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ℓ -DEGREE TURÁN DENSITY

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ABSTRACT. Let H_n be a k -graph on n vertices. For $0 \leq \ell < k$ and an ℓ -subset T of $V(H_n)$, define the degree $\deg(T)$ of T to be the number of $(k - \ell)$ -subsets S such that $S \cup T$ is an edge in H_n . Let the minimum ℓ -degree of H_n be $\delta_\ell(H_n) = \min\{\deg(T) : T \subseteq V(H_n) \text{ and } |T| = \ell\}$. Given a family $\mathcal{F}$ of k -graphs, the ℓ -degree Turán number $\text{ex}_\ell(n, \mathcal{F})$ is the largest $\delta_\ell(H_n)$ over all $\mathcal{F}$ -free k -graphs H_n on n vertices. Hence, $\text{ex}_0(n, \mathcal{F})$ is the Turán number. We define ℓ -degree Turán density to be

$$\pi_\ell^k(\mathcal{F}) = \limsup_{n \rightarrow \infty} \frac{\text{ex}_\ell(n, \mathcal{F})}{\binom{n-\ell}{k}}.$$

In this paper, we show that for $k > \ell > 1$, the set of $\pi_\ell^k(\mathcal{F})$ is dense in the interval $[0, 1)$. Hence, there is no “jump” for ℓ -degree Turán density when $k > \ell > 1$. We also give a lower bound on $\pi_\ell^k(\mathcal{F})$ in terms of an ordinary Turán density.

1. INTRODUCTION

A k -uniform hypergraph, or a k -graph for short, is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set of vertices and the edge set $E(H)$ is a set of k -subsets of $V(H)$. The notation H_n indicates that $|V(H_n)| = n$.

Given a family $\mathcal{F}$ of k -graphs, a k -graph H_n is $\mathcal{F}$ -free if it contains no copy of any member of $\mathcal{F}$. The Turán number $\text{ex}(n, \mathcal{F})$ is the maximum number of edges in H_n for all $\mathcal{F}$ -free k -graphs H_n (with n vertices). Katona, Nemetz and Simonovits [11] showed that the Turán density $\pi(\mathcal{F}) = \lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{F}) / \binom{n}{k}$ exists. The celebrated Erdős–Simonovits–Stone theorem [5, 7] implies that for $k = 2$ and any family $\mathcal{F}$ of 2-graphs, $\pi(\mathcal{F}) = \min \left\{ 1 - \frac{1}{\chi(F)-1} : F \in \mathcal{F} \right\}$, where $\chi(G)$ is the chromatic number of G . For $k \geq 3$, determining $\pi(\mathcal{F})$ is known to be hard.

Recall that the Turán number $\text{ex}(n, \mathcal{F})$ studies the maximum number of edges in an $\mathcal{F}$ -free k -graph of order n . One natural variant is the so-called Dirac-type condition, which studies the largest possible minimum degree for $\mathcal{F}$ -free k -graphs. For a k -graph H and an ℓ -subset T of $V(H)$,

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let the *neighbourhood* of T be $N^H(T) = \{S \in \binom{V(H)}{k-\ell} : T \cup S \in E(H)\}$ and let the *degree* of T be $\deg^H(T) = |N^H(T)|$. (Here, $\binom{V}{k}$ is the set of k -subsets of V .) The *minimum ℓ -degree* $\delta_\ell(H)$ is $\min\{\deg^H(T) : T \in \binom{V(H)}{\ell}\}$. Hence, if $\ell \geq k$, then $\delta_\ell(H) = 0$ unless H is complete and $k = \ell$. Recently, there has been an increasing interest in Dirac-type conditions for perfect matchings and Hamiltonian cycles. We recommend [17] for a survey on perfect matchings and Hamiltonian cycles in k -graphs.

Note that $\delta_0(H) = |E(H)|$. Thus, $\text{ex}(n, \mathcal{F})$ is the maximum $\delta_0(H_n)$ over all $\mathcal{F}$ -free k -graphs H_n . In this paper, we generalize $\text{ex}(n, \mathcal{F})$ for the minimum ℓ -degree. Define *ℓ -degree Turán number* $\text{ex}_\ell(n, \mathcal{F})$ to be the maximum $\delta_\ell(H_n)$ over all $\mathcal{F}$ -free k -graphs H_n . We further define the *ℓ -degree Turán density* of $\mathcal{F}$ to be

$$\pi_\ell(\mathcal{F}) = \limsup_{n \rightarrow \infty} \frac{\text{ex}_\ell(n, \mathcal{F})}{\binom{n}{k-\ell}}.$$

As mentioned by Mubayi and Zhao [15] for the case $\ell = k - 1$, one should divide by $\binom{n-k}{k-\ell}$ instead of $\binom{n}{k-\ell}$ in the definition above. Since ℓ and k are fixed and n tends to infinity, this does not affect the value of $\pi_\ell(\mathcal{F})$.

Note that for $\ell = 0$, we have $\text{ex}(n, \mathcal{F}) = \text{ex}_0(n, \mathcal{F})$ and $\pi_0(\mathcal{F}) = \pi(\mathcal{F})$. For $\ell = 1$, it is not difficult to deduce that the limit $\pi_1(\mathcal{F}) = \lim_{n \rightarrow \infty} \text{ex}_1(n, \mathcal{F}) / \binom{n}{k-1} = \pi(\mathcal{F})$. The exact value of $\pi(\mathcal{F})$ is determined only for a few families $\mathcal{F}$ of k -graphs. We recommend [18, 12] for surveys on hypergraph Turán problems.

By a simple averaging argument, we know that for $0 \leq \ell \leq \ell' < k$ and k -graphs H_n , if $\delta_{\ell'}(H_n) \geq \alpha \binom{n}{k-\ell'}$ then $\delta_\ell(H_n) \geq (\alpha - o(1)) \binom{n}{k-\ell}$. Thus, $\pi_\ell(\mathcal{F}) \geq \pi_{\ell'}(\mathcal{F})$ for $0 \leq \ell \leq \ell' < k$. Recall that for a k -graph H and an ℓ -subset T of $V(H)$, $N^H(T) = \{S \in \binom{V(H)}{k-\ell} : T \cup S \in E(H)\}$. Thus, $N^H(T)$ can be viewed as a $(k - \ell)$ -graph. Often $N^H(T)$ is called the *link-graph* of T . Define the *ℓ -link family* $\mathcal{L}_\ell(H)$ of H to be the family of the $N^H(T)$ for all ℓ -subsets T of $V(H)$. For example, $\mathcal{L}_\ell(K_t^k) = \{K_{t-\ell}^{k-\ell}\}$, where K_t^k is the complete k -graphs on t vertices. Also, $\mathcal{L}_1(K_4^3 - e) = \{K_3, P_3\}$, where $K_4^3 - e$ is the unique 3-graph on 4 vertices with 3 edges and P_3 is the path of length 2. For a family $\mathcal{F}$ of k -graphs, $\mathcal{L}_\ell(\mathcal{F})$ is simply the union of $\mathcal{L}_\ell(F)$ for all $F \in \mathcal{F}$. Using a probabilistic construction, we show that $\pi_\ell(\mathcal{F}) \geq \pi(\mathcal{L}_{\ell-1}(\mathcal{F}))$.

