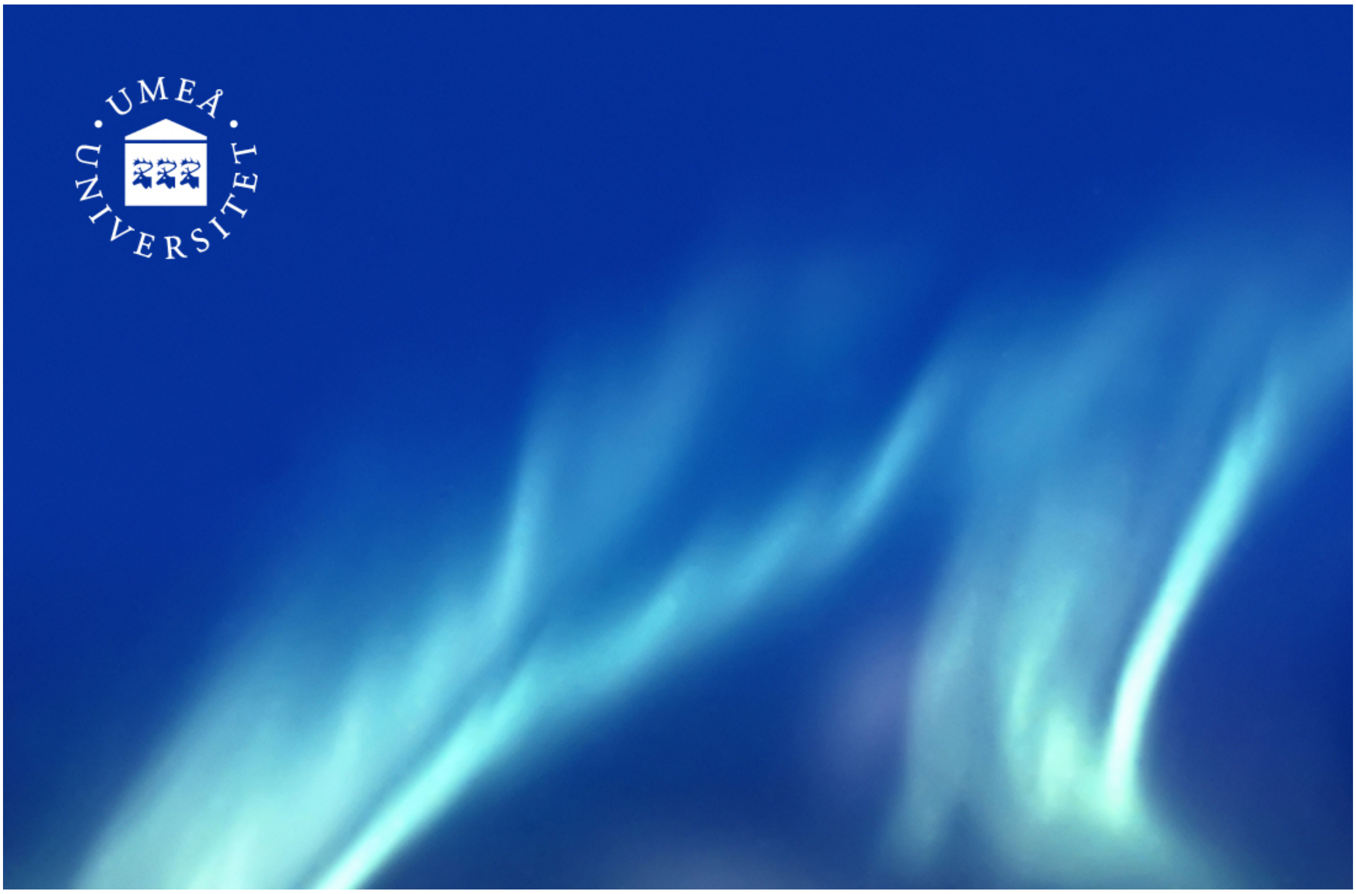

Evaluating $\int_0^\infty f(x)dx$ and $\int_a^b f(x)dx$ using residue calculus

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Abstract

In this essay we use complex analysis, in particular modern residue calculus, to compute certain Riemann integrals.

Sammanfattning

I den här uppsatsen använder vi komplex analys, då särskilt modern residykalkyl, för att beräkna vissa Riemann-integraler.

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1. Introduction

To evaluate an integral even from the freshman year can be immensely problematic. On the contrary to differentiations algorithmically computations, the evaluation of Riemann integrals of the type

$$\int_a^b f(x)\,dx$$

depends highly on the type of the function $f : \mathbb{R} \to \mathbb{R}$. For example if f is a rational function we need to use partial fractions, but if we shall integrate $\sqrt{1+x^2}$ we proceed by using an inverse trigonometric substitution. In this essay we shall attack the problem of evaluating Riemann integrals of functions of one real variable in another manner. We shall complexify the integrand and choose a clever curve, so that we can employ the machinery of complex analysis and Cauchy's residue calculus.

In the beginning of the 19th century, the French mathematician Augustin-Louis Cauchy single-handedly introduced and laid the foundation of complex analysis. Here we shall use the concept of complex curve integrals, which today is viewed as an application of the so called Riemann-Stieltjes integral (see [4] for details). The basic tool at our disposal is the famous Cauchy's residue theorem (Theorem 3.2) which states the following: Let Γ be a simple, closed and sufficiently smooth curve in an open set $G \subseteq \mathbb{C}$, and let f be a function that is holomorphic except for a finite number of isolated singularities inside Γ. Denote these singularities with $a_1, \ldots, a_n$. Then we have that

$$\oint_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f(z)\,,$$

where $\operatorname{Res}_{z=a_k} f(z)$ is the residue of f at a_k. In every introduction to complex function theory one uses the residue theorem to calculate some Riemann integrals of functions of one real variable. The aim of this essay is to look at two more modern and far-fetched applications.

The first aim of this essay is by following [8] to arrive at the following theorem:

Theorem A: *Let $\hat{\mathbb{C}}$ denote the Riemann sphere, and assume that the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfies the following conditions*

1. *The function f is holomorphic in the complex plane, except for a finite amount of isolated singularities where those on the positive real axis are denoted $a_1, a_2, \ldots, a_m$ and the rest are denoted $z_1, z_2, \ldots, z_n$.*
2. *The function f has an odd principal part for $a_1 \ldots, a_m$.*
3. $\lim_{z\to 0} zf(z)\log(z) = \lim_{z\to\infty} zf(z)\log(z) = 0.$

Then

$$P.V.^* \int_0^\infty f(x)dx = -\sum_{k=1}^{n} \operatorname*{Res}_{z=-z_k} f(-z)\log(z) - \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z)\log(z)$$

where P.V.* is the modified principal value defined in Definition 3.6.

Recall that to use complex function theory to evaluate real integrals we need to introduce some kind of principle value. Furthermore, if the integral itself exists then it equals to the principle value (see section 3 for details). It should be noted the possibility of an isolated singularity occurring at $z = \infty$, and therefore we spend some extra attention on this particular case in section 2.

As in [5] we use Theorem A to arrive at our second, and final, goal of this essay:

Theorem B: *Let $\hat{\mathbb{C}}$ denote the Riemann sphere, and assume that the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfies the following conditions*

(1) *The function f has no singularities on $\{a, b\}$, $-\infty < a < b < \infty$.*

(2) *The function f is holomorphic in the extended plane, except for a finite amount of isolated singularities where those on the interval (a, b) are denoted $a_1, a_2, \ldots, a_m$ and the rest are denoted $z_1, z_2, \ldots, z_n$.*

(3) *The singularities $a_1 \ldots, a_m$ are simple poles.*

Then

$$P.V.^* \int_a^b f(x)dx = -\left(\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \log\left(\frac{z-a}{z-b}\right) + \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z) \log\left(\frac{z-a}{b-z}\right) \right).$$

This essay is organized as follows. In section 2 we give the necessary background on isolated singularities and residues. Then in section 3 we collect some results, such as the Cauchy's residue theorem, that is needed for the proofs of our two main theorems. In section 4 and section 5, we give proofs of Theorem A and Theorem B, respectively. This essay then ends in section 6 with some applications of Theorem A and Theorem B.

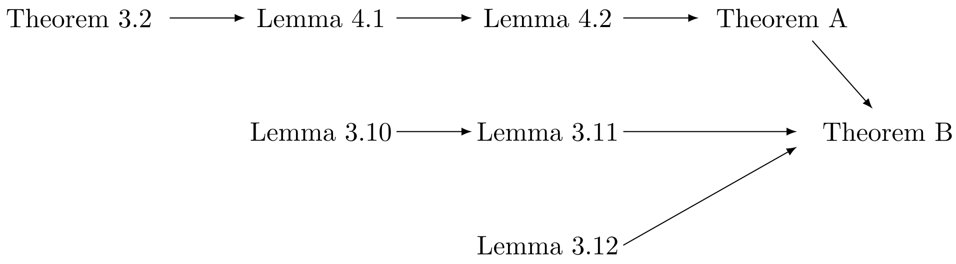


FIGURE 1.1. The flowchart for lemmas and theorems.

The reader is assumed to be familiar with basic knowledge of complex analysis, and for further information we refer to [1, 3, 6]. For the interested reader we refer to [7] for the history of Cauchy's work on complex function theory. The inspiration and mathematics for this essay have been mostly [2, 5, 8], in no particular order.

2. Isolated singularities and residues

The purpose of this section will be to guide the reader from the concept of isolated singularities to the concept of residues, and finish with a couple of examples of the more advanced notion of singularities and residues at infinity. With the exception of the latter, the material is at the level of an introductory course in complex analysis. We will briefly walk through the four definitions leading up to, and including, the residue.

Definition 2.1. Let $A \subseteq \mathbb{C}$ be a nonempty and open set. We say that $f : A \to \mathbb{C}$ has an *isolated singularity at* $z_0 \in A$ if it is holomorphic in an open punctuated disc $D = \{z : |z - z_0| < r\} \setminus \{z_0\}$ for some $r > 0$, but not on $D \cup \{z_0\}$.

Theorem 2.2. *Let* $D = \{z : r < |z - z_0| < R\}$ *for some* $R > r > 0$*. Assume that the function* $f : D \to \mathbb{C}$ *is holomorphic. Then* f *can be represented with the series*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j, \tag{2.1}$$

where a_j *are constants, see p. 185 in* [1]*, which we call the* Laurent series of f around the point z_0.

Proof. See e.g. p. 184 in [1]. □

For those looking for a more in depth explanation of the Laurent series we refer to [1, 3].

Definition 2.3. Let $D = \{z : r < |z - z_0| < R\}$ for some $R > r > 0$. Assume that the function $f : D \to \mathbb{C}$ is holomorphic. Let (2.1) be the Laurent series of f around z_0, then

(1) If $a_j = 0$ for all $j < 0$, we say that z_0 is a *removable singularity* of f.

