An analysis of a shared mating in $V_2$

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Abstract. In this master thesis we investigate, from a topological point of view and without applying Thurston’s Theorem, why the mating of the so-called basilica polynomial \( f_{-1}(z) = z^2 - 1 \) and the dendrite \( f_i(z) = z^2 + i \) is shared with the mating of \( f_{-1} \) and the dendrite \( f_{-i}(z) = z^2 - i \). Both these matings equal the rational map \( R_3(z) = \frac{3}{z^3 + 2z} \).

Defined in the thesis are for both matings homeomorphic changes of coordinates \( \psi_{-1}^{\pm} \) from the set \( L = K_{f_{-1}} \cup \left( \bigcup_{n=0}^{\infty} f_{-1}^n(z_{\alpha}) \right) \) to the Fatou and Julia set of \( R_3 \). Here \( K_{f_{-1}} \) is the filled Julia set of \( f_{-1} \) and \( z_{\alpha} \) is the \( \alpha \)-fixed point of \( K_{f_{-1}} \).

Sammanfattning. I detta examensarbete undersöker vi, från en topologisk synvinkel och utan applicering av Thurstons teorem, varför matcningen av det så kallade basilikapolynomet \( f_{-1}(z) = z^2 - 1 \) och dendriten \( f_i(z) = z^2 + i \) är delad med matcningen av \( f_{-1} \) och dendriten \( f_{-i}(z) = z^2 - i \). Båda dessa matchningar är lika med den rationella avbildningen \( R_3(z) = \frac{3}{z^3 + 2z} \).

Definierat i examensarbetet är för båda matchningarna homöomorfa koordinatbyten \( \psi_{-1}^{\pm} \) från mängden \( L = K_{f_{-1}} \cup \left( \bigcup_{n=0}^{\infty} f_{-1}^n(z_{\alpha}) \right) \) till Fatou- och Julia mängden av \( R_3 \). Här är \( K_{f_{-1}} \) den ifyllda Julia mängden av avbildningen \( f_{-1} \) och \( z_{\alpha} \) den \( \alpha \)-fixerade punkten i \( K_{f_{-1}} \).
Preface

This master thesis has been conducted at the Centre for Mathematical Sciences, at Lund University, under the supervision of senior lecturer Magnus Aspenberg. Lisa Hed, senior lecturer at the Department of Mathematics and Mathematical Statistics of Umeå University, has audited and approved the thesis.

The findings were presented at Umeå University on the 14th of November 2014.

If printed, the pages 20, 36, 38, 42, 44, 49 and 51 should for best result be done so in color.

For a reader already familiar with basic complex dynamics theory chapter 2 may be ignored. For a reader already familiar with basic theory on mating complex maps chapter 4 may be ignored.

Special gratitude is given Magnus Aspenberg for extraordinary mathematical mentoring and my beloved Emelie for her unconditional support.
1 Introduction

Throughout the late 1910s Gaston Julia and Pierre Fatou initiated what today is known as complex dynamics. A subfield of complex dynamics is the theory of mating complex maps which dates back to the 1980s when A. Douady and J. Hubbard proposed a way of parameterizing the space of rational maps of degree 2 by pairs of quadratic polynomials.

The idea is to take two quadratic polynomials \( f_{c_1}(z) = z^2 + c_1 \) and \( f_{c_2}(z) = z^2 + c_2 \), where \( c_1 \) and \( c_2 \) do not belong to conjugate limits of the Mandelbrot set \( \mathcal{M} \). If the Julia sets \( J(f_{c_1}) \) and \( J(f_{c_2}) \) are locally connected, one can glue these sets along their boundaries in reverse order and thereby obtain a new Julia set for a rational map of degree 2. This rational map is then said to be the mating of \( f_{c_1} \) and \( f_{c_2} \). The mating conjecture states that this is possible whenever \( c_1 \) and \( c_2 \) do not belong to conjugate limits of \( \mathcal{M} \).

\( V_2 \) is the class of rational maps of degree 2 that have a super-attracting periodic cycle of period 2. This thesis deals only with post critically finite quadratic maps and the matings investigated belong to \( V_2 \). For this case Tan Lei, Rees and Shishikura has already concluded the basic theory. However, here we put a special case under the scope. This special case is also included in the work by V. Timorin ([Ti2]), where he proves existence of matings by using laminations (see also [Du]). Several results from the paper [AsYa] by Yampolsky and Aspenberg, where post critically infinite matings with the basilica and non-renormalizable quadratic polynomials are analyzed, will be considered.

Comparison of the mating between the basilica polynomial \( f_{-1}(z) = z^2 - 1 \) and \( f_i(z) = z^2 + i \), and the mating between \( f_{-1}(z) \) and \( f_{-i}(z) = z^2 - i \) is the origin of the problem specification. The mating of both these pairs of polynomials equals the quadratic rational map \( R_3(z) = \frac{3}{z^2 + 2} \). We refer to \( R_3(z) \) as a so called shared mating, i.e. a mating of two, at least partially separated, pairs of polynomials.