Proposition 1.1. *Let $2 \leq \ell < k$ be integers and let $\mathcal{F}$ be a family of k -graphs. Then $\pi_\ell(\mathcal{F}) \geq \pi(\mathcal{L}_{\ell-1}(\mathcal{F}))$.*

Corollary 1.2. *For $2 \leq \ell < k \leq t$, $\pi_\ell(K_t^k) \geq \pi_1(K_{t-\ell+1}^{k-\ell+1})$. In particular, $\pi_{k-1}(K_t^k) \geq (t - k) / (t - k + 1)$ and $\pi_{k-1}(K_{k+1}^k) \geq 1/2$.*

Note that for $k = 3$, the corollary above gives $\pi_2(K_4^3) \geq 1/2$. In fact, Czygrinow and Nagle [3] conjectured that equality holds. For $k = 4$, the corollary also implies that $\pi_3(K_5^4) \geq 1/2$. Here, we also give a different construction to show that $\pi_3(K_5^4) \geq 1/2$, which is due to Giraud [10].

Example 1.3. Let M be an $m \times m$ 0/1-matrix. Define the hypergraph $H(M)$ with $2m$ vertices corresponding to the rows and columns of M . Any 4-set consisting of exactly 3 rows or 3 columns is an edge. Also any 4-set of 2 rows and 2 columns inducing a 2×2 submatrix with odd sum is an edge.

It is easy to verify that $H(M)$ is K_5^4 -free and $\delta_3(H(M)) \geq (1/2 - o(1))|V(H(M))|$. When M is the Hadamard matrix and replacing the -1 's by 0's, $H(M)$ was originally used to show that $\pi(K_5^4) \geq 11/16$. Sidorenko [18] conjectured that $\pi(K_5^4) = 11/16$. The best known upper bound is $\pi(K_5^4) \leq \frac{1753}{2380} \approx 0.73655 \dots$ given by the second author [13]. We believe that it would be very interesting to get an answer to the following conjecture.

Conjecture 1.4. For all $k \geq 2$, $\pi_{k-1}(K_{k+1}^k) = 1/2$.

Note that $(\text{ex}(n, \mathcal{F})/\binom{n}{k})_{n \in \mathbb{N}}$ is a decreasing sequence for all families $\mathcal{F}$ of k -graphs. Mubayi and Zhao [15] commented that they could not prove that $(\text{ex}_{k-1}(n, \mathcal{F})/n)_{n \in \mathbb{N}}$ (or $(\text{ex}_{k-1}(n, \mathcal{F})/(n - k + 1))_{n \in \mathbb{N}}$) is a decreasing sequence for $k \geq 3$. Here, we give an explicit example to show that the sequence is not necessarily monotone. With the aid of computer, we determined the exact values of $\text{ex}_2(n, K_4^3)$ for $n \leq 11$. The exact values are, with the pairs being $(n, \text{ex}_2(n, K_4^3))$, $\{(5, 2), (6, 3), (7, 4), (8, 4), (9, 5), (10, 5), (11, 6)\}$. Thus, neither $(\text{ex}_2(n, K_4^3)/n)_{n \in \mathbb{N}}$ nor $(\text{ex}_2(n, K_4^3)/(n - 2))_{n \in \mathbb{N}}$ is a decreasing sequence. Although $(\text{ex}_{k-1}(n, \mathcal{F})/n)_{n \in \mathbb{N}}$ is not monotone, Mubayi and Zhao [15] showed that this sequence does indeed converge to $\pi_{k-1}(\mathcal{F})$. We generalize their result and show that $\pi_\ell(\mathcal{F}) = \lim_{n \rightarrow \infty} \text{ex}_\ell(n, \mathcal{F})/\binom{n}{k-\ell}$ for all $k > \ell \geq 0$ and all families $\mathcal{F}$ of k -graphs. This observation has been made before ([12] page 118) but that we have not found a written proof of it.

Proposition 1.5. Given integers $k > \ell \geq 0$, $\pi_\ell(\mathcal{F}) = \lim_{n \rightarrow \infty} \text{ex}_\ell(n, \mathcal{F})/\binom{n}{k-\ell}$ for all families $\mathcal{F}$ of k -graphs.

It is not surprising that for a k -graph F , $\pi_\ell(F)$ also satisfies the so-called “supersaturation” phenomenon discovered by Erdős and Simonovits [6]. Informally speaking, given a k -graph F of order f and $\varepsilon > 0$, there exist $\delta > 0$ and N such that every k -graph H_n with $n \geq N$ and $\delta_\ell(H_n) \geq (\pi_\ell(F) + \varepsilon)\binom{n}{k-\ell}$ contains $\delta\binom{n}{f}$ copies of F . By the supersaturation phenomenon, “blowing-up F ” does not change the value of $\pi_\ell(F)$. Given an integer s and a k -graph F , the s -blow-up $F(s)$ of F is the k -graph (V', E') such that V' is obtained by replacing each $v_i \in V(F)$ by a vertex set V_i of size s , and $E' = \{u_1 \dots u_k : u_j \in V_{i_j}, v_{i_1} \dots v_{i_k} \in E(F)\}$. Given a family of k -graphs, let $\mathcal{F}(s) = \{F(s) : F \in \mathcal{F}\}$.

Proposition 1.6 (Supersaturation). Let $k > \ell \geq 1$ be integers. Let $\mathcal{F}$ be a family of k -graphs. For any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon, \mathcal{F}) > 0$ and N such that every k -graph H_n with $n > N$ and $\delta_\ell(H_n) > (\pi_\ell(\mathcal{F}) + \varepsilon)\binom{n}{k-\ell}$ contains $\delta\binom{n}{f}$ copies of F for some $F \in \mathcal{F}$ on f vertices. Moreover, $\pi_\ell(\mathcal{F}) = \pi_\ell(\mathcal{F}(s))$ for all integers $s \geq 1$.

1.1. Jumps. For $0 \leq a < b \leq 1$, we denote by $[a, b)$ and (a, b) the intervals $\{c : a \leq c < b\}$ and $\{c : a < c < b\}$, respectively. For $k > \ell \geq 0$, let

$$\Pi_\ell^k = \{\pi_\ell(\mathcal{F}) : \mathcal{F} \text{ is a family of } k\text{-graphs}\}.$$

The Erdős-Simonovits-Stone theorem implies that $\Pi_0^2 = \{0, 1/2, 2/3, \dots\}$. The constant $\alpha \in [0, 1)$ is a *jump for k* , if there exists a constant $\delta > 0$ such that no family $\mathcal{F}$ of k -graphs satisfies $\pi(\mathcal{F}) \in (\alpha, \alpha + \delta)$. Hence, every $\alpha \in [0, 1)$ is a jump for $k = 2$. For $k \geq 3$, every $\alpha \in [0, k!/k^k)$ is a jump for k by a result of Erdős [4]. However, Frankl and Rödl [9] showed that $1 - t^{1-k}$ is not a jump for $t > 2k$ and $k \geq 3$. Later, other non-jump values were discovered, e.g. [8, 16]. On the other hand, the first example of jumps in $[k!/k^k, 1)$ for $k \geq 3$ was only given recently by Baber and Talbot [2].

We generalize the concept of jump as follows. Given $k \geq \ell \geq 0$, the constant $\alpha \in [0, 1)$ is a π_ℓ^k -*jump*, if there exists a constant $\delta > 0$ such that no family $\mathcal{F}$ of k -graphs satisfies $\pi_\ell(\mathcal{F}) \in (\alpha, \alpha + \delta)$. Equivalently, $\alpha \in [0, 1)$ is a π_ℓ^k -jump if there exists a constant $\delta > 0$ such that $(\alpha, \alpha + \delta) \cap \Pi_\ell^k = \emptyset$. Recall that $\pi_0(\mathcal{F}) = \pi_1(\mathcal{F})$ for every family $\mathcal{F}$ of k -graphs, so for $k = 2$ any $\alpha \in [0, 1)$ is a π_ℓ^2 -jump. Mubayi and Zhao [15] showed that no $\alpha \in [0, 1)$ is a π_{k-1}^k -jump for all $k \geq 3$. We generalize their construction to show that no $\alpha \in [0, 1)$ is a π_ℓ^k -jump for all $k > \ell > 1$.