(2) If $a_{-m} \neq 0$ for some positive integer m but $a_j = 0$ for all $j < -m$, we say that z_0 is a *pole of order* m for f.

(3) If $a_j \neq 0$ for an infinite number of negative values of j, we say that z_0 is an *essential singularity* of f.

Notation 2.4. A pole of order 1 is called a simple pole.

Notation 2.5. We denote the extended complex plane, also known as the Riemann sphere, as $\hat{\mathbb{C}}$. Likewise we denote the extended real line as $\hat{\mathbb{R}}$.

Definition 2.6. Let $A \subseteq \mathbb{C}$ be an open and nonempty set. Assume that the function $f : A \to \hat{\mathbb{C}}$ is holomorphic except possibly in the neighborhood around a point $z_0 \in A$ which may be an isolated singularity. Let f have the Laurent series as seen in (2.1) around the point z_0. Then the coefficient a_{-1} of $\frac{1}{z-z_0}$ is called the *residue of* f *at* z_0 and is denoted by

$$\operatorname*{Res}_{z=z_0} f.$$

Equivalently, as seen at page 5 in [5], with Γ as a circular curve containing only one isolated singularity, we can write the residue as

$$\operatorname*{Res}_{z=z_0} f = \frac{1}{2\pi i} \oint_{\Gamma} f(z) dz \tag{2.2}$$

which will be used more often than the coefficient representation itself. We now proceed to present two lemmas to evaluate removable singularities and poles. The result concerning the removable singularities is trivial compared to the results for the poles, though we will prove both since they are good examples of when the coefficient representation is advantageous.

Lemma 2.7. *Let $A \subseteq \mathbb{C}$ be an open and nonempty set. Assume that the function $f : A \to \mathbb{C}$ has a removable singularity at $z_0 \in A$ or is holomorphic in $D = \{z : |z - z_0| < \delta\}$ for some $\delta > 0$. Then*

$$\operatorname*{Res}_{z=z_0} f = 0. \tag{2.3}$$

Proof. We have from the definition of removable singularities and from holomorphicity that all terms $a_j = 0$ for $j < 0$ in the Laurent expansion for f around z_0. Since the definition of the residue is

$$\operatorname*{Res}_{z=z_0} f = a_{-1}$$

it follows that

$$\operatorname*{Res}_{z=z_0} f = 0.$$

□

Lemma 2.8. *Let $A \subseteq \mathbb{C}$ be an open and nonempty set. Assume that the function $f : A \to \hat{\mathbb{C}}$ is holomorphic in A with the exception of the neighborhood around a pole of order m at $z_0 \in A$. Then*

$$\operatorname*{Res}_{z=z_0} f = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Big[(z - z_0)^m f(z) \Big]. \tag{2.4}$$

Proof. Starting with the Laurent expansion for f around z_0,

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \ldots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)^1} + a_0 + a_1(z - z_0) + \ldots,$$

we multiply by $(z - z_0)^m$, which yields

$$(z - z_0)^m f(z) = a_{-m} + \ldots + a_{-2}(z - z_0)^{m-2} + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + a_1(z - z_0)^{m+1} + \ldots.$$

We then differentiate $m - 1$ times to conclude

$$\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] = (m-1)!\ a_{-1} + m!\ a_0(z - z_0) + \frac{(m+1)!}{2} a_1(z - z_0)^2 + \ldots.$$

Hence

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] = (m-1)!\ a_{-1}.$$

Thus, dividing by $(m-1)!$, this is equivalent to (2.4). □

We have so far not mentioned essential singularities much, and with good reason. Their residues are not as easy to calculate as poles or removable singularities.

An extra observant reader has noticed that the definitions 2.3 and 2.6 have excluded the possibility of a singularity and residue at infinity and we will promptly address that special case. As implied by the admission of a special case, we cannot approach a singularity at infinity the same way as for the rest.

Definition 2.9. Assume that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a function. We then say that

(1) $f(z)$ is holomorphic at ∞ if $f(1/w)$ is holomorphic at $w = 0$,

(2) $f(z)$ has a removable singularity at ∞ if $f(1/w)$ has a removable singularity at $w = 0$,

(3) $f(z)$ has a pole of order m at ∞ if $f(1/w)$ has a pole of order m at $w = 0$,

(4) $f(z)$ has an essential singularity at ∞ if $f(1/w)$ has an essential singularity at $w = 0$.

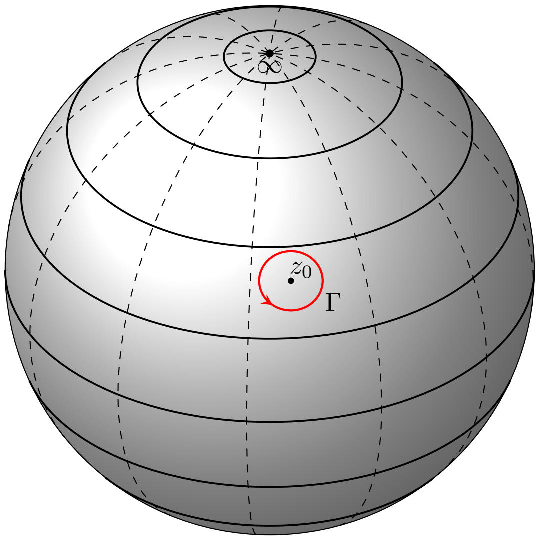


FIGURE 2.1. The Riemann sphere with the curve Γ for the residue at $z_0 \in \mathbb{C}$

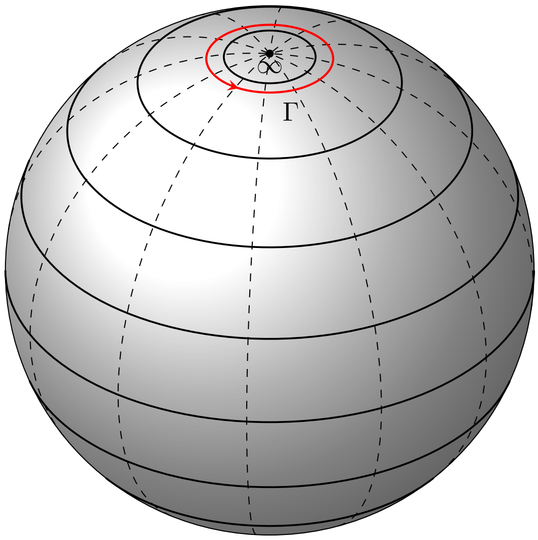


FIGURE 2.2. The Riemann sphere with the curve Γ for the residue at $z_0 = \infty$

We remind ourselves of how the extended plane is homeomorphic to the Riemann sphere. Thus we have, for a residue of a point in the finite plane, that the curve looks like as presented in Figure 2.1. When producing a similar smooth closed curve with the point $z = \infty$ lying inside it, we get Figure 2.2. With that motivation we write

$$\operatorname*{Res}_{z=\infty} f = \frac{1}{2\pi i} \oint_{\Gamma} f(z)dz \tag{2.5}$$

where Γ is defined as $\Gamma = \{z : z = Re^{i\theta}, 0 \leq \theta < 2\pi\}$ traversed from $\theta = 2\pi$ to $\theta = 0$ and $R > 0$ is chosen such that all singularities, with the exception of the possible singularity $z = \infty$, lies within the disc $|z| < R$. The reader is encouraged to take note of the change of orientation of the curve compared to the ordinary notation. To continue, we note the transformation we observed in Definition 2.9 and present the following lemma.

Lemma 2.10. *Let $A \subseteq \hat{\mathbb{C}}$ be an open, nonempty set such that $\{z : |z| > M\} \in A$ for some $M > 0$. Assume that the function $f : A \to \hat{\mathbb{C}}$ is holomorphic except in the neighborhood of an isolated singularity at ∞. Then*

$$\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} -\frac{1}{z^2} f\left(\frac{1}{z}\right). \tag{2.6}$$

Proof. We take the transformation $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$g(w) = 1/w = z$$

which maps $g(\infty) = 0$. The transformation g taken in polar coordinates, representing

$$z = Re^{i\theta} \tag{2.7}$$

$$w = re^{i\gamma}, \tag{2.8}$$

corresponds to

$$R = \frac{1}{r} \tag{2.9}$$

$$\theta = -\gamma. \tag{2.10}$$

We also have that

$$dz = iRe^{i\theta} d\theta \tag{2.11}$$

and

$$dw = ire^{i\gamma} d\gamma. \tag{2.12}$$

Using (2.5), (2.7) and (2.11), we get

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{2\pi i} \oint_{\Gamma} f(z)dz = -\frac{1}{2\pi i} \int_0^{2\pi} iRe^{i\theta} f\left(Re^{i\theta}\right) d\theta.$$

Further, we use (2.9) and (2.10) to show

$$-\frac{1}{2\pi i}\int_0^{2\pi} iRe^{i\theta} f\left(Re^{i\theta}\right) d\theta = -\frac{1}{2\pi i}\int_0^{2\pi} i\frac{1}{r}e^{-i\gamma} f\left(\frac{1}{r}e^{-i\gamma}\right) d\gamma.$$

The definition of the residue in the finite plane, (2.8), (2.12), and the mapping g gives that

$$-\frac{1}{2\pi i}\int_0^{2\pi} i\frac{1}{r}e^{-i\gamma} f\left(\frac{1}{r}e^{-i\gamma}\right) d\gamma = -\frac{1}{2\pi i}\oint_{\Gamma^*} \frac{1}{w^2} f\left(\frac{1}{w}\right) dw = \operatorname*{Res}_{z=0} \frac{1}{w^2} f\left(\frac{1}{w}\right).$$

□

We will finish this section with examples demonstrating the different concepts.

2.1. **Examples.** First off we will begin with a polynomial and a rational function. This because these families of functions are generally easy to visualize and most mathematicians are familiar with their behavior. The polynomial is easy enough to construct a general demonstration of while we will limit us to more specific rational functions.

Example 2.11. Take a polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree n such that

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n.$$

The polynomial is holomorphic in the finite plane, i.e. lacking any isolated singularity. Though if we are to expand the domain to include infinity, such that $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we gain an isolated singularity at infinity. Observing $P\left(\frac{1}{z}\right)$,

$$P\left(\frac{1}{z}\right) = a_0 + a_1\frac{1}{z} + a_2\frac{1}{z^2} + \ldots + a_n\frac{1}{z^n},$$

we see that following directly from the Definition 2.3 and Definition 2.9, the polynomial P have a pole of order n at infinity. Now proceed to calculate the residue at infinity, by Lemma 2.10 we get

$$\begin{aligned}\operatorname*{Res}_{z=\infty} P(z) = \operatorname*{Res}_{z=0} -\frac{1}{z^2}P\left(\frac{1}{z}\right) &= \operatorname*{Res}_{z=0} -\frac{1}{z^2}\left(a_0 + a_1\frac{1}{z} + a_2\frac{1}{z^2} + \ldots + a_n\frac{1}{z^n}\right)\\ &= \operatorname*{Res}_{z=0} -\left(a_0\frac{1}{z^2} + a_1\frac{1}{z^3} + a_2\frac{1}{z^4} + \ldots + a_n\frac{1}{z^{n+2}}\right) = 0.\end{aligned}$$

Now, since we have taken a general polynomial, we have demonstrated that any polynomial of degree n have a pole of order n at infinity, with the residue 0. □

With the polynomial done, we turn to a very simple rational function.