As mentioned above, the existence of these type of post critically finite shared matings is in the literature already and these results rest to a large extend on Thurston’s famous theorem for post-critically finite branched coverings of the Riemann sphere. The goal here is to, from a topologically point of view, derive an explicit explanation to why \( R_3(z) \) is shared. We will partially construct a continuous map \( \psi \) from the filled Julia set \( K_{f_{-1}} \) of the basilica into the Riemann sphere such that

\[ \psi \circ f_{-1}(z) = R_3 \circ \psi(z) \]

for all \( z \in K_{f_{-1}} \). It turns out that there are two canonical ways of doing this, which then suggests that indeed \( R_3(z) \) is a shared mating.
2 Introductory theory

In this chapter a walk-through of general complex dynamics theory underlying the thesis and the theory of mating is presented. For a reader already familiar with complex dynamics it is safe to here, without loss of understanding in upcoming chapters, skip ahead to chapter 3.

The mating of two maps can in a pictorial conception be thought of as the process of, on the Riemann sphere, gluing the exterior of the boundaries of two filled Julia sets to one another in reverse order. In understanding what this means some underlying theory of complex dynamics need be presented as it will in this chapter.

2.1 The Fatou and Julia sets

In section 2.1 all definitions, lemmas and theorems are as given by J. Milnor in [Mil].

Before defining the filled Julia set we need to elaborate and define the idea of normal families, Fatou sets and Julia sets.

The definition of Normal families is as follows.

**Definition 2.1.1 (Normal families)** A collection $\mathcal{F}$ of maps from a Riemann surface $S$ to a Riemann surface $T$ will be called normal if every infinite sequence of maps from $\mathcal{F}$ contains either a subsequence which converges locally uniformly or a subsequence which diverges locally uniformly from $T$. □

For determining in which domain a map belongs to a normal family the indispensable Montel’s Theorem states as follows.

**Theorem 2.1.1 (Montel’s Theorem)** Let $S$ be a Riemann surface and let $\mathcal{F}$ be a collection of holomorphic maps $f : S \to \hat{\mathbb{C}}$ which omit three different values. That is, assume that there are distinct points $a, b, c \in \hat{\mathbb{C}}$ so that $f(S) \subset \hat{\mathbb{C}} \setminus \{a, b, c\}$ for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family. □

In the cases of this thesis $S = \hat{\mathbb{C}}$. At this stage one might wonder what good this classification can do. In a moment however the benefits will emerge. But first a necessary notation. Throughout the thesis the cases in question will be concerning iteration steps of maps. A convenient notation when dealing with iterations of a map is $f^n$ which is defined as

$$f^n(z) \equiv \underbrace{f \circ f \circ \cdots \circ f}_n(z).$$

Known as the $n$-fold iterate or the $n$th image of $z$ for $f$ is $f^n(z)$. If

$$g(z) \equiv f^{(-1)}(z)$$

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2.1 The Fatou and Julia sets

The \textit{n}\textsuperscript{th} preimage (or just the preimage) of \(z\) for \(f\) is defined as \(g^{\circ n}(z)\) and can also be written as \(f^{\circ(-n)}(z)\). The \textit{immediate preimage} of \(z\) for \(f\) is given by \(f^{\circ(-1)}(z)\).

The point for defining two important sets is now reached.

\textbf{Definition 2.1.2 (Fatou and Julia set)} Let \(S\) be a compact Riemann surface, let \(f : S \to S\) be a non constant holomorphic mapping, and let \(f^{\circ n} : S \to S\) be its \(n\)-fold iterate. The domain of normality for the collection of iterates \(\{f^{\circ n}\}\) is called the \textit{Fatou} set for \(f\), and its complement is called the \textit{Julia} set. Given a mapping \(f\) its Julia set is customary referred to as \(J(f)\) and the Fatou set \(S \setminus J(f)\).

\(\Box\)

Remark: The Julia set \(J(f)\) can also be written as \(J_f\).

At first glance the definition of the Julia and Fatou sets can seem almost out the blue and quite insipid. However it is not the definition that is the focal point. The benefits are in the properties of the Julia and Fatou sets which one acquires when defining them as in definition 2.1.2. Some especially interesting properties of the Julia and Fatou sets are

\[z \in J(f) \Rightarrow f^{\circ n}(z) \in J(f)\]

and

\[w \in \mathbb{C} \setminus J(f) \Rightarrow f^{\circ n}(w) \in \mathbb{C} \setminus J(f)\]

for all \(n\)-fold iterates.

A recurring expression is the \textit{Fatou component}, which for nonlinear rational maps simply corresponds to a connected component of the Fatou set.

For periodic orbits J. Milnor has made the following definition.

