. That is, that the Cauchy-Riemann equations hold is not alone sufficient to ensure holomorphicity. For example, we need existence of the first order partial derivatives of u and v in the given domain, as well as their continuity.

For a holomorphic function $f : A \to \mathbb{C}$ in a non-empty open set $A \subset \mathbb{C}$, defined by $f(z) = u(x, y) + iv(x, y)$, differentiation of f with respect to z gives

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big) .$$

Applying Cauchy-Riemanns' equations (2.3) to this expression yields the following result for all holomorphic functions.

Proposition 2.15. *If $f : A \to \mathbb{C}$ is a holomorphic function in a non-empty open set $A \subset \mathbb{C}$, then*

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \ .$$

Now, we have sorted out some basic theory for holomorphic functions and turn to the concept of continuously homotopic curves and the domains in which these so called "deformable" curves exist.

Definition 2.16. Let $D \subset \mathbb{C}$ be an open, connected set. A closed, smooth curve $\gamma_0 \subset D$ is *continuously homotopic* to the closed, smooth curve $\gamma_1 \subset D$ if there exists a continuous function $z : [0, 1] \times [0, 1] \to D$ such that

I. For every fixed $s \in [0, 1]$,

$$z_s(t) = z(s, t)$$

is a parametrization of a closed, piecewise smooth curve in D.

II. The function $z_0(t) = z(0, t)$ is a parametrization of γ_0.

III. The function $z_1(t) = z(1, t)$ is a parametrization of γ_1.

A closed, smooth curve γ_0 is also continuously homotopic to a curve γ_1 in a given domain if, in the domain, γ_0 is continuously homotopic to a single point and γ_1 is also continuously homotopic to the same point.

Definition 2.17. Let $D \subset \mathbb{C}$ be an open, connected domain. If every closed, piecewise smooth curve in D is continuously homotopic to a point, then D is said to be *simply-connected.*

In practice, the concept of continuously homotopic curves is of great importance to the theory of holomorphic functions since it lays ground for complex integration and path independence in simply-connected domains. We will return to this in detail when considering complex curve integrals in section 4.

3. The existence of the Riemann-Stieltjes integral

The purpose of this section is to show the existence of the Riemann-Stieltjes integral. We follow the method from Hille in [8], using Riemann-Stieltjes sums to show existence of an element J which can be approximated to the Riemann-Stieltjes integral. This as opposed to the more common method of using upper and lower Riemann-Stieltjes sums and defining the Riemann-Stieltjes integral as the value where the infimum of the upper sum and the supremum of the lower sum coincide. Hille's method proves much technically pretty and provides a clearer overview of the proof, and we therefore consider it favourable.

Before we begin, we need to introduce some required conditions for the existence of the Riemann-Stieltjes integral, such as the domain and functions involved, as well as define the Riemann-Stieltjes sums with which we are to approximate our integral. We have chosen to show the existence of the Riemann-Stieltjes integral for functions defined on a non-empty open subset of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. This since existence may then be proved by the same method for both real and complex valued functions.

Consider a function $g : [a, b] \to \mathbb{K}$ of bounded variation. For every partition $\pi = \{t_k\}$ as defined in Definition 2.1, let $L_\pi(g)$ be a sum corresponding to π and g such that

$$L_\pi(g) = \sum_{k=1}^{n} |g(t_k) - g(t_{k-1})| \ .$$

since g is a function of bounded variation, there is a constant L such that

$$L_\pi(g) \le L \ .$$

Since each sum $L_\pi(g)$ is bounded, the set of sums $\{L_\pi(g)\}$ will also be bounded, and Definition 2.2 gives the total variation of g on $[a, b]$ as

$$V_a^b(g) = \sup_\pi L_\pi(g) \ ,$$

where g is of bounded variation. For a point $c \in [a, b]$ such that $a < c < b$, Theorem 2.3 gives the following property for the total variation of g

$$V_a^b(g) = V_a^c(g) + V_c^b(g) \ .$$

For the continuous function $f : [a, b] \to \mathbb{K}$, the notation of modulus of continuity applies

$$\mu(\delta, f) = \max\left\{|f(t_1) - f(t_2)| : t_1, t_2 \in [a, b], |t_1 - t_2| \le \delta\right\} \ . \tag{3.1}$$

For $\delta > 0$ we have that $\mu(\delta, f)$ is a monotonically increasing function satisfying

$$\lim_{\delta \to 0^+} \mu(\delta, f) = 0$$

with the exception when f is constant and $\mu(\delta, f) = 0$. This follows from the evaluation of the limit where we have $f(t_1) - f(t_2)$ with $|t_1 - t_2|$ tending towards zero.

For a partition $\pi = \{t_k\}$, consider a point s_k in every subinterval $[t_{k-1}, t_k]$ of $[a, b]$. Let $\sigma = \{s_k\}$ be the set of all points $s_k \in [t_{k-1}, t_k]$, $k = 1, 2, ..., n$.

Definition 3.1. Let π be a partition of an interval $[a, b]$, and let $\sigma = \{s_k\}$ be a point set where $s_k \in [t_{k-1}, t_k] \subseteq [a, b]$, $k = 1, 2, ..., n$. We define the

Riemann-Stieltjes sum corresponding to π, σ, f and g as

$$S_{\pi,\sigma}(f,g) = \sum_{k=1}^{n} f(s_k)\big(g(t_k) - g(t_{k-1})\big) \ ,$$

where $f : [a,b] \to \mathbb{K}$ is a continuous function and $g : [a,b] \to \mathbb{K}$ is a function of bounded variation. The norm of the partition π is given by

$$\|\pi\| = \max_k (t_k - t_{k-1}) \ .$$

Now, we want to prove the following theorem for Riemann-Stieltjes sums.

Theorem 3.2. *Let $[a,b]$ be an interval and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $f : [a,b] \to \mathbb{K}$ be a continuous function and $g : [a,b] \to \mathbb{K}$ a function of bounded variation. There exists exactly one element $J \in \mathbb{R}$ such that*

$$|J - S_{\pi,\sigma}(f,g)| \leq 2\mu(\delta, f) V_a^b(g) \ , \tag{3.2}$$

for any $\delta > 0$ such that $\|\pi\| \leq \delta$. Here, $S_{\pi,\sigma}(f,g)$ is a Riemann-Stieltjes sum, $V_b^a(g)$ is the total variation of g, and $\mu(\delta, f)$ is the modulus of continuity.

Proof. This proof is based on evaluating the distance between two Riemann-Stieltjes sums. The value on the right hand side in equation (3.2) is reached by alternating the partition corresponding to a Riemann-Stieltjes sum. We begin with a fixed partition, then refine it, and then create a relation between the partition and its refinement. The final step of defining the element J is made by turning to the world of topology and Cantor's intersection theorem (Theorem 2.11).