Example 2.12. Take the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $f(z) = \frac{1}{z}$. Defining the function

$$g(z) = f\left(\frac{1}{z}\right) = z$$

we can, through Definition 2.9, conclude that f is holomorphic at infinity. We also note that

$$f(\infty) = 0.$$

But we have that

$$\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} -\frac{1}{z^2} f\left(\frac{1}{z}\right) = \operatorname*{Res}_{z=0} -\frac{1}{z}.$$

We observe that the Laurent expansion for $-\frac{1}{z}$ around $z = 0$ is $-\frac{1}{z}$ itself and since $a_{-1} = -1$ we conclude

$$\operatorname*{Res}_{z=0} -\frac{1}{z} = -1.$$

Which lastly, through Lemma 2.10, gives us

$$\operatorname*{Res}_{z=\infty} f(z) = -1.$$

That is, despite $f(\infty) = 0$, the function still has a residue of -1 there. □

Now we will observe a more complex rational function and investigate where it has isolated singularities and also what the residue is at each. This will further demonstrate what the earlier examples did, but also utilize Lemma 2.8. Further, this example demonstrates the difference of a simple pole in the finite plane and one at infinity.

Example 2.13. Take the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$f(z) = \frac{z^3}{(z+1)(z-1)}.$$

Immediately we observe that we have two simple poles at $z = \pm 1$. Applying Lemma 2.8 to these two simple poles yields

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{z^3}{(z+1)} = \frac{1}{2}$$

and

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1} (z+1) f(z) = \lim_{z\to -1} \frac{z^3}{(z-1)} = \frac{1}{2}.$$

Obviously, nothing dramatic. However, investigating whether f has a singularity at infinity we start with observing how $g(w) = f(1/w)$ looks like

$$g(w) = \frac{\left(\frac{1}{w}\right)^3}{\left(\frac{1}{w}+1\right)\left(\frac{1}{w}-1\right)} = \frac{\left(\frac{1}{w}\right)^3}{\left(\frac{1}{w}\right)^2 - 1} = \frac{1}{w - w^3} = \frac{1}{w(1-w^2)}.$$

We see that g has a simple pole at $w = 0$, which combined with Definition 2.9 leads us to conclude that f has another simple pole, this time at infinity. We find that Lemma 2.10 gives us

$$\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} -\frac{1}{z^2} f\left(\frac{1}{z}\right) = \operatorname*{Res}_{z=0} -\frac{1}{z^2} g(z) = \operatorname*{Res}_{z=0} \frac{1}{z^2} \frac{1}{z(z^2-1)} = \operatorname*{Res}_{z=0} \frac{1}{z^3(z^2-1)}.$$

We have from Lemma 2.8 that

$$\operatorname*{Res}_{z=0} \frac{1}{z^3(z^2-1)} = \lim_{z\to 0} \frac{1}{(2)!} \frac{d^2}{dz^2} \left[z^3 \frac{1}{z^3(z^2-1)} \right] = \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2} \frac{1}{z^2-1} = \lim_{z\to 0} \frac{1}{2} \frac{6z^2+2}{(z^2-1)^3} = -1.$$

Concluding, f has three simple poles. One at $z = 1$, one at $z = -1$ and lastly one at $z = \infty$, which have the residues $\frac{1}{2}$, $\frac{1}{2}$ and -1 respectively. □

The following example will demonstrate a function with essential singularity at infinity with residue zero.

Example 2.14. Take the function $f(z) = e^z$. With the help of the Maclaurin series

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \dots,$$

which is equivalent to the Laurent series evaluated at $z = 0$, we immediately see that it has no singularities in the finite plane. Again using the alternative representation of the residue at infinity, we get

$$\begin{aligned} \operatorname*{Res}_{z=\infty} e^z = \operatorname*{Res}_{z=0} -\frac{1}{z^2} e^{\frac{1}{z}} &= \operatorname*{Res}_{z=0} -\frac{1}{z^2} \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \frac{1}{24z^4} + \dots \right) \\ &= \operatorname*{Res}_{z=0} - \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2z^4} + \frac{1}{6z^5} + \frac{1}{24z^6} + \dots \right) = 0. \end{aligned}$$

This means that, from Definition 2.9, f have an essential singularity in $z = \infty$, which have the residue 0. □

In Example 2.15 and Example 2.16 we characterize those functions that are holomorphic on $\widehat{\mathbb{C}}$, and those who only have one pole at $\hat{\mathbb{C}}$.

Example 2.15. Take a function $f : \hat{\mathbb{C}} \to \mathbb{C}$ which is holomorhic on the extended plane. Since f is holomorphic at ∞, it is bounded for $|z| > M$, for some $M > 0$. By continuity, f is also bounded for $|z| \leq M$. Consequently, f is a bounded entire function. Hence f is constant, by Liouville's theorem. □

Example 2.16. Take a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is holomorphic except for a pole at one point. If f has a pole of order m at $z_0 \in \mathbb{C}$, then the Laurent series for f around z_0,

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

converges for all $z \neq z_0$. We also note, since $z_0 \neq \infty$ we have that f is holomorphic, and thus bounded, at ∞. This leads to the fact that the sum in the Laurent series needs to be bounded at ∞, therefore we need $a_n = 0$ for $n > 0$. Therefore the most general form for such a function f is

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{a_{-1}}{z-z_0} + a_0. \tag{2.13}$$

If the pole, of order m, occurs at $z = \infty$, then $f(1/w)$ has a pole, of order m, at the origin and can be expressed in the form

$$f\left(\frac{1}{w}\right) = \frac{a_{-m}}{w^m} + \frac{a_{-m+1}}{w^{m-1}} + \ldots + \frac{a_{-1}}{w} + \sum_{n=0}^{\infty} a_n w^n.$$

Since $f(z)$ is bounded near $z = 0$, it follows that $f(1/w)$ is bounded for large $|w|$, and as before we conclude that $a_n = 0$ for $n > 0$. We then get

$$f(z) = a_{-m}z^m + a_{-m+1}z^{m-1} + \ldots + a_{-1}z + a_0. \tag{2.14}$$

That is, f is a polynomial. Equation (2.13) and (2.14) categorize all functions which are holomorphic in $\hat{\mathbb{C}}$ with the exception of one pole. □

Lastly we end with another function with an essential singularity. Upon looking at the convergence along the real line, we observe a convergence towards 0 at a rate of $O(z^2)$. But this is, of course, insufficient.

Example 2.17. Take the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

$$f(z) = \frac{\sin z}{z^2}.$$

We notice that f converges to 0 as $z \to \infty$, along the real line. That is not the case outside of the real line however. It, in fact, diverges and produces an essential singularity at infinity. Define the function $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

$$g(z) = f\left(\frac{1}{z}\right) = z^2 \sin\frac{1}{z}.$$

Then Lemma 2.10 gives us

$$\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} -\frac{1}{z^2} g(z) = \operatorname*{Res}_{z=0} - \sin\frac{1}{z}.$$

Using the Maclaurin series, which again is equivalent to the Laurent series evaluated at $z = 0$, for $\sin z$, we get

$$\operatorname*{Res}_{z=0}\left(-\frac{1}{z} + \frac{1}{3!z^3} - \frac{1}{5!z^5} + \ldots\right) = -1.$$

Thus we conclude that the function $\frac{\sin z}{z^2}$ has an essential singularity with the residue -1 at infinity. □

We introduce a central definition before presenting a well known theorem, which we also will prove as a warm up before the first main theorem.

Definition 3.1. A directed, closed, piecewise smooth curve is called a *contour*.

Theorem 3.2 (Cauchy's Residue Theorem)**.** *Let $A \subseteq \mathbb{C}$ be an open, nonempty and simply connected set. Let $G \subset A$ be an open and simply connected set bounded by the contour $\Gamma \subset A$. Assume that the function $f : A \to \mathbb{C}$ is holomorphic, and continuous on Γ, with the exception for a finite amount of isolated singularities. Let $a_1, \dots, a_n$ denote the singularities of f contained in G. Then*

$$\oint_\Gamma f(z)dz = 2\pi i \sum_{\upsilon=1}^{n} \operatorname*{Res}_{z=a_\upsilon} f(z).$$

Proof. Choose δ such that $|a_i - a_j| > 2\delta$, for $i \neq j$, $i, j = 1, 2, \dots, n$. Let $K_\upsilon = \{z : |z - a_\upsilon| < \delta\}$ and let $k_\upsilon = \partial K_\upsilon$ for $\upsilon = 1, \dots, n$. Let $l_1, \dots, l_n$ be lines connecting Γ to k_1, k_1 to k_2 ... k_{n-1} to k_n. Let C be the contour which traverses Γ, k_υ and l_υ, for $\upsilon = 1, \dots, n$, such that it traverses each l_υ twice, once in each direction. C is illustrated in Figure 3.1.

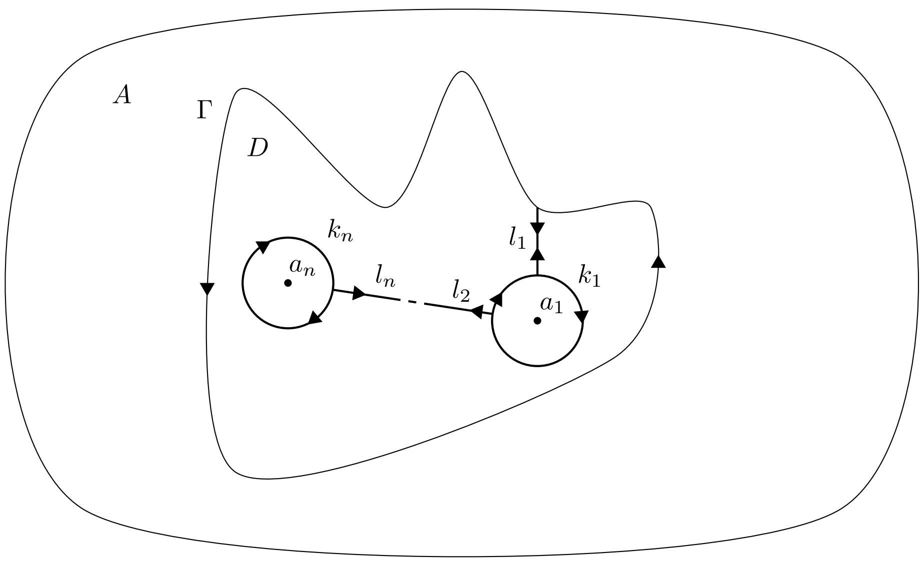


FIGURE 3.1. The contour C, with each part denoted, traversed counter clockwise.