Theorem 1.7. *For all integers $k > \ell > 1$, no $\alpha \in [0, 1)$ is a π_ℓ^k -jump. In particular, Π_ℓ^k is dense in $[0, 1)$.*

This answers a question of Mubayi, Pikhurko and Sudakov [14]. Furthermore, we would like to ask the question whether $\Pi_\ell^k = [0, 1)$.

We prove Propositions 1.5 and 1.6 in the next section. Proposition 1.1 is proved in Section 3. Finally, we prove Theorem 1.7 in Sections 5 and 6.

2. SUPERSATURATION

Our aim for this section is to prove Propositions 1.5 and 1.6. For $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, 2, \dots, n\}$. Given a k -graph H and $U \subseteq V(H)$, write $G[U]$ to be the induced k -subgraph of H on U . We will need Azuma's inequality (see e.g. [1]).

Theorem 2.1 (Azuma's inequality). *Let $\{X_i : i = 0, 1, \dots\}$ be a martingale and $|X_i - X_{i-1}| \leq c_i$. Then for all positive integers N and $\lambda > 0$,*

$$\mathbb{P}(X_N \leq X_0 - \lambda) \leq \exp\left(\frac{-\lambda^2}{2 \sum_{i=1}^N c_i^2}\right).$$

The following lemma is important.

Lemma 2.2. *Let $k > \ell \geq 1$. Given $\alpha, \varepsilon > 0$ with $\alpha + \varepsilon < 1$, let $M(k, \ell, \varepsilon)$ be the smallest integer such that*

$$\binom{m}{\ell} \exp\left(-\frac{\varepsilon^2 m}{8(k-\ell)^2}\right) \leq 1/2 \text{ and } \frac{\binom{m}{k-\ell} - \binom{m-\ell}{k-\ell}}{\binom{m}{k-\ell} - \binom{m-\ell}{k-\ell}/2} \leq \varepsilon$$

for all $m > M(k, \ell, \varepsilon)$. If $n \geq m \geq M(k, \ell, \varepsilon)$ and H is a k -graph on $[n]$ with $\delta_\ell(H) \geq (\alpha + \varepsilon) \binom{n}{k-\ell}$, then the number of m -subsets S of $[n]$ satisfying $\delta_\ell(H[S]) > \alpha \binom{m}{k-\ell}$ is at least $\frac{1}{2} \binom{n}{m}$.

Proof. Given an ℓ -subset T of $[n]$, we call an m -subset S of $[n]$ *bad for T* if $T \subseteq S$ and $|N(T) \cap \binom{S}{k-\ell}| \leq \alpha \binom{m}{k-\ell}$. An m -subset S of $[n]$ is called *bad* if it is bad for some T . In the next claim, we count the number of m -subsets S that is bad for a given T .

Claim 2.3. *Let ϕ_T be the number of m -subsets S that are bad for an ℓ -subset T of $[n]$. Then*

$$\phi_T \leq \binom{n-\ell}{m-\ell} \exp\left(-\frac{\varepsilon^2 m}{8(k-\ell)^2}\right).$$

Proof of claim. Observe that

$$\phi_T = \left| \left\{ S' \in \binom{[n] \setminus T}{m-\ell} : \left| N(T) \cap \binom{S'}{k-\ell} \right| \leq \alpha \binom{m}{k-\ell} \right\} \right|.$$

Let X be the random variable $|N(T) \cap \binom{S'}{k-\ell}|$, where S' is an $(m-\ell)$ -subset of $[n] \setminus T$ picked uniformly at random. We consider the vertex exposure martingale on S' . Let Z_i be the i th exposed vertex in S' . Define $X_i = \mathbb{E}(X | Z_1, \dots, Z_i)$. Note that $\{X_i : i = 0, 1, \dots, m-\ell\}$ is a martingale and $X_0 \geq (\alpha + \varepsilon) \binom{m-\ell}{k-\ell} \geq (\alpha + \varepsilon/2) \binom{m}{k-\ell}$. Moreover, $|X_i - X_{i-1}| \leq \binom{i-1}{k-\ell-1}$. Thus, by Theorem 2.1 taking $\lambda = \varepsilon \binom{m}{k-\ell}/2$ and $c_i = \binom{i-1}{k-\ell-1}$, we have

$$\begin{aligned} \mathbb{P}\left(X_m \leq \alpha \binom{m-\ell}{k-\ell}\right) &\leq \mathbb{P}(X_m \leq X_0 - \lambda) \leq \exp\left(\frac{-\varepsilon^2 \binom{m}{k-\ell}^2}{8 \sum_{i=1}^{m-\ell} \left(\binom{i-1}{k-\ell-1}\right)^2}\right) \\ &\leq \exp\left(\frac{-\varepsilon^2 \binom{m}{k-\ell}^2}{8m \binom{m-1}{k-\ell-1}^2}\right) = \exp\left(-\frac{\varepsilon^2 m}{8(k-\ell)^2}\right). \end{aligned}$$

Thus, the claim follows. $\square$

Therefore, the number of bad m -subsets is at most

$$\begin{aligned} \sum_{T \in \binom{[n]}{\ell}} \phi_T &\leq \binom{n}{\ell} \binom{n-\ell}{m-\ell} \exp\left(-\frac{\varepsilon^2 m}{8(k-\ell)^2}\right) \\ &= \binom{n}{m} \binom{m}{\ell} \exp\left(-\frac{\varepsilon^2 m}{8(k-\ell)^2}\right) \leq \frac{1}{2} \binom{n}{m}. \end{aligned}$$

Hence, the proof of the lemma is completed. $\square$

Corollary 2.4. *Let $k > \ell \geq 1$. Given $\varepsilon > 0$, let $M(k, \ell, \varepsilon)$ be the integer as defined in Lemma 2.2. Then*

$$\frac{\text{ex}_\ell(n, \mathcal{F})}{\binom{n}{k-\ell}} - \frac{\text{ex}_\ell(m, \mathcal{F})}{\binom{m}{k-\ell}} \leq \varepsilon$$

for all $n \geq m \geq M(k, \ell, \varepsilon)$ and all families $\mathcal{F}$ of k -graphs.

Proof. Let H_n be an $\mathcal{F}$ -free graph with $\delta_\ell(H_n) = \text{ex}_\ell(n, \mathcal{F})$. Without loss of generality, we may assume that $\text{ex}_\ell(n, \mathcal{F}) > \varepsilon \binom{n}{k-\ell}$ or else there is nothing to prove. By Lemma 2.2, there exists an m -subset S such that $\delta_\ell(H[S]) \geq \left(\frac{\text{ex}_\ell(n, \mathcal{F})}{\binom{n}{k-\ell}} - \varepsilon \right) \binom{m}{k-\ell}$. Note that $H[S]$ is $\mathcal{F}$ -free and so $\delta_\ell(H[S]) \leq \text{ex}_\ell(m, \mathcal{F})$. $\square$

Proof of Proposition 1.5. Define a_n to be $\text{ex}_\ell(n, \mathcal{F}) / \binom{n}{k-\ell}$. Given $\varepsilon > 0$, Corollary 2.4 implies that there exists an integer M such that $a_n - a_m < \varepsilon$ for all $n \geq m \geq M$. Thus the sequence of a_n converges. $\square$

From Corollary 2.4, we also deduce the following statement, which will be useful.