First, consider the left hand side of inequality (3.2) for two arbitrary Riemann-Stieltjes sums with a fixed partition on the given interval. Let $\pi = \{t_k\}_{k=1}^n$ be a partition of the interval $[a,b]$, and let $\|\pi\| \leq \delta$, for $\delta > 0$. For every subinterval $[t_{k-1}, t_k]$ of $[a,b]$, consider two arbitrary points $s_{k,1}$ and $s_{k,2}$ and form the sets $\sigma_1 = \{s_{k,1}\}$ and $\sigma_2 = \{s_{k,2}\}$, $k = 1, 2, ..., n$. Now we can estimate the distance between these two sums

$$\begin{aligned}
|S_{\pi,\sigma_1}(f,g) - S_{\pi,\sigma_2}(f,g)| &= \Big| \sum_{k=1}^{n} \big(f(s_{k,1}) - f(s_{k,2})\big)\big(g(t_k) - g(t_{k-1})\big) \Big| \\
&\leq \sum_{k=1}^{n} |f(s_{k,1}) - f(s_{k,2})||g(t_k) - g(t_{k-1})| \ .
\end{aligned}$$

Since $\|\pi\| \leq \delta$ is given, we have that $|s_{k,1} - s_{k,2}| \leq \delta$ for every $k = 1, 2, ..., n$ and therefore the inequality $|f(s_{k,1}) - f(s_{k,2})| \leq \mu(\delta, f)$ holds. For $k = 1, 2, ..., n$ we thus have

$$\begin{aligned}
|S_{\pi,\sigma_1}(f,g) - S_{\pi,\sigma_2}(f,g)| &\leq \mu(\delta, f) \sum_{k=1}^{n} |g(t_k) - g(t_{k-1})| \\
&\leq \mu(\delta, f) V_a^b(g) \ .
\end{aligned} \tag{3.3}$$

Secondly, consider two partitions π_1 and π_2 of $[a,b]$ such that $\|\pi_1\| \leq \delta$ and π_2 is set as the total refinement of π_1. Let $\sigma_1 = \{s_{k,1}\}$ be a point set to the Riemann-Stieltjes sum $S_{\pi_1,\sigma_1}(f,g)$ constructed as before with the corresponding partition π_1. Take a subinterval $[t_0, t_1]$ of $[a, b]$ where $t_0, t_1 \in \pi_1$. The first term of $S_{\pi_1,\sigma_1}(f, g)$ will be

$$T_1 = f(s_1)\big(g(t_1) - g(t_0)\big) \ .$$

Now, partition $[t_0, t_1]$ into m subintervals

$$t_0 = t'_0 < t'_1 < ... < t'_m = t_1 \ .$$

T_1 can then be written as

$$T_1 = \sum_{j=1}^{m} f(s_1)\big(g(t'_j) - g(t'_{j-1})\big) \ .$$

We can choose the same points t'_j, $j = 1, 2, ..., m$, in π_2 and write

$$T_2 = \sum_{j=1}^{m} f(s'_j)\big(g(t'_j) - g(t'_{j-1})\big)$$

for the part of $S_{\pi_2,\sigma_2}(f, g)$ over $[t_0, t_1]$. Here, the $s'_j \in [t'_{j-1}, t'_j]$ are arbitrary.

With this we can write

$$\begin{aligned} |T_1 - T_2| &= \Big| \sum_{j=1}^{m} \big(f(s_1) - f(s'_j)\big)\big(g(t'_j) - g(t'_{j-1})\big) \Big| \\ &\le \sum_{j=1}^{m} |f(s_1) - f(s'_j)||g(t'_j) - g(t'_{j-1})| \ . \end{aligned}$$

Since $\|\pi_1\| \le \delta$, $|s_1 - s'_j| \le \delta$ will follow and therefore $|f(s_1) - f(s'_j)| \le \mu(\delta, f)$ so that

$$|T_1 - T_2| \le \mu(\delta, f) \sum_{j=1}^{m} |g(t'_j) - g(t'_{j-1})| \le \mu(\delta, f) V_{t_0}^{t_1}(g) \ . \tag{3.4}$$

If this is repeated for every subinterval $[t_{k-1}, t_k]$ of $[a, b]$, $k = 1, 2, ..., n$, equation (3.4) yields

$$|S_{\pi_1,\sigma_1}(f, g) - S_{\pi_2,\sigma_2}(f, g)| \le \mu(\delta, f) \sum_{k=1}^{n} V_{t_{k-1}}^{t_k}(g) = \mu(\delta, f) V_a^b(g) \ . \tag{3.5}$$

Now, consider two arbitrary Riemann-Stieltjes sums to the partitions π_1 and π_2 with the corresponding point sets σ_1 and σ_2. Let the union of the sets of division points of π_1 and π_2 define a partition π_3, which will be a refinement of both π_1 and π_2. Independently consider an arbitrary point set σ_3 with one point in between every pair of consecutive points in π_3. From equation (3.5) we get

$$\begin{aligned} &|S_{\pi_1,\sigma_1}(f, g) - S_{\pi_2,\sigma_2}(f, g)| = \\ &\qquad = |S_{\pi_1,\sigma_1}(f, g) - S_{\pi_3,\sigma_3}(f, g) + S_{\pi_3,\sigma_3}(f, g) - S_{\pi_2,\sigma_2}(f, g)| \\ &\qquad \le |S_{\pi_1,\sigma_1}(f, g) - S_{\pi_3,\sigma_3}(f, g)| + |S_{\pi_2,\sigma_2}(f, g) - S_{\pi_3,\sigma_3}(f, g)| \\ &\qquad \le \mu(\delta, f) V_a^b(g) + \mu(\delta, f) V_a^b(g) = 2\mu(\delta, f) V_a^b(g) \ . \end{aligned} \tag{3.6}$$

Finally, for $\delta > 0$, we define $H(\delta)$ as the closure of the set of our Riemann-Stieltjes sums $S_{\pi,\sigma}(f, g)$ with $\|\pi\| \le \delta$, i.e.

$$H(\delta) = \overline{\{S_{\pi,\sigma}(f, g) : \|\pi\| \le \delta\}} \ .$$

It then follows from equation (3.6) that the distance between any two Riemann-Stieltjes sums in $H(\delta)$ is

$$d(H(\delta)) \leq 2\mu(\delta, f)V_a^b(g) \ .$$

Since $\mu(\delta, f)$ is a positive, monotone increasing function for all non-constant functions f, which tends to zero with δ, we have that $H(\delta)$ will be a non-empty subset of $\mathbb{K}$ and

$$\lim_{\delta \to 0^+} d(H(\delta)) = 0 \ .$$

We now consider Cantor's intersection theorem (Theorem 2.11) which says that for the complete metric space X, a decreasing sequence $\{F_n\}$ of non-empty, closed subsets of X, satisfying the limit

$$\lim_{n \to \infty} d(F_n) = 0$$

we have that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one element.

$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a complete metric space due to the completeness of both $\mathbb{R}$ and $\mathbb{C}$, and $H(\delta)$ is a closed, non-empty subset of $\mathbb{K}$ such that $d(H(\delta)) \to 0$ as δ tends towards zero. Set $H_n = H(\frac{1}{n})$. We can then apply Theorem 2.11 which says that there exists exactly one element J in $\bigcap_{n=1}^{\infty} H_n$ that satisfies the inequality

$$|J - S_{\pi,\sigma}(f, g)| \leq 2\mu(\delta, f)V_b^a(g) \ .$$

□

Definition 3.3. Let $[a, b]$ be an interval. Let $f : [a, b] \to \mathbb{K}$ be a continuous function and let $g : [a, b] \to \mathbb{K}$ be a function of bounded variation. We define the element $J \in \mathbb{R}$ in

$$|J - S_{\pi,\sigma}(f, g)| \leq 2\mu(\delta, f)V_b^a(g) \ ,$$

for any $\delta > 0$ such that $\|\pi\| \leq \delta$, as the *Riemann-Stieltjes integral* of f with respect to g over the interval $[a, b]$ and denote it by

$$J = \int_a^b f(t)dg(t) \ . \tag{3.7}$$

The main properties of the integral in equation (3.7) are presented in the following theorem.