We have that the function f is holomorphic in the region

$$D = G \setminus \bigcup_{\upsilon=1}^{n} K_\upsilon.$$

Hence, since G is the interior of C traversed counter clockwise, by Cauchy's integral theorem (see e.g. [5, p. 2]) we have that

$$\oint_C f(z)dz = 0.$$

So what we have done here is essentially, as stated earlier, to reach in and take out all of the areas around the singularities with the help of the connecting lines $l_1, l_2, \ldots, l_n$. We traverse those lines twice, each time from a new direction so that the integrals cancel out. That gives us

$$\oint_\Gamma f(z)dz + \sum_{v=1}^{n} \oint_{k_v} f(z)dz = 0.$$

Subtracting the series and reversing the orientation of the integration gives

$$\oint_\Gamma f(z)dz = \sum_{v=1}^{n} \oint_{k_v} f(z)dz. \tag{3.1}$$

Equation (2.2) gives that

$$\oint_{k_v} f(z)dz = 2\pi i \operatorname*{Res}_{z=a_v} f(z)$$

and applying that to (3.1) yields

$$\oint_\Gamma f(z)dz = 2\pi i \sum_{v=1}^{n} \operatorname*{Res}_{z=a_v} f(z).$$

□

We now need to define the principal value. It will give us the opportunity to approach a divergent point if it diverges towards the positive real infinity on one side and towards the negative real infinity on the other, at the same rate. We present the following notations.

Notation 3.3. Given $\delta > 0$ and $a \in \mathbb{R}$, we denote the set $\{z : |z-a| = \delta, \ \text{Im } z \geq 0\}$ as $C_{\delta,a}$. Adding an orientation to $C_{\delta,a}$, traversing from $a-\delta$ to $a+\delta$, we denote the oriented curve as $C^-_{\delta,a}$. $C_{\delta,a}$ traversed from $a+\delta$ to $a-\delta$ is likewise denoted $C^+_{\delta,a}$.

Notation 3.4. Assume that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a holomorphic function except for in a finite amount of isolated singularities. We number the singularities lying on the interior of the interval $I = [a,b] \subseteq \hat{\mathbb{R}}$ as

$$a_1 < a_2 < \ldots < a_n,$$

and the rest as

$$a_{n+1}, \ldots, a_m.$$

Let $\delta > 0$ be such that

$$|a_i - a_j| > 2\delta, \text{ for all } i \neq j \text{ , } i,j = 1,2,\ldots,m,$$

i.e. choose $\delta > 0$ such that a maximum of one singularity lies within a circle with radius δ and center a_i, $i = 0,1,\ldots m$ and such that none of these circles intersect one another.
Set ρ such that

$$|\rho| > |a_i| + \delta \text{ for all } i = 1, 2, \ldots, m.$$

If $a = -\infty$, then we set $a_0 = -\rho$, otherwise we set $a_0 = a + \delta$. If $b = \infty$, then we set $a_{m+1} = \rho$, otherwise we set $a_{m+1} = b - \delta$. We denote the union of intervals

$$(a_0, a_1 - \delta) \cup (a_1 + \delta, a_2 - \delta) \cup \cdots \cup (a_{n+1} + \delta, a_n - \delta) \cup (a_n + \delta, a_{m+1})$$

as $L(f, I)$ (see Figure 3.2).

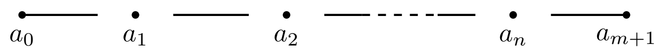


FIGURE 3.2. The lines constituting $L(f, I)$.

In a similar manner we denote the curve

$$L(f, I) \cup C_{\delta,a_1} \cup C_{\delta,a_2} \cup \cdots \cup C_{\delta,a_n}$$

as $K(f, I)$ (see Figure 3.3).

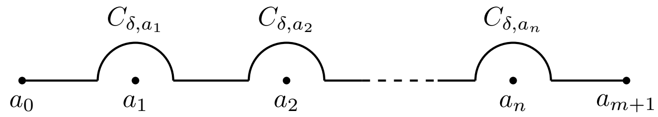


FIGURE 3.3. The lines and half circles constituting $K(f, I)$.

Definition 3.5. We define *the principal value of the integral of a function f over* $[a, b] \subseteq \hat{\mathbb{R}}$ as

$$P.V. \int_a^b f(x)dx = \lim_{\substack{\rho \to \infty, \\ \delta \to 0^+}} \int_{L(f,[a,b])} f(x)dx.$$

Definition 3.6. We define the *modified principal value of a function f over* $[a, b] \subseteq \hat{\mathbb{R}}$ as

$$P.V.^* \int_a^b f(x)dx = \lim_{\substack{\rho \to \infty, \\ \delta \to 0^+}} \int_{K(f,[a,b])} f(z)dz.$$

Obviously the only difference between the principal value and the modified principal value is the indentation, which has some desirable properties. What we want to do now is to investigate when the modified principal value exists. It exists when the ordinary Riemann integral does, but beyond that we need a special requirement.

Definition 3.7. Let $A \subseteq \mathbb{C}$ be a nonempty and open set. Assume that the function $f : A \to \hat{\mathbb{C}}$ has an isolated singularity at $a \in A$. We say that f *has an odd principal part for a* if the Laurent expansion for f

$$f(z) = \sum_{j=-\infty}^{\infty} s_j (z - a)^m$$

around a has $s_j = 0$ for all even $j < -1$.

Lemma 3.8. *Assume that the functions $f : [a,b] \to \mathbb{C}$ and $M : [a,b] \to \mathbb{R}$ are continuous. If $|f(t)| \leq M(t)$ for all $t \in [a,b]$, then*

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b M(t)dt.$$

Proof. See e.g. proposition 2.1.7 at p. 32-33 in [3]. □

This, in short, means that one can use dominating functions to limit integrals, which is what we will need to process the integral of the holomorphic part of the Laurent expansion. We now have everything necessary to show when we can apply the principal value to an integral.

Lemma 3.9. *Let $I = [a,b] \subseteq \hat{\mathbb{R}}$, and let $A \subseteq \hat{\mathbb{C}}$ be a nonempty and open set such that I can be viewed as a subset of A. Assume that the function $f : A \to \hat{\mathbb{C}}$ is holomorphic with the exception for a finite amount of isolated singularities on I. Additionally, assume that the isolated singularities lying on the interior of I have an odd principal part. Denote the isolated singularities lying on the interior of I as $a_1, \ldots, a_n$. Then*

$$P.V. \int_a^b f(x)dx = P.V.^* \int_a^b f(x)dx + \pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f(x).$$

Proof. We take the Laurent expansion around the point a_k

$$f(z) = \sum_{j=-\infty}^{\infty} s_{j,a_k}(z - a_k)^j \tag{3.2}$$

and split it over three different parts

$$\sum_{j=-\infty}^{\infty} s_{j,a_k}(z - a_k)^j = g(z) + h(z) + u(z) \tag{3.3}$$

such that

$$g(z) = \sum_{j=0}^{\infty} s_{j,a_k}(z - a_k)^j$$

and

$$u(z) = \sum_{j=-\infty}^{-2} s_{j,a_k}(z - a_k)^j.$$

Which leaves us with

$$h(z) = \frac{s_{-1,a_k}}{z - a_k}.$$

We start with evaluating the integral of g over C^+_{δ,a_k}. First we utilize z represented with polar coordinates over the half circle C_{δ,a_k}, we have

$$z = a_k + \delta e^{i\theta} \tag{3.4}$$

and

$$\frac{dz}{d\theta} = i\delta e^{i\theta}. \tag{3.5}$$

This gives us

$$\int_{C^+_{\delta,a_k}} g(z)dz = \int_0^\pi g(a_k + \delta e^{i\theta})i\delta e^{i\theta} d\theta$$

and we note that g is holomorphic at a_k and therefore bounded in the neighborhood of a_k, which means that $|g(a_k + \delta e^{i\theta})|$ also is bounded on the curve, for some $\delta > 0$. We further notice that $|ie^{i\theta}|$ also is bounded for $\theta \in [0, \pi]$. Thus, we have the following inequality

$$|g(a_k + \delta e^{i\theta})ie^{i\theta}| \leq M$$

for $\delta < R$. Lemma 3.8 then gives that

$$\left|\int_{C^+_{\delta,a_k}} g(z)dz\right| = \left|\int_0^\pi g(a_k + \delta e^{i\theta})i\delta e^{i\theta} d\theta\right| \leq \delta \int_0^\pi M dz = \delta M \pi$$

and as we can see the right-hand side tends to zero, as $\delta \to 0^+$. Hence,

$$\lim_{\delta \to 0^+} \int_{C^+_{\delta,a_k}} g(z)dz = 0. \tag{3.6}$$

Now evaluating the integral of h over C^+_{δ,a_k}. Applying (3.4) and (3.5) to the integral of h gives us

$$\int_{C^+_{\delta,a_k}} h(z)dz = \int_{C^+_{\delta,a_k}} \frac{s_{-1,a_k}}{z - a_k} dz = \int_0^\pi \frac{s_{-1,a_k}}{\delta e^{i\theta}} \delta i e^{i\theta} d\theta = is_{-1,a_k} \int_0^\pi d\theta = \pi i s_{-1,a_k}.$$

From the definition of the residue we get that

$$\int_{C^+_{\delta,a_k}} h(z)dz = \pi i \operatorname*{Res}_{z=a_k} f(z). \tag{3.7}$$

Lastly we evaluate the integral of u over C^+_{δ,a_k}. We define $\zeta = -j$ for $j = -2, -3, \ldots$ and approach the integral term wise. Again, we integrate using polar coordinates and applying (3.4) and (3.5) to the integral of one term from u, yielding

$$\begin{aligned}
\int_{C^+_{\delta,a_k}} \frac{s_{-\zeta,a_k}}{(z - a_k)^\zeta} dz &= \int_0^\pi \frac{s_{-\zeta,a_k}}{(a_k + \delta e^{i\theta} - a_k)^\zeta} i\delta e^{i\theta} d\theta = \frac{is_{-\zeta,a_k}}{\delta^{\zeta-1}} \int_0^\pi e^{-i(\zeta-1)\theta} d\theta \\
&= \frac{is_{-\zeta,a_k}}{\delta^{\zeta-1}} \left[\frac{e^{-i(\zeta-1)\theta}}{-i(\zeta-1)}\right]_0^\pi = \frac{s_{-\zeta,a_k}}{\delta^{\zeta-1}(1-\zeta)} \left[e^{-i(\zeta-1)\theta}\right]_0^\pi \\
&= \frac{s_{-\zeta,a_k}}{\delta^{\zeta-1}(1-\zeta)}((-1)^{\zeta-1} - 1) = \frac{s_{-\zeta,a_k}}{\delta^{\zeta-1}(\zeta-1)}((-1)^\zeta + 1).
\end{aligned}$$