Corollary 2.5. *Let $k > \ell \geq 1$ be integers. Let $\mathcal{F}$ be a family of k -graphs. For every $\varepsilon > 0$, there exists a finite family $\mathcal{F}' \subseteq \mathcal{F}$ with $\pi_\ell(\mathcal{F}) \leq \pi_\ell(\mathcal{F}') \leq \pi_\ell(\mathcal{F}) + \varepsilon$.*

Proof. Clearly, $\pi_\ell(\mathcal{F}) \leq \pi_\ell(\mathcal{F}')$ for $\mathcal{F}' \subseteq \mathcal{F}$. Let $\pi = \pi_\ell(\mathcal{F})$ and choose $m = m(\varepsilon)$ such that

$$\frac{\text{ex}_\ell(m, \mathcal{F})}{\binom{m}{k-\ell}} < \pi + \frac{\varepsilon}{2} \text{ and } m \geq M(k, \ell, \varepsilon/2),$$

where $M(k, \ell, \varepsilon)$ is the integer as defined in Lemma 2.2. Define $\mathcal{F}'$ to be the family of k -graphs in $\mathcal{F}$ of order at most m . Then $\text{ex}_\ell(m, \mathcal{F}) = \text{ex}_\ell(m, \mathcal{F}')$. By Corollary 2.4,

$$\frac{\text{ex}_\ell(n, \mathcal{F}')}{\binom{n}{k-\ell}} < \frac{\text{ex}_\ell(m, \mathcal{F}')}{\binom{m}{k-\ell}} + \frac{\varepsilon}{2} = \frac{\text{ex}_\ell(m, \mathcal{F})}{\binom{m}{k-\ell}} + \frac{\varepsilon}{2} \leq \pi + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \pi + \varepsilon.$$

Therefore, $\pi_\ell(\mathcal{F}') = \lim_{n \rightarrow \infty} \frac{\text{ex}_\ell(n, \mathcal{F}')}{\binom{n}{k-\ell}} \leq \pi + \varepsilon$. $\square$

Proof of Proposition 1.6. Given $\varepsilon > 0$, M and ℓ are assumed to be sufficiently large integers throughout this proof. By Corollaries 2.4 and 2.5, there exists a finite family $\mathcal{F}' \subseteq \mathcal{F}$ such that

$$\text{ex}_\ell(M, \mathcal{F}') / \binom{M}{k-\ell} < \pi_\ell(\mathcal{F}') + \varepsilon/4 \leq \pi_\ell(\mathcal{F}) + \varepsilon/2.$$

Let H_n be a graph with $\delta_\ell(H_n) > (\pi_\ell(\mathcal{F}) + \varepsilon) \binom{n}{k-\ell}$ and $n \geq M$. By Lemma 2.2, the number of M -subsets S satisfying $\delta_\ell(H[S]) \geq (\pi_\ell(\mathcal{F}) + \varepsilon/2) \binom{M}{k-\ell} > \text{ex}_\ell(M, \mathcal{F}')$ is at least $\binom{n}{M}/2$. Thus, there exists a copy of a

member of $\mathcal{F}'$ in $H[S]$ and so for some $F \in \mathcal{F}'$ on f vertices, there are at least

$$\frac{\binom{n}{M}/2}{|\mathcal{F}'|\binom{n-f}{M-f}} = \frac{\binom{n}{f}}{2|\mathcal{F}'|\binom{M}{f}} \geq \delta \binom{n}{f}$$

copies of F , where we set $\delta^{-1} = 2|\mathcal{F}'| \max_{F' \in \mathcal{F}'} \left\{ \binom{M}{|V(F')|} \right\}$.

Define an f -graph H' on the same vertex set of H such that an edge of H' corresponds to a copy of F in H . Hence, H' contains at least $\delta \binom{n}{f}$ edges. By [4], we know that there exists a copy of ℓ -blow-up $K_f^f(L)$ of an edge K_f^f in H' . Each edge $e = (v_1, \dots, v_f)$ in $K_f^f(L)$ corresponds to an embedding of F in H . Also, each embedding of F induced an colouring of e in $K_f^f s$, namely a permutation of $V(F)$. A result in Ramsey theory says that if ℓ is large enough, there exists a monochromatic $K_f^f(s)$ in $K_f^f(L)$. Hence, $K_f^f(s)$ corresponds to a copy of $F(s)$ in H and so $\pi_\ell(\mathcal{F}) = \pi_\ell(\mathcal{F}(s))$. $\square$

3. A LOWER BOUND ON $\pi_\ell(\mathcal{F})$

Given k -graphs F and H , we say that $\phi : V(F) \rightarrow V(H)$ is *homomorphism* if it preserves edges, i.e. $\phi(e) \in E(H)$ for all $e \in E(F)$. Given a family $\mathcal{F}$ of k -graphs, we say that H is $\mathcal{F}$ -hom-free if there is no homomorphism from F to H for all $F \in \mathcal{F}$. Given $0 \leq \ell < k$, define $\text{ex}_{\text{hom},\ell}(n, \mathcal{F})$ to be the largest $\delta_\ell(H_n)$ in $\mathcal{F}$ -hom-free k -graphs H_n on n vertices. We further define $\pi_{\text{hom},\ell}(\mathcal{F}) = \limsup_{n \rightarrow \infty} \frac{\text{ex}_{\text{hom},\ell}(n, \mathcal{F})}{\binom{n}{k-\ell}}$. Note that H is $\mathcal{F}$ -hom-free if and only if $H(s)$ is $\mathcal{F}$ -free for all $s \in \mathbb{N}$, where we recall $H(s)$ is the s -blow-up of H . By ‘blowing-up’, $\pi_{\text{hom},\ell}(\mathcal{F}) = \pi_\ell(\mathcal{F})$.

Proposition 3.1. *Let $0 \leq \ell < k$. Given a finite family $\mathcal{F}$ of k -graphs, we have $\pi_{\text{hom},\ell}(\mathcal{F}) = \pi_\ell(\mathcal{F})$.*

Proof. Clearly, $\pi_{\text{hom},\ell}(\mathcal{F}) \leq \pi_\ell(\mathcal{F})$. Let $\mathcal{G}$ be the family of all k -graphs G such that there is a homomorphism from F to G for some $F \in \mathcal{F}$. So $\pi_{\text{hom},\ell}(\mathcal{F}) = \pi_\ell(\mathcal{G})$. Let $\varepsilon > 0$. By Corollary 2.5, there exists a finite family $\mathcal{G}' \subseteq \mathcal{G}$ with $\pi_\ell(\mathcal{G}') \leq \pi_\ell(\mathcal{G}) + \varepsilon/2$. By Proposition 1.6, $\pi_\ell(\mathcal{G}') = \pi_\ell(\mathcal{G}'(s))$ for all $s \in \mathbb{N}$. Let s and N be sufficiently large integers. For $n \geq N$, let H_n be a k -graph with $\delta_\ell(H_n) > (\pi_{\text{hom},\ell}(\mathcal{F}) + \varepsilon) \binom{n}{k-\ell} > \text{ex}_\ell(n, \mathcal{G}'(s))$. Hence H_n contains a copy of $G(s)$ for some $G \in \mathcal{G}'$, which then implies that H_n contains a member of $\mathcal{F}$. Therefore $\pi_{\text{hom},\ell}(\mathcal{F}) \geq \pi_\ell(\mathcal{F})$. $\square$

We are going to bound $\pi_\ell(\mathcal{F})$ from below by $\pi(\mathcal{L}_{\ell-1}(\mathcal{F}))$ proving Proposition 1.1.