Theorem 3.4. *Let $A \subset \mathbb{K}$ be a non-empty open set and let $[a, b]$ be an interval. For a continuous function $f : A \to \mathbb{K}$ and a function $g : [a, b] \to A$ of bounded variation, the following properties for the Riemann-Stieltjes integral of f with respect to g apply:*

I. *The integral is linear both with respect to the integrand f and with respect to the integrator g such that*

$$\int_a^b [\alpha f_1(t) + \beta f_2(t)]dg(t) = \alpha \int_a^b f_1(t)dg(t) + \beta \int_a^b f_2(t)dg(t) \ ,$$

$$\int_a^b f(t)d[\alpha g_1(t) + \beta g_2(t)] = \alpha \int_a^b f(t)dg_1(t) + \beta \int_a^b f(t)dg_2(t) \ ,$$

where $\alpha, \beta \in \mathbb{K}$.

II. *The integral is additive with respect to the interval of integration, that is for $a < c < b$*

$$\int_a^b f(t)dg(t) = \int_a^c f(t)dg(t) + \int_c^b f(t)dg(t) \ .$$

III. *The Riemann-Stieltjes integral can be estimated by the following inequality:*

$$\left| \int_a^b f(t)dg(t) \right| \leq \max_{t \in [a,b]} |f(t)| \cdot V_a^b(g) \ . \tag{3.8}$$

IV. *If the integrator g has a continuous derivative g', the following equality applies*

$$\int_a^b f(t)dg(t) = \int_a^b f(t)g'(t)dt \ .$$

where the integral on the right hand side is an ordinary Riemann integral.

Properties (I)-(III) in Theorem 3.4 follow directly from the definition of the Riemann-Stieltjes integral as a limit of sums. For further details on proofs of these properties we refer to Krantz [11], and for details on the proof of property (IV) we refer to Wikström [18].

Before turning to some applications of the Riemann-Stieltjes integral, note that since Definition 3.3 is valid for both complex and real valued functions, the properties of Theorem 3.4 will apply to both complex and real valued functions f and g.

4. Some applications in complex analysis

The purpose of this section is to define the curve integral in the complex plane, and the main focus will then be Cauchy's great integral theorem. We begin by considering the complex curve integral, which is actually a special case of the Riemann-Stieltjes integral. Whilst the Riemann-Stieltjes integral simply requires that the parametrization function which generate a curve for integration is of bounded variation, the complex curve integral also requires that the parametrization function is continuous.

After defining the complex curve integral, we turn to Cauchy's original work on complex integration leading up to his complete final proof of the integral theorem. We then present a general version of the theorem along with two modern proofs, one with applied real analysis and one with a homotopy approach.

Let $A \subset \mathbb{C}$ be a non-empty open set. Consider a continuous function of bounded variation $z : [a, b] \to A$. Let z be a parametrization which happens to generate an oriented curve γ in the complex plane,

$$\gamma = \{z = z(t) : t \in [a, b]\} \ .$$

If $f : A \to \mathbb{C}$ is a continuous function, then $(f \circ z) : [a, b] \to A$ will be continuous on $[a, b]$. The Riemann-Stieltjes integral of $(f \circ z)$ with respect to z then exists according to Theorem 3.2, and we define the curve integral in the complex plane as the Riemann-Stieltjes integral.

Definition 4.1. Let $z : [a, b] \to A$ be a continuous parametrization function of bounded variation, which generates a curve γ on $[a, b]$. Furthermore, let $A \subset \mathbb{C}$ be a non-empty open set and let $f : A \to \mathbb{C}$ be a continuous function. Then $(f \circ z) : [a, b] \to \mathbb{C}$ is a continuous function and we define the *curve integral in the complex plane* as the Riemann-Stieltjes integral of $(f \circ z)$ with respect to z

$$\int_\gamma f(z)dz = \int_a^b (f \circ z)(t)dz(t) \ .$$

This integral will be completely independent of the chosen parametrization. According to [8] and [11], properties for the complex curve integral follow without change from the corresponding properties in Theorem 3.4 for the Riemann-Stieltjes integral. We note however, that an estimation of the complex curve integral as in equation (3.8), the length of the curve $l(\gamma)$ is used instead of the total variation $V_a^b(g)$.

Now, we turn back in time to consider the actual work of the French mathematician Augustin-Louis Cauchy who first presented the basic idea to his integral theorem in *Mémoire sur la théorie des intégrales définies* from his 1814 memoires [10]. There, he treated relations between first order partial derivatives of real valued functions, and considered the simple case of complex integration along the boundary of a rectangle. In *Mèmoire sur les intégrales définies, prises entre des limites imaginaires* [3] of his 1825 memoires he continued to develop this work on a definite integral between complex limits

$$\int_{x_0+iy_0}^{X+iY} f(z)dz \ , \tag{4.1}$$

where $[x_0, y_0]$ and $[X, Y]$ were the boundary points of the rectangle around which integration was considered. He studied the two line integrals obtained by separating the real and imaginary parts in his definite integral and eventually reached the following expressions

$$\int_{x_0}^{X} f(x+iy_0)dx + i\int_{y_0}^{Y} f(X+iy)dy$$

$$\int_{x_0}^{X} f(x+iY)dx + i\int_{y_0}^{Y} f(x_0+iy)dy \ ,$$

which he called the extreme values of the integral in equation (4.1). Cauchy stated that equality between these extreme values would occur as long as f was finite for all values of x and y on and within any chosen boundaries between the points (x_0, y_0) and (X, Y). If opposite occurred, that f had singularities for some value of x and y on the boundaries, the difference between the extreme values would be infinite or indefinite. The case when f was finite for all x and y on the boundaries, but might have singularities within the boundaries, would generate a finite difference between the extreme values which Cauchy denoted by $\triangle$. To summarize

$$\int_{x_0}^{X} f(x+iy_0)dx+i\int_{y_0}^{Y} f(X+iy)dy = \int_{x_0}^{X} f(x+iY)dx+i\int_{y_0}^{Y} f(x_0+iy)dy+\triangle \ ,$$

where $\triangle$ represented the result of possible singularities of f within the specified boundaries. Cauchy's $\triangle$-notation corresponds to what we today call residues and denote by $2\pi i\sum_{k=1}^{n} \operatorname{Res}(z_k)$, z_k being singularities of f, with $k = 1, 2, ..., n$ for some n.

Cauchy continued by investigating what happened when letting $f(x+iy)$ vanish when sending each of the boundary values of the rectangle towards ±infinity. He finally reached that $\triangle = 0$, which implied that his definite integral would ultimately equal zero. However, his method had certain limitations. Cauchy's assumption that the integrals would vanish with their integrand functions and that he could therefore evaluate limits under the integral sign was considered true in general, but there existed cases where the opposite occurred, which required certain modifications to the integrand functions. So basically, Cauchy's proof in [3] was insufficient since his results did not apply for all cases.