However, by the assumption of odd principal part we have that $s_{-\zeta,a_k} = 0$ for even $\zeta > 0$. But we have for the odd terms that $(-1)^\zeta + 1 = 0$. Which gives us that

$$\int_{C^+_{\delta,a_k}} \frac{s_{-\zeta,a_k}}{(z-a_k)^\zeta} dz = 0 \tag{3.8}$$

for all $\zeta > 1$, i.e. for all $-j < -1$. Thus (3.8) gives that for the integral of u we have

$$\int_{C^+_{\delta,a_k}} u(z)dz = 0. \tag{3.9}$$

Now applying (3.3), (3.6), (3.7) and (3.9) to the modified principle value we have

$$\begin{aligned} P.V.^* \int_a^b f(x)dx &= \lim_{\substack{\rho\to\infty,\\ \delta\to 0^+}} \int_K f(x)dx = \lim_{\substack{\rho\to\infty,\\ \delta\to 0^+}} \int_L f(x)dx - \sum_{k=1}^n \int_{C^-_{\delta,a_k}} f(z)dz \\ &= \lim_{\substack{\rho\to\infty,\\ \delta\to 0^+}} \int_L f(x)dx - \sum_{k=1}^n \int_{C^-_{\delta,a_k}} g(z) + h(z) + u(z)dz = \lim_{\substack{\rho\to\infty,\\ \delta\to 0^+}} \int_L f(x)dx + \sum_{k=1}^n \pi i \operatorname*{Res}_{z=a_k} f(z) \\ &= \lim_{\substack{\rho\to\infty,\\ \delta\to 0^+}} \int_L f(x)dx + \pi i \sum_{k=1}^n \operatorname*{Res}_{z=a_k} f(z) = P.V. \int_a^b f(x)dx + \pi i \sum_{k=1}^n \operatorname*{Res}_{z=a_k} f(z). \end{aligned}$$

□

We shall need the following well known lemma.

Lemma 3.10. *Let $A \subseteq \mathbb{C}$ be open and nonempty. Assume that the function $f : A \to \mathbb{C}$ is holomorphic in $D = \{z : |z - z_0| < r\}$, for some $r > 0$, and $f(z_0) = w_0$. Then the equation $f(z) = w$ has a unique solution $z = F(w)$ and f is holomorphic at $w = w_0$, if and only if $f'(z_0) \neq 0$.*

Proof. See e.g. p.74 in [5]. □

Since we will be applying a transformation to our integral, the following theorem is needed in order to evaluate how the residues are affected. Unfortunately, while the residues are easily rewritten they do not necessarily preserve the odd property which we want. Instead we will have to limit ourselves, later on, to the simple poles.

Lemma 3.11. *Let $A \subseteq \hat{\mathbb{C}}$ be open and nonempty. Assume that the function $H : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, for a point ξ is holomorphic and $H'(\xi) \neq 0$, or that H has a simple pole at ξ. Assume that for the function $G : A \to \hat{\mathbb{C}}$, the point $\omega = H(\xi) \in A$ is an isolated singularity for G. If h is the inverse of H in a neighborhood of ω, then*

$$\operatorname*{Res}_{z=\omega} G(H(z)) = \operatorname*{Res}_{w=H(\omega)} G(w)h'(w). \tag{3.10}$$

In this proof, we deal with the cases $\zeta \neq \infty, \omega \neq \infty$ as these requirements guarantee the use of lemma 3.10, which is not the case for $\zeta = \infty, \omega = \infty$. This is a conscious omission and the readers should be aware of this.

Proof. Lemma 3.10 gives, due to the assumptions on H, the existence of a local inverse function h which maps a neighborhood of $w = \omega$ onto a neighborhood of $z = \zeta$ bijectively. By definition

$$\operatorname*{Res}_{z=\zeta} G(H(z)) = \frac{1}{2\pi i} \oint_{\Gamma} G(H(z))dz \tag{3.11}$$

where Γ is a positively oriented positively oriented smooth closed curve surrounding ζ. But using $h(w) = z$ and $\frac{dz}{dw} = \frac{dh(w)}{dw} = h'(w)$ we get

$$\oint_{\Gamma} G(H(z))dz = \oint_{\Gamma^*} G(w)h'(w)dw \tag{3.12}$$

where Γ^*, the image of Γ under H, is a positively oriented smooth closed curve surrounding the point $\omega = H(\zeta)$. However,

$$\operatorname*{Res}_{w=H(\zeta)} G(w)h'(w) = \frac{1}{2\pi i} \oint_{\Gamma^*} G(w)h'(w)dw. \tag{3.13}$$

From (3.11),(3.12) and (3.13) we get the case for $\zeta \neq \infty, \omega \neq \infty$. □

The following lemma will demonstrate some properties of a transformation which will be used later on.

Lemma 3.12. *For $a, b \in \mathbb{R}$, the Möbius transformation $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$,*

$$T(z) = \frac{z-a}{b-z} = w, \tag{3.14}$$

maps the line (a, b) onto the line $(0, \infty)$.

Proof. We know that the class of lines and circles is mapped onto itself, see e.g. p. 188 in [3], and we have that

$$\begin{aligned} T(a) &= \frac{a-a}{b-a} = 0 \\ T(b) &= \frac{b-a}{b-b} = \infty. \end{aligned} \tag{3.15}$$

Assuming these two points are mapped onto a circle, we can conclude that it would have an infinite radius. A circle with an infinite radius is a line. Therefore the line (a, b) is mapped onto the line $(0, \infty)$. □

4. The first theorem

Utilizing the properties of singularities with an odd principal part we can deal with some poles and even essential singularities lying on the line of integration. Something worth noting is that we can approach simple poles without any special consideration due to it lacking all even terms in the denominator. Later on we will need to restrict our result to simple poles only, or use a very specific requirement, but for now we are free to do as we please. This lemma will have little resemblance to the final product due to some smart choices of functions which have some desirable properties but all that is required will be an intermediate result to correct that. The sources from this section have been p. 130-133 in [5], p. 175 in [2] and in [8], where the details have been carefully and explicitly explained and the results combined.

Lemma 4.1. *Assume that the functions $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfy the following conditions*

(1) *The function Q is holomorphic in the complex plane.*

(2) *The function F is holomorphic in the complex plane, except for a finite number of isolated singularities where those on the positive real axis are denoted $a_1, a_2, \ldots, a_m$ and the rest are denoted $z_1, z_2, \ldots, z_n$.*

(3) *The function F has an odd principal part for $a_1 \ldots, a_m$.*

(4) $\lim_{z\to 0} zF(-z)Q(\log(z)) = \lim_{z\to\infty} zF(-z)Q(\log(z)) = 0.$

then we have that

$$P.V.^* \int_0^\infty F(x)(Q(\log(x+\pi i)) - Q(\log(x-\pi i)))dx$$
$$= -2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=-z_k} F(-z)Q(\log(z)) - 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z)Q(\log(z+\pi i)).$$

Proof. Take a numbering of all isolated singularities, $s_1, s_2, \ldots, s_{m+n-1}, s_{m+n}$ and number $s_0 = 0$. Choose $\delta > 0$ and $R > 0$ such that $|s_i - s_j| > 2\delta$ for $i \neq j$ and such that $|s_i| + \delta < R$, $i, j = 0, 1, \ldots, m+n$. Let $\Gamma_0 = \{z : |z| = R\}$ and $\gamma_0 = \{z : |z| = \delta\}$, see Figure 4.1.

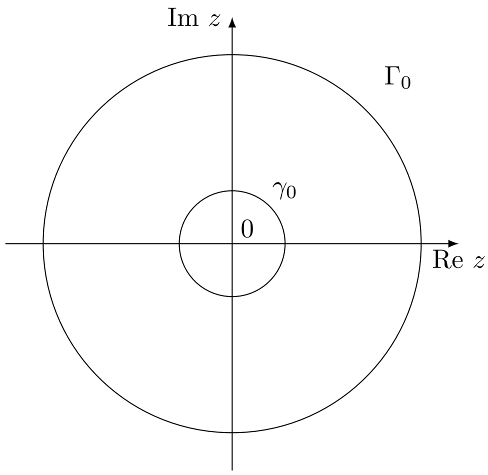


Figure 4.1. The circles Γ_0 and γ_0.

Let $\gamma_k = C^-_{\delta,a_k}, k = 1, \ldots, m$ and let $\gamma^*_k, k = 1, \ldots, m$ be the corresponding reflections across the real axis. Let $l_k = [a_k + \delta, a_{k+1} - \delta], k = 0, 1, \ldots, m$, where $a_0 = 0$, $a_{m+1} = R + \delta$ and let l be the directed line segment from $\delta e^{i\theta}$ to $Re^{i\theta}$, where θ is chosen so that no singularities of F are on l, and so that $\pi/2 < \theta \leq \pi$. Let Γ be the positively oriented portion of the circle Γ_0 defined by $\Gamma = \{z : z = Re^{it}, 0 \leq t \leq \theta\}$ and let Γ^* be the negatively oriented portion of this circle defined by $\Gamma^* = \{z : z = Re^{it}, -\pi/2 - \theta \leq t \leq 0\}$. Similarly, let γ be the negatively oriented portion of the circle γ_0: $\gamma = \{z : z = \delta e^{it}, 0 \leq t \leq \theta\}$ and let γ^* be the positively oriented portion of that circle: $\gamma^* = \{z : z = \delta e^{it}, -\pi/2 - \theta \leq t \leq 0\}$. See Figure 4.2.