Proof of Proposition 1.1. Let $2 \leq \ell < k$ be integers and let $\mathcal{F}$ be a family of k -graphs. Set $\mathcal{L} = \mathcal{L}_{\ell-1}(\mathcal{F})$. We may assume that $\pi = \pi(\mathcal{L}) > 0$, or else we are done. Pick a constant $0 < \varepsilon < \pi/3$ and a sufficiently large integer

$m \geq \varepsilon^{-1}$ such that there exists an $\mathcal{L}$ -hom-free $(k - \ell + 1)$ -graph G_m on $[m]$ with $e(G_m) \geq (\pi - \varepsilon) \binom{m}{k - \ell + 1}$ by Proposition 3.1.

Now we define a k -graph H on $[n]$ as follows. For every ℓ -subset S of $[n]$, let X_S be the independent uniform random variable on $[m]$. The k -set $\{i_1, \dots, i_k\}$ is an edge in H with $i_1 < \dots < i_k$ if and only if

$$\left\{ X_{\{i_1, \dots, i_{\ell-1}, i_{\ell-1+j}\}} : j \in [k - \ell + 1] \right\} \text{ is an edge in } G_m. \quad (\dagger)$$

Let S be an ℓ -subset of $[n]$. Suppose that R is a $(k - \ell)$ -subset of $[n] \setminus S$ and $R \cup S = \{i_1, \dots, i_k\}$ with $i_1 < \dots < i_k$. Recall that for an ℓ -set S' , $X_{S'}$ is the independent uniform random variable on $[m]$. By $(\dagger)$, the probability of $R \cup S$ is an edge in H is equal to the probability that a random $(k - \ell + 1)$ -subset S'' of $[m]$ forms an edge in G_m , where the elements of S'' are chosen independently uniformly at random with replacements. Therefore

$$\mathbb{P}(R \in N^H(S)) \geq \frac{e(G_m)}{\binom{m}{k - \ell + 1}} - O(1/m) \geq \pi - \varepsilon - O(1/m) \geq \pi - 3\varepsilon/2$$

as m is large. Thus, $\mathbb{E}(\deg^H(S)) \geq (\pi - 3\varepsilon/2) \binom{n - \ell}{k - \ell} \geq (\pi - 2\varepsilon) \binom{n}{k - \ell}$.

We now show that with high probability $\deg^H(S) \geq (\pi - 3\varepsilon) \binom{n}{k - \ell}$ by considering the following martingale. For each $1 \leq i \leq n$, define Z_i to be the collection of $X_{S'}$ such that S' is an ℓ -subset of $[n]$ and the largest element of S' is i . So $Z_1, \dots, Z_i$ can be viewed as exposing the first i vertices. Define $Y_i = \mathbb{E}(\deg^H(S) | Z_1, \dots, Z_i)$. Note that $\{Y_i : i = 0, 1, \dots, n\}$ is a martingale with

$$Y_0 = \mathbb{E}(\deg^H(S)) \geq (\pi - 2\varepsilon) \binom{n}{k - \ell}$$

and $Y_n = \deg^H(S)$. Moreover, $|Y_i - Y_{i-1}| \leq \binom{i-1}{k - \ell - 1}$. Thus, by Theorem 2.1 taking $\lambda = \varepsilon \binom{n}{k - \ell}$ and $c_i = \binom{i-1}{k - \ell - 1}$, we have

$$\begin{aligned} \mathbb{P}\left(\deg^H(S) \leq (\pi - 3\varepsilon) \binom{n}{k - \ell}\right) &\leq \mathbb{P}(Y_n \leq Y_0 - \lambda) \\ &\leq \exp\left(\frac{-\varepsilon^2 \binom{n}{k - \ell}^2}{2 \sum_{i=1}^n \left(\binom{i-1}{k - \ell - 1}\right)^2}\right) \leq \exp\left(\frac{-\varepsilon^2 \binom{n}{k - \ell}^2}{2n \binom{n-1}{k - \ell - 1}^2}\right) = \exp\left(-\frac{\varepsilon^2 n}{2(k - \ell)^2}\right). \end{aligned}$$

Therefore, $\delta_\ell(H) \geq (\pi - 3\varepsilon) \binom{n}{k - \ell}$ with high probability for large n .

Next, we show that H is $\mathcal{F}$ -free. Let F be a member of $\mathcal{F}$ say with $|V(F)| = f$. Let $T = \{i_1, \dots, i_f\}$ be an f -subset of $[n]$ with $i_1 < \dots < i_f$. By considering the map $\phi(i_j) = X_{\{i_1, \dots, i_{\ell-1}, i_{\ell+j}\}}$ for $j \in [k - \ell + 1]$, the linked graph $N^H(\{i_1, \dots, i_{\ell-1}\})$ is homomorphic to G_m . Since G_m is $\mathcal{L}$ -hom-free, $N^H(\{i_1, \dots, i_{\ell-1}\})$ is $\mathcal{L}_{\ell-1}(F)$ -free and so $H[T]$ is F -free. Therefore, H is $\mathcal{F}$ -free and so $\text{ex}_\ell(n, \mathcal{F}) \geq (\pi - 3\varepsilon) \binom{n}{k - \ell}$. Hence, $\pi_\ell(\mathcal{F}) \geq \pi$ as required. $\square$

4. JUMPS

From this section onward, unless stated otherwise we simply say a jump to mean a π_ℓ^k -jump, where k and ℓ are integers with $k > \ell > 1$. The following proposition shows the equivalent statement for α being a jump.

Proposition 4.1. *Let $k > \ell > 1$ be integers. Let $0 \leq \alpha < 1$ and $0 < \delta \leq 1 - \alpha$. The following statements are equivalent.*

- (S1) *Every family of k -graphs $\mathcal{F}$ satisfies $\pi_\ell(\mathcal{F}) \notin (\alpha, \alpha + \delta)$.*
- (S2) *Every finite family of k -graphs $\mathcal{F}$ satisfies $\pi_\ell(\mathcal{F}) \notin (\alpha, \alpha + \delta)$.*
- (S3) *For every $\varepsilon > 0$ and every $M \geq k - 1$, there exists an integer N such that, for every k -graph H_n with $n \geq N$ and $\delta_\ell(H_n) \geq (\alpha + \varepsilon) \binom{n}{k-\ell}$, we can find a subhypergraph $H'_m \subseteq H_n$ with $\delta_\ell(H'_m) \geq (\alpha + \delta - \varepsilon) \binom{m}{k-\ell}$ for some $m \geq M$.*

Proof. Trivially, (S1) $\Rightarrow$ (S2).

(S2) $\Rightarrow$ (S1). Assume that there exist $0 \leq \alpha < 1$ and $\delta > 0$ such that no finite family of k -graphs $\mathcal{F}$ satisfies $\pi_\ell(\mathcal{F}) \in (\alpha, \alpha + \delta)$. Suppose that (S1) does not hold, so there exists a family of k -graph $\mathcal{F}$ such that $\pi_\ell(\mathcal{F}) \in (\alpha, \alpha + \delta)$. Let $\varepsilon = (\alpha + \delta - \pi_\ell(\mathcal{F}))/2$. By Corollary 2.5, there exists a finite family $\mathcal{F}' \subseteq \mathcal{F}$ with $\alpha < \pi_\ell(\mathcal{F}) \leq \pi_\ell(\mathcal{F}') \leq \pi_\ell(\mathcal{F}) + \varepsilon < \alpha + \delta$, a contradiction.