> *nous avons supposé que les intégrales comprises dans cette formule s'èvanouissaint avec les fonctions qu'elles renferment. C'est ce qui a lieu en géneral. Néanmoins le contraire arrive dans un petit nombre de cas particuliers, et alors les équations que nous avons établies dovient être modifiées.*

However, in *Sur les intégrales qui s'étendent à tous les points d'une courbe fermée* [4] from 1847, he proved some remarkable theorems that provides the necessary conditions for the existence of definite integrals of complex functions along closed curves, allowing him to provide a complete proof of his integral theorem. The results are presented in Theorem 4.2 below. In order to preserve a sense of originality to Cauchy's work, great effort has been made to hold to Cauchy's original notations and peculiar language in the translation from French.

Theorem 4.2.

I. *The position of a variable point P is determined by the rectangular or polar coordinates, or coordinates of other nature,*

$$x, y, z, ...$$

which vary continuously along with the position of the point. Let S be an area measured in a given plane or on a given surface, and let S be bounded by a single closed curve. Let the point P travel around the curve bounding S in a given direction, and set s as this positively oriented arc. Finally, let k be a function of the variables $x, y, z, ...$ with derivatives relative to s, and set (S) as the value acquired by the integral

$$\int k ds$$

when the variable point P has travelled one revolution along the curve s around the area S. If by straight lines or curves on a plane or on a given surface, the area S is divided into subareas

$$A, B, C, ...$$

such that

$$(A), (B), (C), ...$$

are the corresponding (S) to S for each subarea A, or B, or C, ..., we get

$$S = A + B + C + ...$$

and furthermore

$$(S) = (A) + (B) + (C) + ...$$

in which the function k is defined as finite and continuous on every point of every contour.

II. *Let the conditions from part (I) apply, then set*

$$k = XD_s x + YD_s y + ZD_s z + ...$$

where $X, Y, Z, ...$ are functions of $x, y, z, ...$ chosen such that the sum

$$Xdx + Ydy + Zdz + ...$$

is an exact differential, and let the surface S vary with varying insensible degrees of the curve which bounds S. Such changes will not affect the value of the integral (S) when k is finite and continuous at each successive point on the curve.

III. *Let the conditions from part (II) apply. Suppose that the function k ceases to be finite and continuous at the single points*

$$P', P'', P''', ...$$

situated in the interior of S. Let $a, b, c, ...$ be very small area elements of S, each enclosing one of the points of discontinuity. Then the equation

$$(S) = (a) + (b) + (c) + ...$$

provides an expression for the integral (S) as a sum of single integrals over each of the area elements $a, b, c, ...$ In the particular case when k

is finite and continuous for all points in the interior of S, the single integrals will vanish simply leaving

$$(S) = 0 \ .$$

The final result in (III) of Theorem 4.2 corresponds to Cauchy's integral theorem, and his original proof is presented below.

Proof of Theorem 4.2 (III). Let S be a planar surface. Then x and y are reduced to two rectangular or polar coordinates, or coordinates of other nature, which determine the position of a point in the plane of the surface S. If we let X and Y be two continuous functions of x and y, and if we assume

$$k = XD_s x + YD_s y \ ,$$

then we have

$$(S) = \pm \int \int (D_y X - D_x Y) \, dxdy \ ,$$

where the double integral extends to all points of the surface S. Let x and y be two measurable lengths from an arbitrary point P which corresponds to the coordinates x and y, directing the lengths in a manner to obtain positively increasing coordinates. The $\pm$ sign is determined by considering if the orientation of rotation around S generates an increasing s or not.

In the particular case where the sum

$$Xdx + Ydy$$

is an exact differential, we get

$$D_y X = D_x Y \ ,$$

and the equation which determines the value of (S) is reduced to the simple case of

$$(S) = 0 \ .$$

□

Cauchy's method includes separating the real and imaginary parts of the definite integral in equation (4.1), and considering a simple closed curve with positive orientation as his path of integration. He considers the particular case when $k = Xdx + Ydy + Zdz + ...$ is an exact differential, which means that $(X, Y, Z, ...)$ represents a conservative vector field with potential $\int kds$, where the integral is independent of the path of integration. Thus, Cauchy deals with a simply-connected domain. No real argument is given for why he is able to rewrite the definite integral along a simple closed curve as a double integral over the area enclosed by the curve. The justification however, lies in the fact that k is made up of continuous functions in a bounded domain.

The notations Cauchy uses to express derivatives, $D_s x$ and $D_s y$, refer to the total differentials $\frac{dx}{ds}$ and $\frac{dy}{ds}$ respectively. He considers total differentials throughout Theorem 4.2 and makes no comment on the difference between total and partial differentials. From now on, we will however distinguish between total and partials differentials such as $\frac{d}{ds}$ and $\frac{\partial}{\partial s}$ respectively.

Now, let us turn to the general version of Cauchy's integral theorem.

Theorem 4.3 (Cauchy's integral theorem)**.** *Let D be a simply-connected domain in the complex plane, and let $f : D \to \mathbb{C}$ be a holomorphic function. Then for any simple, closed curve $\gamma \subset D$ generated by a continuously differentiable parametrization we have that*

$$\int_\gamma f(z)dz = 0 \ . \tag{4.2}$$

The real analysis approach to this theorem is based on Cauchy's method of rewriting the curve integral along a closed curve into a double integral over the area enclosed by the chosen curve. Therefore, we introduce Green's theorem. This is done in full awareness that it requires the additional condition that the integrand function f has a continuous first order derivative. We are also aware of the discussions leading up to Goursat's more general proof, but will stick to the straightforward proof that Green's theorem entails.

Let $f : D \to \mathbb{C}$ be a holomorphic function and set $f(z) = u(x, y) + iv(x, y)$, separating the real and imaginary parts of f. The definite curve integral of f along any closed curve $\gamma \subset D$, generated by a continuously differentiable parametrization can then be written as

$$\int_\gamma f(z)dz = \int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy) \ .$$

Here $u, v : \tilde{D} \to \mathbb{R}$ are two harmonic functions which satisfy Cauchy-Riemanns' equations, and $\tilde{D} \subset \mathbb{R}^2$ is the corresponding set of D when $\mathbb{C}$ is identified with $\mathbb{R}^2$ chosen so that $(x, y) \in \tilde{D}$ if and only if $z = x + iy \in D$.

Theorem 4.4 (Green's Theorem)**.** *Let $D \subset \mathbb{C}$ be a simply-connected domain and let a continuously differentiable parametrization generate a positively oriented, simple closed, piecewise smooth curve γ in D. Let $u(x, y)$ and $v(x, y)$ be two continuous real-valued functions with continuous first order partial derivatives. Then,*

$$\int_\gamma \big(u(x, y)dx + v(x, y)dy\big) = \int\int_{A_\gamma} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy \ , \tag{4.3}$$

where A_γ is a regular region in D corresponding to the area enclosed by γ.

Proof. Since A_γ is a regular region, it is both x- and y-simple. Let A_γ be defined by $a \le x \le b$ and $y_1(x) \le y \le y_2(x)$, where $y = y_1(x)$ is oriented from left to right and $y = y_2(x)$ is oriented from right to left. Consider the last term in the integral of equation (4.3). Since $u(x, y)$ is continuous on a bounded regular domain we can rewrite this term as

$$-\int\int_{A_\gamma} \frac{\partial u}{\partial y} dxdy = -\int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial u}{\partial y} dy = \int_a^b \Big(-u(x, y_2(x)) + u(x, y_1(x))\Big) dx \ .$$

Also, $dx = 0$ on the vertical axes of A_γ, so

$$\int_\gamma u(x, y)dx = \int_a^b \big(u(x, y_1(x)) - u(x, y_2(x))\big) dx$$

Similarly,

$$\int_\gamma v(x, y)dy = \int\int_{A_\gamma} \frac{\partial v}{\partial x} dxdy \ .$$

Combining these two results completes the proof. □

Now, we are ready to compute a real analysis proof of Theorem 4.3.