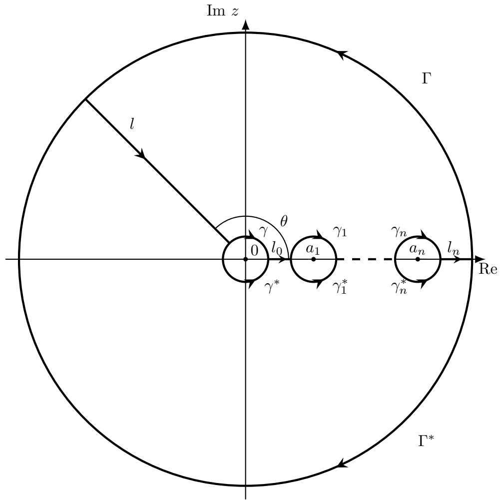
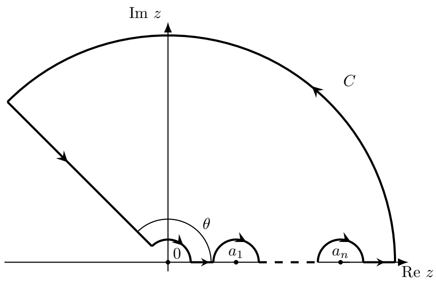
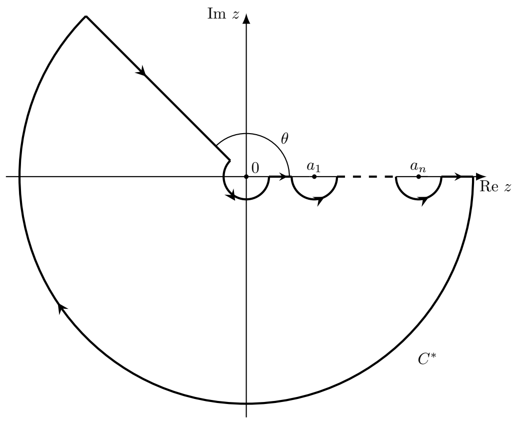


FIGURE 4.2. The arcs $\Gamma, \Gamma^*, \gamma, \gamma^*, \gamma_1, \gamma^*_1, \ldots, \gamma_n, \gamma^*_n$ and the lines $l, l_0, \ldots, l_n$.



(A) The positively oriented curve C.



(B) The negatively oriented curve C^*

FIGURE 4.3. The curves C and C^*.

Let

$$C = \Gamma \cup l_n \cup \gamma_n \cup \cdots \cup \gamma_1 \cup l_0 \cup \gamma \cup l \text{ and } C^* = \Gamma^* \cup l_n \cup \gamma^*_n \cup \cdots \cup \gamma^*_1 \cup l_0 \cup \gamma^* \cup l.$$

The curves C and C^* are both illustrated in Figure 4.3. Then C is positively oriented and C^* is negatively oriented. Now we define L and L^* as the following, where we take careful note of where they are defined,

$$L^*(z) = \log(z) \text{ if } -\pi/2 < \arg z \leq \pi,$$

and

$$L(z) = \log(z) \text{ if } -\pi < \arg z < \pi/2.$$

We then have that $L^*(z) = L(z)$ in the right half plane. Thus we also see that $L^*(-z) = L(-z)$ in the left half plane. Note also that

$$L^*(-z) = \log(z + \pi i), L(-z) = \log(z - \pi i) \text{ if } z \in \mathbb{R}. \tag{4.1}$$

Now define

$$G(z) = F(z)Q(L(-z))$$

and

$$G^*(z) = F(z)Q(L^*(-z)).$$

For residues at the points z_k, $k = 1, \ldots, m$, within C we have, through reflection in the imaginary axis, that

$$\operatorname*{Res}_{z=z_k} G(z) = \operatorname*{Res}_{z=z_k} F(z)Q(L(-z)) = \operatorname*{Res}_{z=z_k} F(z)Q(\log(-z)) = -\operatorname*{Res}_{z=-z_k} F(-z)Q(\log(z)). \tag{4.2}$$

Likewise we have for z_k^* within C^*

$$\operatorname*{Res}_{z=z_k^*} G^*(z) = \operatorname*{Res}_{z=z_k^*} F(z)Q(L^*(-z)) = \operatorname*{Res}_{z=z_k^*} F(z)Q(\log(-z)) = -\operatorname*{Res}_{z=-z_k^*} F(-z)Q(\log(z)). \tag{4.3}$$

If we now take the two positively oriented curves C, C^* and integrate over them with G and G^* respectively.

$$\begin{aligned}
&\oint_C G(z)dz + \oint_{C^*} G^*(z)dz \\
&\qquad = \sum_{k=0}^{n} \int_{l_k} G(z)dz + \sum_{k=1}^{n} \int_{\gamma_k} G(z)dz + \int_{\Gamma} G(z)dz + \int_{l} G(z)dz + \int_{\gamma} G(z)dz \\
&\quad - \left(\sum_{k=0}^{n} \int_{l_k} G^*(z)dz + \sum_{k=1}^{n} \int_{\gamma_k^*} G^*(z)dz + \int_{\Gamma^*} G^*(z)dz + \int_{l} G^*(z)dz + \int_{\gamma^*} G^*(z)dz \right).
\end{aligned} \tag{4.4}$$

Since we have that $G(z) = G^*(z)$ on l, due to l lying in the left half plane, the integrals over l cancel out. We also have that, through Cauchy's residue theorem,

$$\int_{\gamma_k^*} G^*(z)dz - \int_{\gamma_k} G^*(z)dz = 2\pi i \operatorname*{Res}_{z=a_k} G^*(z).$$

Rewritten we have

$$\int_{\gamma_k^*} G^*(z)dz = \int_{\gamma_k} G^*(z)dz + 2\pi i \operatorname*{Res}_{z=a_k} G^*(z).$$

Summing this result and taking the integrals over the lines $l_0, \ldots, l_n$ we get

$$\sum_{k=0}^{n} \int_{l_k} G^*(z)dz + \sum_{k=1}^{n} \int_{\gamma_k^*} G^*(z)dz = \sum_{k=0}^{n} \int_{l_k} G^*(z)dz + \sum_{k=1}^{n} \left(\int_{\gamma_k} G^*(z)dz + 2\pi i \operatorname*{Res}_{z=a_k} G^*(z) \right).$$

We see that the summed integrals are equivalent to the integrals over the curve $K(G^*, [0, \infty])$.

$$\sum_{k=0}^{n} \int_{l_k} G^*(z)dz + \sum_{k=1}^{n} \int_{\gamma_k^*} G^*(z)dz = \int_{K(G^*,[0,\infty])} G^*(z)dz + 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} G^*(z).$$

Now, using the definition of G^* and (4.1), we get

$$\begin{aligned} &\sum_{k=0}^{n} \int_{l_k} G^*(z)dz + \sum_{k=1}^{n} \int_{\gamma_k^*} G^*(z)dz \\ &\qquad = \int_{K(G^*,[0,\infty])} F(z)Q(\log(z + \pi i))dz + 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z)Q(\log(z + \pi i)). \end{aligned} \tag{4.5}$$

Likewise for G we have that the summed integration forms the integration over the curve $K(G, [0, \infty])$ and combined with (4.1) we get

$$\sum_{k=0}^{n} \int_{l_k} G(z)dz + \sum_{k=1}^{n} \int_{\gamma_k} G(z)dz = \int_{K(G,[0,\infty])} F(z)Q(\log(z - \pi i))dz. \tag{4.6}$$

Combining (4.5), (4.6) and (4.4) (while reminding that the integrals over l cancel out each other) we get

$$\oint_C G(z)dz + \oint_{C^*} G^*(z)dz$$

$$= \int_{K(G,[0,\infty])} F(z)Q(\log(z-\pi i))dz + \int_\Gamma G(z)dz + \int_\gamma G(z)dz$$

$$-\left(\int_{K(G^*,[0,\infty])} F(z)Q(\log(z+\pi i))dz + 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z)Q(\log(z+\pi i)) + \int_{\Gamma^*} G^*(z)dz + \int_{\gamma^*} G^*(z)dz\right)$$

$$= \int_{K(M,[0,\infty])} F(z)(Q(\log(z-\pi i)) - Q(\log(z+\pi i)))dz - 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z)Q(\log(z+\pi i))$$

$$+ \int_\Gamma G(z)dz + \int_\gamma G(z)dz - \int_{\Gamma^*} G^*(z)dz - \int_{\gamma^*} G^*(z)dz, \quad (4.7)$$

where $M(z) = F(z)(Q(\log(z-\pi i)) - Q(\log(z+\pi i)))$.We now restate our assumption that

$$\lim_{z\to 0} zF(-z)Q(\log(z)) = \lim_{z\to\infty} zF(-z)Q(\log(z)) = 0$$

and note that $L^*(-z) = \log(-z)$ if z is not in the first quadrant and $L(-z) = \log(-z)$ if z is not in the fourth quadrant. Since Γ lies only in the first and second quadrant, we have that on Γ, as $|z| \to \infty$,

$$\lim_{|z|=\rho\to\infty} G(z) = \lim_{|z|=\rho\to\infty} F(z)Q(L(-z)) = \lim_{|z|=\rho\to\infty} F(z)Q(\log(-z)) = \lim_{|z|=\rho\to\infty} \frac{0}{z} = 0.$$

We gain a similar result for G^* on Γ^*, since it does not lie in the first quadrant,

$$\lim_{|z|=\rho\to\infty} G^*(z) = \lim_{|z|=\rho\to\infty} F(z)Q(L^*(-z)) = \lim_{|z|=\rho\to\infty} F(z)Q(\log(-z)) = \lim_{|z|=\rho\to\infty} \frac{0}{z} = 0$$

and again, similarly for G and G^* on γ and γ^* respectively,

$$\lim_{|z|\to 0^+} G(z) = \lim_{|z|\to 0^+} F(z)Q(L(-z)) = \lim_{|z|\to 0^+} F(z)Q(\log(-z)) = \lim_{|z|\to 0^+} \frac{0}{z} = 0$$

and

$$\lim_{|z|\to 0^+} G^*(z) = \lim_{|z|\to 0^+} F(z)Q(L^*(-z)) = \lim_{|z|\to 0^+} F(z)Q(\log(-z)) = \lim_{|z|\to 0^+} \frac{0}{z} = 0.$$

Using these four results, and rearranging the terms from (4.7), we get

$$\int_{K(M,[0,\infty])} F(z)(Q(\log(z-\pi i)) - Q(\log(z+\pi i)))dz$$

$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z)Q(\log(z+\pi i)) + \oint_C G(z)dz + \oint_{C^*} G^*(z)dz$$

as $\rho \to \infty, \delta \to 0$ and further, through Cauchy's residue theorem combined with (4.2) and (4.3), we arrive at

$$\int_{K(M,[0,\infty])} F(z)(Q(\log(z-\pi i)) - Q(\log(z+\pi i)))dz$$
$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z)Q(\log(z+\pi i)) + 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=-z_k} F(-z)Q(\log(z)),$$

as $\rho \to \infty, \delta \to 0$. Reversing the order of subtraction within the integral and using the definition of the modified principal value yields the final result

$$P.V.^* \int_0^\infty F(x)(Q(\log(x+\pi i)) - Q(\log(x-\pi i)))dx$$
$$= -2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=-z_k} F(-z)Q(\log(z)) - 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z)Q(\log(z+\pi i)). \quad (4.8)$$

□

What we now want to do is to produce the last intermediate step for the first main theorem. This is done through the addition of a fifth assumption concerning the behaviour of Q.