(S3) $\Rightarrow$ (S1). Suppose that (S3) holds for some $0 \leq \alpha < 1$ and $\delta > 0$. Assume (S1) does not hold for a family $\mathcal{F}$ of k -graphs such that $\pi_\ell(\mathcal{F}) = \alpha + b$ for some $0 < b < \delta$. Set $\varepsilon_0 = \min\{b/2, (\delta - b)/2\}$. Then, there exists an integer $m = m(\varepsilon_0)$ such that

$$\text{ex}_\ell(m, \mathcal{F}) \leq (\pi_\ell(\mathcal{F}) + \varepsilon_0) \binom{m}{k-\ell}. \quad (4.1)$$

Also, there exists an integer $n > N(\varepsilon_0, m)$ such that there exists an $\mathcal{F}$ -free k -graph H_n with $\delta_\ell(H_n) \geq (\pi_\ell(\mathcal{F}) - \varepsilon_0) \binom{n}{k-\ell} \geq (\alpha + \varepsilon_0) \binom{n}{k-\ell}$ and $|V(H_n)| = n$. By (S3), there exists a subgraph $H'_m \subseteq H_n$ such that $\delta_\ell(H'_m) \geq (\alpha + \delta - \varepsilon_0) \binom{m}{k-\ell} > (\pi_\ell(\mathcal{F}) + \varepsilon_0) \binom{m}{k-\ell}$. However, this contradicts (4.1) as H'_m is $\mathcal{F}$ -free.

(S1) $\Rightarrow$ (S3). Suppose that (S1) holds but (S3) does not. There exist $0 < \varepsilon < \delta$, $M \geq r - 1$ and a sequence of k -graphs H_{n_i} such that

- (a') $|V(H_{n_i})| = n_i$ and $\delta_\ell(H_{n_i}) \geq (\alpha + \varepsilon) \binom{n_i}{k-\ell}$,
- (b') $\delta_\ell(H') < (\alpha + \delta - \varepsilon) \binom{|H'|}{k-\ell}$ for all $H' \subseteq H_{n_i}$ with $|V(H')| \geq M$.

Let $\mathcal{F} = \{G : |V(G)| \geq M \text{ and } G \not\subseteq H_{n_i} \text{ for any } i\}$. Note that $\mathcal{F}$ is not empty as $K_M \notin H_{n_i}$ for all i . Note that $\pi_\ell(\mathcal{F}) \geq \alpha + \varepsilon$ by (a') and so $\pi_\ell(\mathcal{F}) \geq \alpha + \delta$ by (S1). Therefore, for every $m \geq M$, there exists an $\mathcal{F}$ -free graph G_m with $\delta_\ell(G_m) \geq (\alpha + \delta - \varepsilon) \binom{m}{k-\ell}$. Since G_m is $\mathcal{F}$ -free, $G_m \subseteq H_{n_i}$ for some i otherwise $G_m \in \mathcal{F}$. This contradicts (b'). $\square$

5. THEOREM 1.7 FOR $\alpha = 0$

Here, we prove 0 is not a jump. Using (S3) in Proposition 4.1, the following statement implies that 0 is not a jump. For every $\delta > 0$, there exist $\varepsilon_0 > 0$ and $M_0 > 0$, such that for every $N \geq k$, there exist $n \geq N$ and a k -graph H_n satisfying

- (i) $\delta_\ell(H_n) \geq \varepsilon_0 \binom{n}{k-\ell}$,
- (ii) $\delta_\ell(H'_m) < (\delta - \varepsilon_0) \binom{m}{k-\ell}$ for every subgraph $H'_m \subseteq H_n$ with $m \geq M_0$.

Hence, in order to prove Theorem 1.7 for $\alpha = 0$, it is enough to show that there exist k -graphs satisfying (i) and (ii).

When $\ell = k - 1$, our construction is identical to that given by Mubayi and Zhao [15]. To illustrate the key idea behind the construction, we consider the following simple case when $k = 3$ and $\ell = 2$. Let H be a 3-graph on $n = pt$ vertices with vertex set $V = V_0 \cup \dots \cup V_{t-1}$ with $V_i \cap V_j = \emptyset$ for $i \neq j$ and $|V_i| = p$ for all i . A 3-set S is an edge in H if and only if either $|S \cap V_i| \leq 1$ for all i or $|S \cap V_i| = 2$ and $|S \cap V_{i+1}| = 1$ for some i . It is easy to check that $\delta_2(H) = n/t$. Let S be a subset of $V(H)$ with $|S| \geq t + 1$. If $|S \cap V_i| \geq 2$ for all i , then $\delta_2(H[S]) = \min\{|S \cap V_i| : i \in [n]\} \leq |S|/t$. Otherwise, $\delta_2(H[S]) \leq 1$. So (i) and (ii) hold by setting $\delta = 3/t$ and $\varepsilon_0 = 1/t$.

For general $k > \ell > 1$, we consider the k -graph $B(p, t, k, \ell)$ defined below.

Definition 5.1. For integers $p \geq 1$ and $t \geq k > \ell > 1$, define $B = B(p, t, k, \ell)$ to be the k -graph (V, E) with the following properties:

- (a) $V = V_0 \cup \dots \cup V_{t-1}$ with $V_i \cap V_j = \emptyset$ for $i \neq j$ and $|V_i| = p$ for all i .
- (b) $E = E_1 \cup E_2$.
- (c) $E_1 = \{S \in \binom{V}{k} : |S \cap V_i| < \ell \text{ for all } i\}$.
- (d) $E_2 = \{S \in \binom{V}{k} : \text{there exists an integer } i \text{ such that } |S \cap V_i| = \ell \text{ and } |S \cap V_{i+j}| = 1 \text{ for each } j \leq k - \ell\}$, (here $V_t = V_0$).

Clearly, $|B| = tp$. We now show that $\delta_\ell(B) = p^{k-\ell}$.

Fact 5.2. For integers $p \geq 1$ and $t \geq k > \ell > 1$, $\delta_\ell(B(p, t, k, \ell)) = p^{k-\ell}$.

Proof. Let T be an ℓ -subset of V . If $T \subseteq V_{i_0}$ for some i_0 , then $\deg(T) = p^{k-\ell}$ by (d). If $|T \cap V_i| < \ell$ for all i , then $\deg(T) \geq p^{k-\ell}$ by (c) and picking one element in each V_i with $V_i \cap T = \emptyset$ as $t \geq k$. $\square$

Proof of Theorem 1.7 for $\alpha = 0$. Let $k > \ell > 1$ and $0 < \delta < (4k^{-1}(k - \ell)^2)^{k-\ell}$. Define ε_0 , t and M_0 to be constants such that

$$\varepsilon_0^{k-\ell} = \frac{\delta}{1 + (4(k - \ell)^2)^{k-\ell}} \leq k^{-(k-\ell)} < 1/2, \quad (5.1)$$

$$t = \left\lfloor \frac{1}{\varepsilon_0} \right\rfloor \geq k, \quad (5.2)$$

$$M_0 = \ell t \geq \frac{\ell}{2(k - \ell)\varepsilon_0}. \quad (5.3)$$

For $p \geq k-1$, set $n = pt$ and let $H_n = B(p, t, k, \ell)$ be the k -graph as defined in Definition 5.1. By Fact 5.2, $\delta_\ell(H_n) = p^{k-\ell}$, so

$$\frac{\delta_\ell(H_n)}{\binom{n}{k-\ell}} = \frac{p^{k-\ell}}{\binom{pt}{k-\ell}} \geq t^{-(k-\ell)} \geq \varepsilon_0^{k-\ell}.$$

To complete the proof, our task is to show that (ii) holds, that is, for every m -subset S of V with $m > M_0$, $\delta_\ell(H_n[S]) \leq (\delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell}$.