Real analysis proof of Cauchy's integral theorem (Theorem 4.3). Let $D \subset \mathbb{C}$ be a simply-connected domain and $f : D \to \mathbb{C}$ a holomorphic function. Assume that f has a continuous first order derivative and set $f(z) = u(x,y) + iv(x,y)$. By Theorem 4.4 the real and imaginary parts of the definite curve integral over the simple closed, continuously differentiable curve γ can be rewritten as double integrals over the area enclosed by γ

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma \big(udx - vdy\big) + i\int_\gamma \big(vdx + udy\big) \\ &= \int\int_{A_\gamma} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\int\int_{A_\gamma} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \ .\end{aligned} \tag{4.4}$$

Since f is holomorphic, Cauchy-Riemanns' equations (2.3) apply. Equation (4.4) thus becomes

$$\int\int_{A_\gamma} \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dxdy + i\int\int_{A_\gamma} \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dxdy = 0 \ .$$

□

The third proof of Cauchy's integral theorem presented in this essay is one based on the idea of a proof in a publication from 1971 by Dixon [5] together with a publication with comments on Dixon's proof by Loeb [12]. It is a proof which combines local properties for holomorphic functions with integration along simple closed curves generated by a continuously differentiable parametrization. We will begin by presenting required conditions for this homotopy proof in three lemmas treating path independence and conditions of holomorphicity for integrals. We also introduce Cauchy's integral formula as a main argument behind the proof.

We begin by showing that the value of the integral in Theorem 4.3 does not depend on the curve that happens to be generated by our chosen parametrization.

Lemma 4.5 (Path independence)**.** *Let $D \subset \mathbb{C}$ be an open, connected domain and let a continuously differentiable parametrization $\gamma : [a,b] \to D$ generate a smooth curve, also denoted by γ. If f and F are holomorphic functions from D to $\mathbb{C}$ such that $F' = f$, then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) \ .$$

Proof. Let γ be defined by $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$, $t \in [a,b]$. Since γ is a smooth curve it has the properties of being continuous on the interval $[a,b]$ as well as having a continuous derivative γ' on $[a,b]$. That is, the boundary limits

$$\lim_{t\to a^+} \gamma'(t) \text{ and } \lim_{t\to b^-} \gamma'(t)$$

both exist. For a small real valued constant $\epsilon > 0$ we can therefore write

$$\gamma(b) - \gamma(a) = \lim_{\epsilon\to 0^+} \big(\gamma(b-\epsilon) - \gamma(a+\epsilon)\big) = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b-\epsilon} \gamma'(t)dt = \int_a^b \gamma'(t)dt \ . \tag{4.5}$$

Let $\tilde{D} \subset \mathbb{R}^2$ be the corresponding set of D when $\mathbb{C}$ is identified with $\mathbb{R}^2$. For a continuously differentiable function $g : \tilde{D} \to \mathbb{R}$ we have that

$$(g \circ \gamma)'(t) = \frac{\partial g}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial g}{\partial y}(\gamma(t))\frac{d\gamma_2}{dt}$$

and by equation (4.5)

$$g(\gamma(b)) - g(\gamma(a)) = \int_a^b (g \circ \gamma)'(t)dt = \int_a^b \left(\frac{\partial g}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial g}{\partial y}(\gamma(t))\frac{d\gamma_2}{dt} \right) dt \ .$$

Now, set $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, where $u, v : \tilde{D} \to \mathbb{R}$. Since f is holomorphic, u and v will satisfy Cauchy-Riemanns' equations. Thus,

$$\begin{aligned}
&F(\gamma(b)) - F(\gamma(a)) = \\
&= \int_a^b \left(\frac{\partial u}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + i\frac{\partial v}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial u}{\partial y}(\gamma(t))\frac{d\gamma_2}{dt} + i\frac{\partial v}{\partial y}(\gamma(t))\frac{d\gamma_2}{dt} \right) dt \\
&= \int_a^b \left(\left(\frac{\partial u}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} - \frac{\partial v}{\partial x}(\gamma(t))\frac{d\gamma_2}{dt} \right) + i\left(\frac{\partial v}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial u}{\partial x}(\gamma(t))\frac{d\gamma_2}{dt} \right) \right) dt \\
&= \int_a^b \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right) \left(\frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt} \right) dt = \int_a^b \frac{\partial F}{\partial x}(\gamma(t))\gamma'(t)dt \ .
\end{aligned}$$

Since F is a holomorphic function we have that $\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x}$ by Proposition 2.15. Therefore, for the primitive function F to f we get

$$F(\gamma(b)) - F(\gamma(a)) = \int_a^b (f \circ \gamma)(t)\gamma'(t)dt \ .$$

From Theorem 3.4 (IV) we have the following property for the complex curve integral

$$\int_\gamma f(z)dz = \int_a^b (f \circ \gamma)(t)\gamma'(t)dt \ .$$

Therefore we finally get that

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) \ .$$

□

Since the curve integral has the property of being additive with respect to path, we can extend the result to piecewise smooth curves. In this case, let the curve γ be a piecewise smooth curve, made up by several smooth curves $\gamma = \gamma_0 + \gamma_1 + ... + \gamma_n$. We get

$$\int_\gamma F(z)dz = \sum_{j=1}^{n} \int_{\gamma_j} F(z)dz = \sum_{j=1}^{n} \big(f(\gamma(z_j)) - f(\gamma(z_{j-1})) \big) = f(\gamma(b)) - f(\gamma(a)) \ ,$$

where z_{j-1} and z_j are the initial and terminal points respectively corresponding to γ_j, $j = 0, 1, ..., n$. This implies that the complex curve integral is independent of any chosen piecewise smooth curve between the points a and b.

The homotopy proof that we will present in this essay is based on so called local Cauchy theory. That is, if the integrand f is holomorphic in a specified domain D and a continuously differentiable parametrization ζ generates a closed curve $\gamma \subset D$, then the integral

$$\int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta$$

is a holomorphic function of z on the whole of D. We specify these conditions in Lemma 4.5.

Lemma 4.6. *Let $D \subset \mathbb{C}$ be a simply-connected domain and let $f : D \to \mathbb{C}$ be a holomorphic function. Furthermore, let φ be the mapping $D \times D \to \mathbb{C}$ defined by*

$$\varphi(\zeta, z) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{if } \zeta \neq z \\ f'(z) & \text{if } \zeta = z \end{cases} \tag{4.6}$$

such that $\varphi(\cdot, z)$ is holomorphic on D for each point $z \in D$. Set $g(z) = \int_\gamma \varphi(\zeta, z) d\zeta$. Then g is holomorphic on D.