Lemma 4.2. *Assume that the functions $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfies the following conditions*

(1) *The function Q is holomorphic in the complex plane.*

(2) *The function F is holomorphic in the complex plane, except for a finite amount of isolated singularities where those on the positive real axis are denoted $a_1, a_2, \ldots, a_m$ and the rest are denoted $z_1, z_2, \ldots, z_n$.*

(3) *The function F has an odd principal part for $a_1 \ldots, a_m$.*

(4) $\lim_{z\to 0} zF(-z)Q(\log(z)) = \lim_{z\to\infty} zF(-z)Q(\log(z)) = 0.$

(5) $Q(z+\pi i) = aQ(z) + b$, *where $a \neq 0$.*

Then

$$P.V.^* \int_0^\infty \left(a - \frac{1}{a}\right) F(x)Q(\log(x)) + b\left(1 + \frac{1}{a}\right) F(x)dx$$
$$= -2\pi i \left(\sum_{k=1}^{m} \operatorname*{Res}_{z=-z_k} F(-z)Q(\log(z)) + \frac{1}{2}\left(a + \frac{1}{a}\right) \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z)Q(\log(z)) - \frac{1}{2} b \left(1 - \frac{1}{a}\right) \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} F(z) \right).$$

Proof. Since we have assumed that

$$Q(z+\pi i) = aQ(z) + b,$$

it follows that

$$Q(z-\pi i) = \frac{Q(z)}{a} - \frac{b}{a}.$$

So we have that, from the left hand side of integral in Lemma 4.1,

$$F(z)(Q(\log(z+\pi i)) - Q(\log(z-\pi i))) = F(z)\left(aQ(\log(z)) + b - \frac{Q(\log(z))}{a} + \frac{b}{a}\right)$$
$$= F(z)\left(Q(\log(z))\left(a - \frac{1}{a}\right) + b\left(1 + \frac{1}{a}\right)\right).$$

That is, we can write the integral as

$$P.V.^{*}\int_0^\infty F(z)(Q(\log(z+\pi i)) - Q(\log(z-\pi i)))dz$$
$$= P.V.^{*}\int_0^\infty F(z)\left(Q(\log(z))\left(a - \frac{1}{a}\right) + b\left(1 + \frac{1}{a}\right)\right)dz \quad (4.9)$$

and from the second series in the right hand, we use that the residues lies on the real line, which means that $\log(z+\pi i) = \log(z-\pi i)$. Thus we split the residue into two halves and rewrite

$$F(z)Q(\log(z+\pi i)) = \frac{F(z)}{2}(Q(\log(z+\pi i)) + Q(\log(z-\pi i)))$$
$$= \frac{F(z)}{2}\left(aQ(\log(z)) + b + \frac{Q(\log(z))}{a} - \frac{b}{a}\right) = \frac{F(z)}{2}\left(Q(\log(z))\left(a + \frac{1}{a}\right) - b\left(1 + \frac{1}{a}\right)\right)$$

Thus we can rewrite

$$\operatorname*{Res}_{z=a_k} F(z)Q(\log(z+\pi i)) = \frac{1}{2}\left(\left(a + \frac{1}{a}\right)\operatorname*{Res}_{z=a_k} F(z)Q(\log(z)) - b\left(1 + \frac{1}{a}\right)\operatorname*{Res}_{z=a_k} F(z)\right). \quad (4.10)$$

Now, combining (4.9), (4.10) and (4.8) we get the desired result.

□

Now we just take the final step to produce the integral where it is only dependent on the function f.

Theorem A. *Assume that the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfies the following conditions*

(1) *The function f is holomorphic in the complex plane, except for a finite amount of isolated singularities where those on the positive real axis are denoted $a_1, a_2, \ldots, a_m$ and the rest are denoted $z_1, z_2, \ldots, z_n$.*

(2) *The function f has an odd principal part for $a_1 \ldots, a_m$.*

(3) $\lim_{z\to 0} zf(z)\log(z) = \lim_{z\to\infty} zf(z)\log(z) = 0.$

Then

$$P.V.^{*}\int_0^\infty f(x)dx = -\sum_{k=1}^{n}\operatorname*{Res}_{z=-z_k} f(-z)\log(z) - \sum_{k=1}^{m}\operatorname*{Res}_{z=a_k} f(z)\log(z). \quad (4.11)$$

Proof. We use Lemma 4.2, setting $Q(z) = z$, $a = 1$ and $b = \pi i$. This gives

$$P.V.^* \int_0^\infty 2\pi i f(x)dx = -2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=-z_k} f(-z)\log(z) - 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z)\log(z)$$

which simplified is

$$P.V.^* \int_0^\infty f(x)dx = -\sum_{k=1}^{n} \operatorname*{Res}_{z=-z_k} f(-z)\log(z) - \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z)\log(z).$$

□

5. The second theorem

The sources from this section have been p. 184-186 in [5], p. 175-176 in [2]. This will be the second main theorem and since we have produced or stated everything necessary, we proceed promptly.

Theorem B. *Assume that the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfies the following conditions*

(1) *The function f has no singularities on $\{a, b\}$, $-\infty < a < b < \infty$.*

(2) *The function f is holomorphic in the extended plane, except for a finite amount of isolated singularities where those on the interval (a, b) are denoted $a_1, a_2, \ldots, a_m$ and the rest are denoted $z_1, z_2, \ldots, z_n$.*

(3) *The singularities $a_1 \ldots, a_m$ are simple poles.*

Then

$$P.V.^* \int_a^b f(x)dx = -\left(\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \log\left(\frac{z-a}{z-b}\right) + \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z) \log\left(\frac{z-a}{b-z}\right)\right). \tag{5.1}$$

A small overview of the proof is essentially that we

(1) Transform the integral over (a, b) so the integral is taken over $(0, \infty)$ instead.

(2) Investigate how the transformation would affect the residues, using Lemma 3.11.

(3) Apply Theorem A.

Proof. We restate the Möbius-transformation T, as presented earlier in (3.14):

$$T(x) = \frac{x-a}{b-x} = t.$$

We also restate the mapping of a and b, from (3.15),

$$T(a) = \frac{a-a}{b-a} = 0,$$
$$T(b) = \frac{b-a}{b-b} = \infty.$$

We know that the transformation T maps $\hat{\mathbb{C}}$ bijectively onto $\hat{\mathbb{C}}$. See e.g. p.186 [3]. Note that there are no singularities at a or b, which would have been moved to 0 and ∞ respectively. The inverse of the transformation, T, and its derivative, is

$$T^{-1}(t) = \frac{a+bt}{1+t} = x, \tag{5.2}$$

$$\frac{dx}{dt} = \frac{-a+b}{(1+t)^2}. \tag{5.3}$$

Applying (3.15), (5.2) and (5.3) to the integral

$$\int_a^b f(x)dx$$

yields

$$\int_a^b f(x)dx = \int_0^\infty f\left(\frac{a+bt}{1+t}\right)\frac{-a+b}{(1+t)^2}dt = \int_0^\infty \frac{b-a}{(1+t)^2} f\left(\frac{a+bt}{1+t}\right)dt.$$

We define

$$F(t) = \frac{b-a}{(1+t)^2} f\left(\frac{a+bt}{1+t}\right) \tag{5.4}$$

which gives us

$$\int_a^b f(x)dx = \int_0^\infty \frac{b-a}{(t+1)^2} f\left(\frac{bt+a}{t+1}\right)dt = \int_0^\infty F(t)dt. \tag{5.5}$$

Before applying Theorem A to the integral in (5.5) we want to rewrite two series of residues. For the first, take the two functions

$$H(z) = \frac{bz+a}{1+z}$$

and

$$G_1(z) = \frac{(b-z)^2}{b-a} f(z) \log\left(\frac{z-a}{z-b}\right).$$

We have that the inverse of H exists since it is a Möbius transformation

$$H^{-1}(z) = \frac{z-a}{b-z} = h(z).$$

Observe that we can find H^{-1} within G_1 and write

$$G_1(z) = \frac{(b-z)^2}{b-a} f(z) \log\left(\frac{z-a}{z-b}\right) = \frac{(b-z)^2}{b-a} f(z) \log\left(-H^{-1}(z)\right).$$

The derivate of h is

$$h'(z) = \frac{d}{dz}\frac{z-a}{b-z} = \frac{b-a}{(b-z)^2}.$$

Now taking $G_1 \circ H$ yields

$$G_1(H(z)) = \frac{\left(b - \frac{bz+a}{1+z}\right)^2}{b-a} f\left(\frac{bz+a}{1+z}\right) \log\left(-H^{-1}(H(z))\right) = \frac{b-a}{(z+1)^2} f\left(\frac{bz+a}{1+z}\right) \log(-z)$$

and multiplying G_1 with h' gives

$$G_1(z)h'(z) = \frac{(b-z)^2}{b-a} f(z) \left(\frac{b-a}{(b-z)^2}\right) \log\left(\frac{z-a}{z-b}\right) = f(z) \log\left(\frac{z-a}{z-b}\right).$$

That is, we have

$$G_1(H(z)) = \frac{b-a}{(z+1)^2} f\left(\frac{bz+a}{1+z}\right) \log(-z) \tag{5.6}$$

and

$$G_1(z)h'(z) = f(z) \log\left(\frac{z-a}{z-b}\right). \tag{5.7}$$

Lastly we note that H is holomorphic everywhere in $\hat{\mathbb{C}}$, except for at $z = -1$ where it has a simple pole. Thus we can apply Lemma 3.11 to (5.6) and (5.7) which gives

$$\operatorname*{Res}_{z=z'_k} \frac{b-a}{(1+z)^2} f\left(\frac{bz+a}{1+z}\right) \log(-z) = \operatorname*{Res}_{z=z_k} f(z) \log\left(\frac{z-a}{z-b}\right), \tag{5.8}$$

where $z'_k = H(z_k)$. We note that in the left hand side of (5.8) there is a fraction which could give the impression of an additional isolated singularity which residue can be evaluated at $z = -1$, despite the behavior of f, this is not the case as demonstrated. The residue will be equivalent to the right hand side evaluated at $z = \infty$. This will be referred to later on. To rewrite the second series of residues, like before, we modify G_1 slightly and set

$$G_2(z) = \frac{(b-z)^2}{b-a} f(z) \log\left(\frac{z-a}{b-z}\right).$$

Again, we observe that the inverse of H is within G_2

$$G_2(z) = \frac{(b-z)^2}{b-a} f(z) \log\left(\frac{z-a}{z-b}\right) = \frac{(b-z)^2}{b-a} f(z) \log\left(H^{-1}(z)\right).$$

Now taking $G_2 \circ H$ yields

$$G_2(H(z)) = \frac{\left(b - \frac{bz+a}{1+z}\right)^2}{b-a} f\left(\frac{bz+a}{1+z}\right) \log\left(H^{-1}H((z))\right) = \frac{b-a}{(z+1)^2} f\left(\frac{bz+a}{1+z}\right) \log(z)$$

and multiplying G_2 with h' gives

$$G_2(z)h'(z) = \frac{(b-z)^2}{b-a} f(z) \left(\frac{b-a}{(b-z)^2}\right) \log\left(\frac{z-a}{z-b}\right) = f(z) \log\left(\frac{z-a}{z-b}\right).$$

That is, we have

$$G_2(H(z)) = \frac{b-a}{(z+1)^2} f\left(\frac{bz+a}{1+z}\right) \log(z) \tag{5.9}$$

and

$$G_2(z)h'(z) = f(z)\log\left(\frac{z-a}{b-z}\right). \tag{5.10}$$

This gives us, again applying Lemma 3.11 to (5.9) and (5.10) that

$$\operatorname*{Res}_{z=z'_k} \frac{b-a}{(1+z)^2} f\left(\frac{bz+a}{1+z}\right) \log(z) = \operatorname*{Res}_{z=z_k} f(z)\log\left(\frac{z-a}{b-z}\right), \tag{5.11}$$

where again $z'_k = H(z_k)$. We now proceed to apply Theorem A to the integral in (5.5), but before that we need to prove that

(1) The function F is holomorphic in the extended plane, except for a finite amount of isolated singularities where those on the positive real axis are denoted $a'_1, a'_2, \ldots, a'_m$ and the rest are denoted $z'_1, z'_2, \ldots, z'_n$.