Let S be an m -subset of V with $m > M_0$ and let $H' = H_n[S]$. Suppose that $\delta_\ell(H') > (\delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell}$. Set $S_i = S \cap V_i$ for all i . Since $m > M_0 = \ell t$, there exists an integer i_0 and an ℓ -subset R_0 of S_{i_0} . By (5.1), we have

$$\begin{aligned} \deg^{H'}(R) &\geq \delta_\ell(H') > (\delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell} \\ &= (4(k-\ell)^2 \varepsilon_0)^{k-\ell} \binom{m}{k-\ell} \geq (2(k-\ell) \varepsilon_0 m)^{k-\ell}, \end{aligned}$$

where we take $\binom{m}{k-\ell} \geq \left(\frac{m}{2(k-\ell)}\right)^{k-\ell}$. Recall Definition 5.1(d) that $N^{H_n}(R) \subseteq V_{i_0+1} \times \cdots \times V_{i_0+k-\ell}$. Hence, there exists an integer $j \leq k-\ell$ such that $|S_{i_0+j}| \geq 2(k-\ell) \varepsilon_0 m$. Since $2(k-\ell) \varepsilon_0 m \geq 2(k-\ell) \varepsilon_0 M_0 \geq \ell$ by (5.3), there is an ℓ -subset R_{i_0+j} of S_{i_0+j} . By repeating this argument to i_0+j , we may conclude that there are at least $t/(k-\ell)$ integers $0 \leq i \leq t-1$ such that $|S_i| \geq 2(k-\ell) \varepsilon_0 m$. Therefore, by (5.2)

$$m \geq \frac{t}{k-\ell} \times 2(k-\ell) \varepsilon_0 m \geq 2 \left(\frac{1}{\varepsilon_0} - 1 \right) \varepsilon_0 m > m$$

as $\varepsilon_0 < 1/2$ by (5.1). This is a contradiction, so $\delta_\ell(H_n[S]) < (\delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell}$ for all m -subset S of V and all $m > M_0$. Therefore, 0 is not a π_ℓ^k -jump for all $k > \ell > 1$. $\square$

6. PROOF OF THEOREM 1.7 FOR $0 < \alpha < 1$

As in Section 5, we are going to construct k -graphs H_n that contradict (S3) in Proposition 4.1. The structure of H_n is based on $B = B(p, t, k, \ell)$. (The formal definition of H_n will be given later.) So $\delta_\ell(H_n) = \deg^{H_n}(T)$ for some ℓ -subset T of V_0 . In order to ensure that $\deg^{H_n}(T) \geq \alpha \binom{n}{k-\ell}$, we modify Definition 5.1(d) so that $N^{H_n}(T)$ contains all $(k-\ell)$ -subsets S of $V_1 \cup \cdots \cup V_a$ with $|S \cap V_i| < \ell$ for all $i \in [a]$.

The following function will turn out to be very useful when calculating $\delta_\ell(H_n)$. For integers a and $k > \ell > 1$, let $\mathcal{A} = \mathcal{A}(a, k, \ell)$ be the set of all ordered a -tuples $(q_1, \dots, q_a)$ in $\{0, 1, \dots, \ell-1\}^a$ such that $\sum q_i = k-\ell$. Given integers $n_1, \dots, n_a$, define the function

$$f(n_1, \dots, n_a) = \sum_{(q_1, \dots, q_a) \in \mathcal{A}} \prod_{i=1}^a \binom{n_i}{q_i}.$$

Thus, $f(n_1, \dots, n_a)$ is the number of ways of choosing $k - \ell$ elements from vertex sets $U_1, \dots, U_a$ with $|U_i| = n_i$ such that at most $\ell - 1$ elements are chosen from each U_i . Write $f(n_0; a)$ for $f(n_1, \dots, n_a)$ if $n_i = n_0$ for all $i \in [a]$. Note that

$$f(n_1, \dots, n_a) \leq \binom{\sum n_i}{k - \ell}.$$

Moreover, from the definition of f , for integers $a, b \geq 1$ with $a/b < 1$ we also have

$$\begin{aligned} f(n_0/b; a) &\geq \binom{\frac{an_0}{b}}{k - \ell} - a \binom{\frac{n_0}{b}}{\ell} \binom{\frac{an_0}{b} - \ell}{k - 2\ell} \\ &\geq \binom{\frac{an_0}{b}}{k - \ell} - \frac{a^{k-2\ell+1}}{b^{k-\ell}} \binom{n_0}{\ell} \binom{n_0 - \ell}{k - 2\ell} \\ &= \binom{\frac{an_0}{b}}{k - \ell} - \left(\frac{a}{b}\right)^{k-\ell} a^{-(\ell-1)} \binom{k - \ell}{\ell} \binom{n_0}{k - \ell}. \end{aligned}$$

Thus, given integers $k > \ell > 1$, a rational $q \in (0, 1)$ and a constant $\varepsilon > 0$ there exist integers $0 < a < b < N_0$ such that $a/b = q$, $a + k < b$ and for all $n' \geq N_0$,

$$q^{k-\ell} - \varepsilon = \left(\frac{a}{b}\right)^{k-\ell} - \varepsilon \leq \frac{f(n'/b; a)}{\binom{n'}{k-\ell}} \leq \frac{\binom{an'/b}{k-\ell}}{\binom{n'}{k-\ell}} \leq \left(\frac{a}{b}\right)^{k-\ell} = q^{k-\ell}.$$

We now give the proof of Theorem 1.7 when $0 < \alpha < 1$, that is, no $\alpha \in (0, 1)$ is a jump.

Proof of Theorem 1.7 for $0 < \alpha < 1$. Note that the set of $q^{k-\ell}$ for rationals $q \in (0, 1)$ is dense in $(0, 1)$. Hence, it is enough to show that $q^{k-\ell}$ is not a jump for all rationals $q \in (0, 1)$. Given $\delta > 0$, let ε_0 , t and M_0 as in (5.1), (5.2) and (5.3) respectively. Pick $0 < a < b < N_0$ such that

$$a/b = q, \quad a + k < b, \quad q^{k-\ell} - \varepsilon_0^{k-\ell} \leq \frac{f(n'/b; a)}{\binom{n'}{k-\ell}} \leq q^{k-\ell} \quad (6.1)$$

for all $n' > N_0$. Set $\varepsilon = (\varepsilon_0/b)^{k-\ell}/4$. Define M to be an integer such that

$$M \geq \max \left\{ 2(k - \ell) \left(\frac{\binom{M_0}{k-\ell}}{\delta - \varepsilon_0^{k-\ell}} \right)^{\frac{1}{k-\ell}}, \frac{b\ell(k - \ell)}{q \ln \left(1 + \frac{1}{2} \left(\frac{\varepsilon_0}{q} \right)^{k-\ell} \right)}, N_0, \frac{b\ell}{1 - q} \right\}. \quad (6.2)$$

For integers $p > k$, set $n = btp$. Let $H = (V, E)$ be the k -graph with the following properties:

- (a') $V = V_0 \cup V_1 \cup \dots \cup V_{b-1}$ with $V_i \cap V_j = \emptyset$ for $i \neq j$ and $|V_i| = tp$ for all i .
- (b') $E = E_1 \cup E_2 \cup \bigcup_{j=0}^{b-1} E'_j$.
- (c') $E_1 = \{S \in \binom{V}{k} : |S \cap V_i| < \ell \text{ for all } i\}$.