Proof. Since f is holomorphic in D the function $\varphi(\cdot, z)$ will, for each $z \in D$, be continuous at z and holomorphic everywhere else in D. If γ is a continuous, piecewise smooth, closed curve in D and $z \notin \gamma$ then

$$g(z) = \int_\gamma \varphi(\zeta, z) d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) \int_\gamma \frac{1}{\zeta - z} d\zeta \ .$$

It follows from the properties of φ that g is holomorphic everywhere in D except on γ. Let z be a fixed point on γ. By Lemma 4.5 we can replace γ with another continuous, piecewise smooth, closed curve Υ such that $z \notin \Upsilon$ without changing the values of g in a surrounding to z. Then g will be holomorphic in the surrounding to z and thus on the whole of D. □

Integrals of the above form are the backbone of one of the main arguments of the homotopy proof of Theorem 4.3. Therefore, we take a closer look at these integrals.

Lemma 4.7. *Let $D \subset \mathbb{C}$ be a simply-connected domain. Furthermore, let a closed curve $C \subset D$ be generated by the continuously differentiable parametrization $\zeta(\theta) = z + re^{i\theta}$, with $r > 0$ and $\theta \in [0, 2\pi]$. Then for $n \in \mathbb{Z}$*

$$\int_C (\zeta - z)^n d\zeta = \begin{cases} 2\pi i & \text{when } n = -1 \\ 0 & \text{when } n \neq -1 \ . \end{cases}$$

Proof. The closed curve C will be the boundary of a disc centred about the point z, of radius $r > 0$. Now, we use the fact that $\zeta(\theta)$ is a differentiable function and apply property (IV) of the Riemann-Stieltjes integral of Theorem 3.4 to the integral

$$\int_C (\zeta - z)^n d\zeta = \int_0^{2\pi} (z + re^{i\theta} - z)^n ire^{i\theta} d\theta = r^{n+1} \int_0^{2\pi} ie^{i(n+1)\theta} d\theta \ .$$

The fact that $\frac{1}{n+1} e^{i(n+1)\theta}$ is a primitive function to $ie^{1(n+1)\theta}$ whrn $n \neq -1$ completes the proof. □

Lemma 4.7 establishes a result for integrals of $\frac{1}{\zeta - z}$ taken over the boundary of a disc centred about the point z. However, we also need to consider the case when evaluating this same integral over any simple, closed, smooth curve γ such that z can be a point either in the interior or exterior of γ.

Lemma 4.8. *Let $D \subset \mathbb{C}$ be a simply-connected domain and let $f : D \to \mathbb{C}$ be a holomorphic function. Let $\gamma \subset D$ be a simple closed, smooth curve generated by a continuously differentiable parametrization. Furthermore, let γ_i denote the interior of γ and γ_e denote its exterior. Then*

$$\int_\gamma \frac{1}{\zeta - z} d\zeta = \begin{cases} 2\pi i & \text{when } z \in \gamma_i \\ 0 & \text{when } z \in \gamma_e \ . \end{cases} \tag{4.7}$$

Proof. By Lemma 4.5 this integral is path independent and therefore the value of the integral will be the same whether taken over γ or over the boundary of a small circle $C \subset D$, positively oriented, with center at c, and with one point in common with γ. The integral can thus be expressed as

$$\int_C \frac{1}{\zeta - z} d\zeta \ ,$$

where C is a closed curve parametrized by $\zeta(\theta) = c + re^{i\theta}$, with $r > 0$ and $\theta \in [0, 2\pi]$. Just as for equation (4.7), the value of this integral will depend on if z lies in the interior or the exterior of C, denoted C_i and C_e respectively.

We evaluate the fraction term as a geometric series for $z \neq c$. For all $z \in C_i$ we will have $|z - c| < |\zeta - c|$ and

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - c} \sum_{n=0}^{\infty} \left(\frac{z - c}{\zeta - c} \right)^n .$$

Similarly, for all $z \in C_e$ we will have $|z - c| > |\zeta - c|$ which gives

$$\frac{1}{\zeta - z} = \frac{-1}{z - c} \sum_{0}^{\infty} \left(\frac{\zeta - c}{z - c} \right)^n .$$

For fixed c, r and $z \notin C$ these series are normally convergent in the variable ζ.

If $z \in C_i$ we have that

$$\int_C \frac{1}{\zeta - z} d\zeta = \int_C \frac{1}{\zeta - c} \sum_{n=0}^{\infty} \left(\frac{z - c}{\zeta - c} \right)^n d\zeta = \sum_{n=0}^{\infty} (z - c)^n \int_C \frac{1}{(\zeta - c)^{n+1}} d\zeta$$

where all the integrals on the right hand side will vanish by Lemma 4.7, except for $n = 0$ when the value will be $2\pi i$.

If $z \in C_e$

$$\int_C \frac{1}{\zeta - z} d\zeta = \int_C \frac{-1}{z - c} \sum_{n=0}^{\infty} \left(\frac{\zeta - c}{z - c} \right)^n d\zeta = -\sum_{n=0}^{\infty} \frac{1}{(z - c)^{n+1}} \int_C (\zeta - c)^n d\zeta \ .$$

Here, the integrals on the right hand side will vanish without exception by Lemma 4.7. □

Now, we have reached the main argument for the homotopy proof of Theorem 4.3.

Theorem 4.9 (Cauchy's integral formula)**.** *Let $D \subset \mathbb{C}$ be a simply-connected domain, and let $f : D \to \mathbb{C}$ be a holomorphic function. Furthermore, let $\gamma \subset D$ be a simple, closed, positively oriented curve generated by a continuously differentiable parametrization. Let γ_i denote the interior of γ. Then for all $z \in \gamma_i$*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \ . \tag{4.8}$$

Proof. The integrand is a holomorphic function of ζ on γ, except at $\zeta = z$. Let the point $z \in \gamma_i$ be given and fixed. Consider the function $\varphi(\zeta, z)$ defined as in equation (4.6) for $\zeta \in D \setminus \{z\}$. Furthermore, let Φ be a holomorphic function such that $\Phi' = \varphi$. Since γ is a closed curve, we will have $\gamma(a) = \gamma(b)$ on some interval $[a, b] \subset D$. Thus, by Lemma 4.5 we have

$$0 = \Phi(\gamma(b)) - \Phi(\gamma(a)) = \int_\gamma \varphi(\zeta, z) d\zeta = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \ ,$$

which gives

$$f(z) \int_\gamma \frac{1}{\zeta - z} d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \ . \tag{4.9}$$

Since $z \in \gamma_i$, the value of the integral on the left hand side in equation (4.9) is $2\pi i$ by Lemma 4.8. Thus,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \ .$$

□

It can be show inductively that repeated differentiations with respect to z under the integral sign in equation (4.8) yields formulas for successive derivatives of f. Under the same conditions as in Theorem 4.9, this gives a generalized Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{4.10}$$

with $n = 1, 2, 3, ...$ For a detailed proof we refer to Theorem 3.1.1 in [7].

As a final step before we present the homotopy proof of Theorem 4.3, we introduce two theorems containing important restrictions on the behaviour of holomorphic functions.

Theorem 4.10 (Cauchy estimates)**.** *Let f be holomorphic inside and on the positively oriented curve C given by $|\zeta - z_0| = r$, generated by a continuously differentiable parametrization. If $|f(\zeta)| \leq M$ for all $\zeta \in C$, then the derivatives of f at z_0 satisfy*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}$$

for $n = 1, 2, 3, ...$

Proof. By Cauchy's generalized integral formula, equation (4.10), we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \ .$$

For $\zeta \in C$, the assumption that $|f(\zeta)| \leq M$ gives

$$\left|\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\right| \leq \frac{M}{r^{n+1}} \ .$$

The length of the curve is $2\pi r$. Thus it follows from Theorem 3.4 (III) for the complex curve integral that

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta\right| \leq \frac{n!}{2\pi}\frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r^n} \ .$$

□

We conclude that if a holomorphic function is bounded in the complex plane by some number, its derivative will vanish, meaning that the function must be constant. This result is known as Liouville's theorem.