(2) The function F has an odd principal part for $a'_1, a'_2, \ldots, a'_m$.

(3) $\lim_{z\to 0} zF(z)\log(z) = \lim_{z\to\infty} zF(z)\log(z) = 0$

We observe that we can use (5.2) to rewrite

$$F(t) = \frac{b-a}{(1+t)^2} f\left(\frac{a+bt}{1+t}\right) = \frac{b-a}{(1+t)^2} f\left(T^{-1}(t)\right).$$

Lemma 3.12 gives that T maps the line (a,b) onto $(0,\infty)$, which in turn gives that the inverse T^{-1} maps $(0,\infty)$ onto (a,b). We also know that Möbius transformations are bijective functions, therefore we have that the singularities of f are moved, with no singularities mapped onto the same values, when taking $f \circ T^{-1}$. With the exception of a finite amount of singularities, where no singularities lies on $\{a,b\}$, f is holomorphic, as stated in the assumptions. Consequently $f \circ T^{-1}$ also has a finite amount of singularities and is holomorphic otherwise, with especially no singularities on $\{0,\infty\}$. We now refer to the note after (5.8), where we remarked that the fraction might give the impression of an additional isolated singularity. To continue, we now see if the function F converges according to (3).

$$\lim_{t\to 0} F(t) = \lim_{t\to 0} \frac{b-a}{(1+t)^2} f\left(\frac{a+bt}{1+t}\right) = C$$

for some constant C since no part of F has a singularity in $t=0$ and that

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \frac{b-a}{(1+t)^2} f\left(\frac{a+bt}{1+t}\right) = O(t^{-2})$$

since $f \circ T^{-1}$ has no singularity in $t=\infty$ and $\frac{b-a}{(1+t)^2}$ is of order t^{-2}. We use that to show

$$\lim_{z\to 0} zF(z)\log(z) = \lim_{z\to 0} zC\log(z) = 0$$

and

$$\lim_{z\to\infty} zF(z)\log(z) = \lim_{z\to\infty} zO(z^{-2})\log(z) = 0.$$

Thus the requirements (1) and (3) are fulfilled. Lastly we investigate how the simple poles are affected by the transformation. If they maintain the odd principal part then the function F satisfies all the requirements. We take the Laurent expansion for one of the simple poles, denoted a_k

$$f(z) = \sum_{j=-1}^{\infty} s_j (z-a_k)^j.$$

We know that the composition of two holomorphic functions are holomorphic, so by approaching term wise we see that it is enough to study the nature of the singularity provided by

$$\frac{s_{-1}}{T^{-1}(z) - a_k}$$

and we get

$$\frac{s_{-1}}{T^{-1}(z) - a_k} = \frac{s_{-1}}{\dfrac{a+bz}{1+z} - a_k} = \frac{s_{-1}}{\dfrac{a+bz-(1+z)a_k}{1+z}} = \frac{s_{-1}(1+z)}{a-1+(b-a_k)z}$$

which we can observe has a simple pole at $z = \frac{a-1}{b-a_k}$. Therefore we have that the simple poles property of odd principal value are preserved by the transformation T^{-1} and that they are mapped onto the positive real line, which was the last thing we needed to demonstrate. Theorem A then gives

$$\begin{aligned} &P.V.^* \int_0^\infty \frac{b-a}{(t+1)^2} f\left(\frac{bt+a}{t+1}\right) dt \\ &= \sum_{k=1}^{n} \operatorname*{Res}_{z=-z_k'} \frac{b-a}{(1-z)^2} f\left(\frac{a-bz}{1-z}\right)\log(z) - \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k'} \frac{b-a}{(z+1)^2} f\left(\frac{bz+a}{z+1}\right)\log(z) \end{aligned} \quad (5.12)$$

where the first n residues are the residues not on the positive real line and the m following residues are those on the real line.

Through reflecting the function in the imaginary axis, we obtain

$$\operatorname*{Res}_{z=-z_k'} \frac{b-a}{(1-z)^2} f\left(\frac{a-bz}{1-z}\right)\log(z) = -\operatorname*{Res}_{z=z_k'} \frac{b-a}{(1+z)^2} f\left(\frac{bz+a}{1+z}\right)\log(-z). \quad (5.13)$$

Thus (5.13) and (5.12) gives us

$$\begin{aligned} &P.V.^* \int_0^\infty \frac{b-a}{(t+1)^2} f\left(\frac{bt+a}{t+1}\right) dt \\ &= -\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k'} \frac{b-a}{(z+1)^2} f\left(\frac{bz+a}{1+z}\right)\log(-z) - \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k'} \frac{b-a}{(z+1)^2} f\left(\frac{bz+a}{z+1}\right)\log(z). \end{aligned} \quad (5.14)$$

Combining (5.8), (5.11) and (5.14) gives us

$$P.V.^* \int_0^\infty \frac{b-a}{(t+1)^2} f\left(\frac{bt+a}{t+1}\right) dt = -\sum_{k=1}^n \operatorname*{Res}_{z=z_k} f(z)\log\left(\frac{z-a}{z-b}\right) - \sum_{k=1}^m \operatorname*{Res}_{z=a_k} f(z)\log\left(\frac{z-a}{b-z}\right). \quad (5.15)$$

Lastly, we combine (5.5) and (5.15) to produce

$$P.V.^* \int_a^b f(x)dx = -\left(\sum_{k=1}^n \operatorname*{Res}_{z=z_k} f(z)\log\left(\frac{z-a}{z-b}\right) + \sum_{k=1}^m \operatorname*{Res}_{z=a_k} f(z)\log\left(\frac{z-a}{b-z}\right)\right). \quad (5.16)$$

□

5.1. **Examples.** We end this essay with some simple examples how to apply Theorem B. We shall integrate a polynomial, a rational function and the normal distribution that occur frequently in statistics. For simplicity we integrate over the unit interval.

Example 5.1. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial given by

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n .$$

If we use the fundamental theorem of calculus, then we immediately arrive at

$$\int_0^1 f(x)dx = \sum_{k=0}^n \frac{a_k}{k+1}.$$

Before we make use of Theorem B, note that

$$\log\left(\frac{1}{1-z}\right) = -\log(1-z) = -\log(1+(-z)) = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots .$$

This Maclaurin series, with the convergence radius $|z| < 1$, together with Theorem B yields that

$$\begin{aligned} P.V^* \int_0^1 f(x)dx &= -\operatorname*{Res}_{z=\infty} f(z)\log\left(\frac{z}{z-1}\right) \\ &= -\operatorname*{Res}_{z=0} -\frac{1}{z^2} f\left(\frac{1}{z}\right)\log\left(\frac{\frac{1}{z}}{\frac{1}{z}-1}\right) = \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)\log\left(\frac{1}{1-z}\right) \\ &= \operatorname*{Res}_{z=0} \frac{1}{z^2}\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \ldots + \frac{a_n}{z^n}\right)\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\right) \\ &= \operatorname*{Res}_{z=0} \left(\frac{a_0}{z^2} + \frac{a_1}{z^3} + \frac{a_2}{z^4} + \frac{a_3}{z^5} + \ldots + \frac{a_n}{z^{n+2}}\right)\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\right) \\ &== a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} + \ldots + \frac{a_n}{n+1} = \sum_{k=0}^n \frac{a_k}{k+1} . \end{aligned}$$

□

We next proceed with integrating a rational function.

Example 5.2. Let $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be two polynomials such that they have no common zeros and Q have no zeros in $[0, 1]$. Set

$$f(z) = \frac{P(z)}{Q(z)}.$$

In this example we could use Theorem B to evaluate

$$P.V.^* \int_0^1 f(x)dx,$$

instead of using the ordinary method of finding a partial fraction decomposition. Let us number the zeroes of Q with $z_1, z_2, \ldots, z_n$, and if necessary set $z_{n+1} = \infty$. By using Theorem B we get that

$$P.V.^* \int_0^1 f(x)dx = -\sum_{k=1}^{n+1} \operatorname*{Res}_{z=z_k} f(z) \log\left(\frac{z}{z-1}\right).$$

□

The last example involves a function known as the standard normal distribution, which is a tool often used within statistics.

Example 5.3. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Since f does not posses a primitive function expressed in elementary functions, the integral $\int_0^1 f(x)\,dx$ is not straight forward. So let us apply Theorem B to evaluate this integral. First we note that the Maclaurin series

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \ldots$$

converges for all z. Using that and again, as in Example 5.1, the Maclaurin series

$$\log\left(\frac{1}{1-z}\right) = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots$$

we get that

$$\begin{aligned} P.V^* \int_0^1 f(x)dx &= -\operatorname*{Res}_{z=\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \log\left(\frac{z}{z-1}\right) = -\operatorname*{Res}_{z=0} -\frac{1}{z^2\sqrt{2\pi}} e^{-\frac{1}{z^2}/2} \log\left(\frac{1}{1-z}\right) \\ &= \operatorname*{Res}_{z=0} \frac{1}{z^2\sqrt{2\pi}} \left(1 - \frac{1}{2}\frac{1}{z^2} + \frac{1}{2!2^2}\frac{1}{z^4} - \frac{1}{3!2^3}\frac{1}{z^6} + \frac{1}{4!2^4}\frac{1}{z^8} - \ldots\right)\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{2}\frac{1}{3} + \frac{1}{2!2^2}\frac{1}{5} - \frac{1}{3!2^3}\frac{1}{7} + \ldots\right) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1^k)}{(1+2k)2^k k!}. \end{aligned}$$

□

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