(d') E_2 is the set of k -subsets S of V such that

$$|S \cap V_i| \begin{cases} = \ell & \text{if } i = i_0, \\ < \ell & \text{if } i_0 + 1 \leq i \leq i_0 + a, \\ = 0 & \text{otherwise,} \end{cases}$$

for some i_0 , where we take $V_{b+j} = V_j$.

(e') $E'_i = E(H[V_i])$, which is isomorphic to $B(p, t, k, \ell)$, for all i .

Note that the definition of the edge set E_2 is different to the one defined in $B(p, t, k, \ell)$. Pick p large such that $n \geq M$. First, we are going to show that

$$\delta_\ell(H) = f(n/b; a) + p^{k-\ell}.$$

Let T be an ℓ -subset of V . If $T \subseteq V_{i_0}$ for some i_0 , then Fact 5.2 implies that $\deg^{H[V_{i_0}]}(T) \geq p^{k-\ell}$ with equality holds for some ℓ -subset T of V_{i_0} . By (d'), the number of $(k-\ell)$ -subsets S of $N^H(T) \setminus \binom{V_{i_0}}{k-\ell}$ is $f(n/b; a)$. Thus, $\deg^H(T) \geq f(n/b; a) + p^{k-\ell}$ with equality holds for some ℓ -subset T of V_{i_0} . If $|T \cap V_i| < \ell$ for all i , then $\deg^H(T) \geq f(n/b; b-\ell) \geq f(n/b; a) + p^{k-\ell}$ by (c') and picking at most $\ell-1$ elements in each V_i with $V_i \cap T = \emptyset$. Hence, $\delta_\ell(H) = f(n/b; a) + p^{k-\ell}$ as claimed. Note by (5.2) and the definition of ε that

$$p^{k-\ell} = \left(\frac{n}{bt}\right)^{k-\ell} \geq \left(\frac{\varepsilon_0}{b}\right)^{k-\ell} \binom{n}{k-\ell} > 2\varepsilon \binom{n}{k-\ell}.$$

Recall that $n \geq M$, so we have by (6.1)

$$\begin{aligned} \delta_\ell(H) &= f(n/b; a) + p^{k-\ell} \\ &> (q^{k-\ell} - \varepsilon) \binom{n}{k-\ell} + 2\varepsilon \binom{n}{k-\ell} = (q^{k-\ell} + \varepsilon) \binom{n}{k-\ell}. \end{aligned}$$

By Proposition 4.1(S3), to complete the proof, our task is to show that for every m -subset S of V with $m > M$, $\delta_\ell(H[S]) \leq (q + \delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell}$. Let S be an m -subset of V with $m > M$ and let $H' = H[S]$. Suppose that $\delta_\ell(H') > (q + \delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell}$. Set $S_i = S \cap V_i$ and $m_i = |S_i|$ for all i .

Claim 6.1. *There exists i_0 such that $m_{i_0} \geq \ell$ and $\sum_{j=i_0+1}^{i_0+a} m_j < qm + b\ell$.*

Proof of claim. If $\sum_{j=i+1}^{i+a} m_j > qm$ for all i , then by averaging we get $|S| > qmb/a = m$, a contradiction. Hence, we may assume without loss of generality that $\sum_{j=1}^a m_j \leq qm$ and $m_0 < \ell$. Let $i_0 < b$ be the largest integer such that $m_{i_0} \geq \ell$. Note that such i_0 exists as $|S| = m \geq M > b\ell$. Then

$$\sum_{j=i_0+1}^{i_0+a} m_j \leq \sum_{j=i_0+1}^{b-1} m_j + |S_0| + \sum_{j=1}^a m_j \leq (b-1)(\ell-1) + qm.$$

Hence, the claim holds. $\square$

Without loss of generality, we may assume that Claim 6.1 holds with $i_0 = 0$. Let R be an ℓ -subset of S_0 . If $|S_0| \geq M_0$, then

$$\deg^{H[S_0]}(R) < (\delta - \varepsilon_0^{k-\ell}) \binom{|S_0|}{k-\ell} \leq (\delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell}$$

as $H[V_0]$ is isomorphic to $B(p, t, k, \ell)$ by (e'). Therefore

$$\deg^{H[S_0]}(R) \leq \max \left\{ (\delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell}, \binom{M_0}{k-\ell} \right\}.$$

By (6.2), $(\delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell} \geq (\delta - \varepsilon_0^{k-\ell}) \left(\frac{m}{2(k-\ell)} \right)^{k-\ell} \geq \binom{M_0}{k-\ell}$. In summary,

$$\deg^{H[S_0]}(R) < (\delta - \varepsilon_0^{k-\ell}) \binom{m}{k-\ell}. \quad (6.3)$$

Recall (d') that if $T \in N^H(R)$ with $T \not\subseteq S_0$, then $T \subseteq S_1 \cup \dots \cup S_a$ and $|T \cap S_i| < \ell$ for all $i \in [a]$. Since $|S_j| = m_j$ and $\sum_{j=1}^a m_j < \frac{am}{b} + b\ell$, the number of $S \in N^{H'}(R) \setminus \binom{V_0}{k-\ell}$ is equal to

$$f(m_1, \dots, m_a) \leq \binom{\sum_{j=1}^a m_j}{k-\ell} < \binom{am/b + b\ell}{k-\ell} \leq \left(q + \frac{b\ell}{m} \right)^{k-\ell} \binom{m}{k-\ell}$$

as $q + b\ell/m \leq q + b\ell/M \leq 1$. Therefore, together with (6.3) and (6.2),

$$\begin{aligned} \deg^{H'}(R) &\leq \deg^{H[S_0]}(R) + \left| N^{H'}(R) \setminus \binom{S_0}{k-\ell} \right| \\ &\leq \left(\delta - \varepsilon_0^{k-\ell} + \left(q + \frac{b\ell}{m} \right)^{k-\ell} \right) \binom{m}{k-\ell} \\ &\leq \left(\delta - \varepsilon_0^{k-\ell} + q^{k-\ell} \exp \left(\frac{b\ell(k-\ell)}{qM} \right) \right) \binom{m}{k-\ell} \\ &\leq \left(\delta - \varepsilon_0^{k-\ell} + q^{k-\ell} \left(1 + \frac{1}{2} \left(\frac{\varepsilon_0}{q} \right)^{k-\ell} \right) \right) \binom{m}{k-\ell} \\ &= \left(q^{k-\ell} + \delta - \frac{1}{2} \varepsilon_0^{k-\ell} \right) \binom{m}{k-\ell} \leq (q^{k-\ell} + \delta - \varepsilon) \binom{m}{k-\ell}. \end{aligned}$$

Therefore, $\delta_\ell(H[S]) < (q^{k-\ell} + \delta - \varepsilon) \binom{m}{k-\ell}$ for all m -subsets S of V and all $m > M$. Hence, $q^{k-\ell}$ is not a π_ℓ^k -jump. $\square$

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