Theorem 4.11 (Liouville's theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. If there exists a positive number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is constant.*

Proof. Suppose that $|f|$ is bounded, that is that there is a constant $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By Theorem 4.10 we get a Cauchy estimate with $n = 1$ for all $z \in \mathbb{C}$

$$|f'(z)| \leq \frac{M}{r} \ .$$

If we evaluate the limit when $r \to \infty$ we get that

$$|f'(z)| = 0$$

since f is holomorphic and independent of r. Hence, we must have $f'(z) = 0$ for all $z \in \mathbb{C}$. That is, $f(z) = K$ for some constant $K \in \mathbb{C}$. □

We are now ready to present the homotopy proof of Theorem 4.3. The most significant difference between this proof and the real analysis one is that we here can set aside the requirement that the integrand function must have a continuous first order derivative.

Homotopy proof of Cauchy's integral theorem (Theorem 4.3). Let $D \subset \mathbb{C}$ be a simply-connected domain, and let φ be the mapping of $D \times D \to \mathbb{C}$ defined as in Lemma 4.6. Then $\varphi(\cdot, z)$ will be holomorphic for every fixed $z \in D$. Furthermore, let E be an open subset of $\mathbb{C}$ containing all points $z \in \gamma_e$. Then $D \cup E = \mathbb{C}$.

Now, define a holomorphic function g in $\mathbb{C}$ as

$$g_1(z) = \int_\gamma \varphi(\zeta, z) d\zeta$$

on D, and as

$$g_2(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

on E. Then, in $D \cap E$, the expressions for g will be equal, $g_1(z) = g_2(z)$. That is, for all $z \in D \cap E$

$$
\begin{aligned}
g_1(z) &= \int_\gamma \varphi(\zeta, z) d\zeta = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \\
&= \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) \int_\gamma \frac{1}{\zeta - z} d\zeta = g_2(z) \ ,
\end{aligned}
$$

since the last integral is zero with $z \in \gamma_e$ by Lemma 4.8.

Thus g is differentiable on both D and E and is thereby an entire function. Since the image of γ is bounded and E contains a neighbourhood of ∞, we get

$$\lim_{z \to \infty} g(z) = 0 \ .$$

Liouville's theorem (Theorem 4.11) therefore implies that g is constant, and furthermore we get that g is equal to zero. Thus, $\int_\gamma \varphi(\zeta, z) d\zeta = 0$ for all $\zeta \in D \setminus \gamma$. For $\zeta \neq z$ we then get

$$\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \int_\gamma \frac{1}{\zeta - z} d\zeta$$

and by Theorem 4.9

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \tag{4.11}$$

for all $z \in D \setminus \gamma$.

Now, for a fixed point $w \in D \setminus \gamma$, applying equation (4.11) to the function $F(\zeta) = f(\zeta)(\zeta - w)$ at $z = w$ then gives

$$
\begin{aligned}
\int_\gamma f(\zeta) d\zeta &= \int_\gamma \frac{f(\zeta)(\zeta - w)}{\zeta - w} d\zeta = \int_\gamma \frac{F(\zeta)}{\zeta - w} d\zeta = 2\pi i F(w) \\
&= 2\pi i f(w)(w - w) = 0 \ .
\end{aligned}
$$

□

5. Some applications in probability theory

This section is dedicated to random processes in probability theory and particularly to the expected value of random variables, which is an essential concept in summarizing important characteristics of probability distributions. We can think of the expected value as a weighted average of all possible values, and it is a location parameter. In this section we will use the Riemann-Stieltjes integral to define the expectation of a random variable and a function of several random variables. The main reason for defining the expected value with the Riemann-Stieltjes integral, instead of with an ordinary Riemann integral, is that it generates a general formula which holds for continuous and discrete random variables as well as for mixtures of the two. Thus it offers a more unified approach to the theory of random variables.

In Section 3 we proved the existence of the Riemann-Stieltjes integral for functions in a non-empty open subset of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In this section, we will begin by considering the real line as our sample space and then extend our results to real valued functions in n dimensions.

We begin by defining some important concepts of probability theory. Our base is a probability space which includes a set of all possible outcomes, called the outcome space Ω, and a probability measure P, which specifies the likelihood of an event occurring.

Definition 5.1. Given a random experiment with an outcome space Ω, we define a *random variable* X as a measurable function that assigns exactly one real number $X(\omega) = x$ to each element $\omega \in \Omega$.

A random variable can either be discrete, only assuming a countable number of real values, or continuous, theoretically assuming an infinite number of real values on finite intervals of Ω. A discrete random variable can for example be a roll of a pair of dice, and a continuous random variable could be the height of a person. We will throughout this section assume that every random variable is defined from the outcome space Ω to a given real dimension.

The probabilities of random variables are measured with the use of a distribution function.

Definition 5.2. For a random variable X we define the *cumulative distribution function* of X as

$$F_X(x) = P(X \leq x) \ .$$

$P(X \leq x)$ represents the probability of the event that the random variable X is equal to or no larger than a given value x. Similarly, the probability of the event that a random variable X belongs to an interval $(x_{i-1}, x_i]$ is given by $P(x_{i-1} < X \leq x_i) = F_X(x_i) - F_X(x_{i-1})$.

In the discrete case the distribution function is the area under the probability function. The probability function is constant on each possible outcome so the distribution function becomes a sum of step functions with jumps at each probability. In the continuous case the distribution function, evaluated at a point x, is the area under an integrable function called the probability density function, from minus infinity up to x. For random variables that are a mixture of discrete and continuous random variables, the distribution function will be a combination the two types such that it will be continuous everywhere except at the points of positive probability, where it will have discontinuous jumps with a height equal to the corresponding probability. Common to all types of distribution functions are that they are non-decreasing functions from zero to one.

We often want to estimate the probability of an event connected to a random variable. This is done by looking at the expectation of a random variable. Recall that the expected value is a weighted average of all possible values, where the weights are the probabilities of certain events occurring. The expectation is a linear operator which by definition exists whenever the expression of the distribution function is finite or converges absolutely.

If the expectation does exist, then the expected value of a random variable X is given by

$$E(X) = \begin{cases} \sum_{i=1}^{\infty} x_i P(X = x_i) & \text{when } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f_X(x) dx & \text{when } X \text{ is continuous} \end{cases} .$$

Here $f_X(x)$ is the probability density function of X. This function is related to the cumulative distribution function by $F_X' = f_X$, if f_X is continuous at x.

The integral above is an ordinary Riemann integral in which integration is performed by summarizing over

$$\lim_{\max|x_i - x_{i-1}| \to 0} \sum_{i=1}^{n} \eta_i f_X(\eta_i)(x_i - x_{i-1}) ,$$

where $\eta_i \in (x_{i-1}, x_i]$ for $i = 1, 2, ..., n$. By recognizing that $f_X(\eta_i)(x_i - x_{i-1})$ is an approximation of the probability $P(x_{i-1} < X \leq x_i)$, we can rewrite this expression as

$$\eta_i f_X(\eta_i)(x_i - x_{i-1}) \approx \eta_i P(x_{i-1} < X \leq x_i) = \eta_i \big(F_X(x_i) - F_X(x_{i-1})\big) .$$

In the limit, this approximation becomes exact. Since all distribution functions have the property of being monotone and bounded on each subinterval $(x_{i-1}, x_i]$, our sum satisfies the conditions for a Riemann-Stieltjes sum and we can write

$$E(X) = \lim_{\max|x_i - x_{i-1}| \to 0} \sum_{i=1}^{n} \eta_i \big(F_X(x_i) - F_X(x_{i-1})\big) .$$

We have thus reached a general definition of the expected value of a random variable.

Definition 5.3. For a random variable X with distribution function F_X, we define the *expectation* of X as

$$E(X) = \int_{-\infty}^{\infty} x dF_X(x) .$$

Expressing the expectation of a random variable by its distribution function thus generates a general formula, which is valid for both continuous and discrete random variables, as well as mixtures of the two.

Now, let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int g(x)dF(x)$ converges. Let Y be a random variable with distribution function $F_Y = P(Y \leq y)$ and expectation $E(Y) = \int y dF_Y(y)$. When Y is a function of another random variable X, we can introduce the following theorem.

Theorem 5.4. *Let X be a random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int g(x)dF_X(x) < \infty$. Then the random variable $Y = g(X)$ has expectation*

$$E(Y) = \int_{-\infty}^{\infty} g(x)dF_X(x) \ . \tag{5.1}$$

Proof. We will prove the special case when g is bijective. A complete proof using measure theory is given in [9], Theorem 4.26. The full proof is however not included since it lies outside the scope of this essay.

The integral on the right hand side in equation (5.1) is a Riemann-Stieltjes integral. Therefore, we assume that the size of $(x_{i-1}, x_i]$ is small and create a Riemann-Stieltjes sum for the left hand side in the expectation of Y

$$\begin{aligned}\sum_{i=1}^{n} \eta_i\big(F_X(y_i) - F_X(y_{i-1})\big) &= \sum_{i=1}^{n} \eta_i P\big(Y \in (y_{i-1}, y_i]\big)\\ &= \sum_{i=1}^{n} \eta_i P\big(g(X) \in (y_{i-1}, y_i]\big)\\ &= \sum_{i=1}^{n} \eta_i P\big(X \in g^{-1}((y_{i-1}, y_i])\big) \ ,\end{aligned}$$

where $\eta_i \in (y_{i-1}, y_i]$.

We define a variable $\xi_i = g^{-1}(\eta_i)$ and note that since g is bijective

$$\begin{aligned}\eta_i \in (y_{i-1}, y_i] &\Leftrightarrow \xi_i = g^{-1}(\eta_i) \in g^{-1}((y_{i-1}, y_i])\\ &\Leftrightarrow g(\xi_i) \in (y_{i-1}, y_i] \ .\end{aligned}$$

Thus for $\xi_i \in g^{-1}((y_{i-1}, y_i])$ we get

$$\sum_{i=1}^{n} \eta_i P\big(X \in g^{-1}((y_{i-1}, y_i])\big) = \sum_{i=1}^{n} g(\xi_i) P\big(X \in g^{-1}((y_{i-1}, y_i])\big) \ .$$

If the $(y_{i-1}, y_i]$ form a partition of an interval in $\mathbb{R}$ such that all subintervals are disjoint and have the entire interval as their union, then each $(x_{i-1}, x_i] = g^{-1}((y_{i-1}, y_i])$ will also form a partition in $\mathbb{R}$. Therefore,

$$\sum_{i=1}^{n} g(\xi_i) P\big(X \in g^{-1}((y_{i-1}, y_i])\big) = \sum_{i=1}^{n} g(\xi_i) P\big(X \in (x_{i-1}, x_i]\big) \ ,$$

where $\xi_i \in g^{-1}((y_{i-1}, y_i])$. The sum on the right hand side above is a Riemann-Stieltjes sum for the integral in equation (5.1). That is,

$$\int_{-\infty}^{\infty} g(x)dF_X(x) = \sum_{i=1}^{n} g(\xi_i) P\big(X \in (x_{i-1}, x_i]\big) = \sum_{i=1}^{n} g(\xi_i)\big(F_X(x_i) - F_X(x_{i-1})\big) \ .$$

□

Now, we want to extend the concept of expectation to random samples of larger dimensions. Therefore, we consider random vectors as opposed to random variables.

Definition 5.5. Given a random experiment with an outcome space Ω, we define a *random vector* $\mathbf{X} = (X_1, X_2, ..., X_n)$ of size n as a function from the sample space to the set of n-dimensional real vectors $\mathbb{R}^n$.

As for the case with random variables in one dimension, random vectors are often described by their joint distribution function.

Definition 5.6. For a random vector $\mathbf{X}$ we define the *joint distribution function* of $\mathbf{X}$ as

$$F_{\mathbf{X}}(x) = P(X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n)$$

for all $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $i = 1, 2, ..., n$.

For a function $g : \mathbb{R}^n \to \mathbb{R}$ such that the existence of the expectation is ensured, we can define a random variable $Y = g(X_1, X_2, ..., X_n)$ with distribution function F_Y. We introduce the following theorem for the expectation, if it exists, of Y in $\mathbb{R}^n$.

Theorem 5.7. *Let* $\mathbf{X} = (X_1, X_2, ..., X_n)$ *be a random vector with distribution function* $F_{\mathbf{X}}$*. For a function* $g : \mathbb{R}^n \to \mathbb{R}$ *such that the expectation exists, the random variable* $Y = g(X_1, X_2, ..., X_n)$ *has expectation*

$$E(Y) = \int g(x_1, ..., x_n) dF_{\mathbf{X}}(x_1, ..., x_n) \ .$$

The proof is similar to the one dimensional case and is not included in this essay. For a detailed proof of Theorem 5.7 we refer to Theorem 4.28 in [9].

An important consequence of Theorem 5.7 is that the expectation of a sum of two random variables now can be formed, regardless of their type: continuous, discrete or a mixture of the two.

Theorem 5.8. *If* X *and* Y *are two random variables and* a *and* b *are real valued constants, then the expectation of* X *and* Y *satisfies*

$$E(aX + bY) = aE(X) + bE(Y) \ .$$

Proof. By definition

$$E(aX + bY) = \int (ax + by) dF(x, y) \ ,$$

which is a Riemann-Stieltjes integral. By property (I) in Theorem 3.4 this integral is equal to

$$a \int x dF(x, y) + b \int y dF(x, y) \ .$$

Using the definition of the expectation again leaves us with

$$aE(X) + bE(Y) \ .$$

□

When we define the expectation of discrete and continuous random variables separately, as sums and ordinary Riemann integrals respectively, it is not possible to define the expectation of $X + Y$ if, for example, X is discrete and Y is continuous. In this section, we have shown that this can be done by using the Riemann-Stieltjes integral, and this is why it provides such a good definition of the expectation of a random variable.

The Riemann-Stieltjes integral may be very useful when constructing proofs in probability theory and mathematical statistics. Nevertheless, we must point out that the application of Riemann-Stieltjes integrals in probability theory is strictly a theoretical application in the sense that one does not evaluate Riemann-Stieltjes integrals when calculating expected values. Furthermore, there is a certain limitation to using Riemann-Stieltjes integrals for the expected value since they cannot handle so called wild functions, like for example the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \ . \end{cases}$$

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