\textbf{Definition 2.1.3 (Multiplier and period for a periodic orbit)} Consider a periodic orbit (or cycle)

\[f : z_0 \to z_1 \to \cdots \to z_{n-1} \to z_n = z_0\]

for a holomorphic map \(f : S \to S\). If the points \(z_1, \ldots, z_n\) are all distinct, then the integer \(n \geq 1\) is called the \textit{period}. The first derivative of the \(n\)-fold iterate \(f^{\circ n}\) at a point of the orbit is called the \textit{multiplier} of the orbit. It is often denoted with \(\lambda\).

\(\Box\)

As in definition 2.1.3, if the Riemann surface \(S\) is an open subset of \(\mathbb{C}\) we have for the multiplier the product formula

\[\lambda = (f^{\circ n})'(z_i) = f'(z_1) \cdot f'(z_2) \cdots f'(z_n)\].

This can be found proven in [Mi1] on page 44.
2.1 The Fatou and Julia sets

Now by definition, for every periodic orbit three possibilities exists. Either it is **attracting**, **repelling** or **indifferent**. These properties depend on the multiplier of the orbit.

\[
\begin{align*}
|\lambda| < 1 & \Rightarrow \text{attracting} \\
|\lambda| = 1 & \Rightarrow \text{indifferent} \\
|\lambda| > 1 & \Rightarrow \text{repelling}
\end{align*}
\]

If \( |\lambda| = 0 \) the orbit will be known as **superattracting**.

Suppose a point \( z \) is iterated \( n \) times under \( f \). We then get the point \( w(n) = f^{\circ n}(z) \). If \( z \) is in a neighbourhood of an attracting periodic orbit, \( w(n) \) will be closer and closer to this orbit as \( n \) increases. For some \( n \), \( w(n) \) will be included in the attracting periodic orbit. A superattracting periodic orbit attracts \( w(n) \) in even less iterations. If \( z \) is in a neighbourhood of a repelling periodic orbit, \( w(n) \) will be pushed away from this orbit as \( n \) increases. For the case of an indifferent periodic orbit further analysis is needed to reveal its behavior.

As previously purposed the meaningfulness of a normal family classification will now through the next lemma unfold.

**Lemma 2.1.1** Every attracting periodic orbit is contained in the Fatou set of \( f \). However, every repelling orbit is contained in the Julia set.

**Proof:** See Lemma 4.6 of [Mi1] on page 45. \( \square \)

It can be shown that all polynomials \( P_m \) have an attracting point at infinity (for proof see appendix). That is, there exists an open connected set (basin of attraction) \( T(P_m) \) for which \( |P_m^{\circ n}(z)| \) tends to infinity for all \( z \in T \) as \( n \to \infty \). Since \( T \) contains an attracting point it must belong to the Fatou set of \( P_m \). The set \( K(P_m) = \mathbb{C} \setminus T \) is known as the **filled Julia set** of \( P_m \) and it is defined as follows.

**Definition 2.1.4 (The Filled Julia set)** For a holomorphic map \( f \) the **filled Julia set** \( K(f) \) is defined as the set of all points \( z \) such that

\[
\lim_{n \to \infty} |f^{\circ n}(z)| < \infty.
\]

\( \square \)

**Remark:** The filled Julia set \( K(f) \) can also be written as \( K_f \).

Next is an important and very useful relation as stated.
Lemma 2.1.2  For any polynomial $f$ of degree at least 2, the filled Julia set $K = K(f) \subset \mathbb{C}$ is compact, with connected complement, with topological boundary $\partial K$ equal to the Julia set $J = J(f)$ and with interior equal to the union of all bounded components $U$ of the Fatou set $\mathbb{C} \setminus J$. Thus $K$ is equal to the union of all such $U$, together with $J$ itself. Any such bounded component $U$ is necessarily simply connected.

Proof: See Lemma 9.4 of [Mi1] on page 95.

Another notation to incorporate is $\overset{o}{K}(f) = K(f) \setminus \partial K(f) = K(f) \setminus J(f)$ which corresponds to the Fatou components of the filled Julia set for the map $f$. The usefulness of Lemma 2.1.2 is best illustrated through practical examples who will present themselves in chapter 5.

The so called “Riemann-Hurwitz Formula” is as follows.

Lemma 2.1.3  Let $T \to S$ be a branched covering map from one compact Riemann surface onto another. Then the number of branch points, counted with multiplicity, is equal to $\chi(S)d - \chi(T)$, where $\chi$ is the Euler characteristic and $d$ is the degree.

Proof: See Lemma 7.2 of [Mi1] on page 70.

2.2 Fixed point, critical point and critical value

If a periodic orbit has period $n = 1$ it will only include one single point

$$z_0 = f(z_0)$$

and be called a fixed point and we write $z_0 = z_{fix}(f)$ or just $z_{fix}$ when no doubt as to which map it belongs is present. Again this point just like orbits composed by several points, is by definition either attracting, repelling or indifferent. As written in [Mi1] but slightly modified a definition for the set of points being attracted by a fixed point can be given by,

Definition 2.2.1 (Basin and immediate basin of attraction)  Suppose that $f : S \to S$ is a holomorphic map from a Riemann surface into itself with an attracting fixed point $z_{fix}$ of multiplier $\lambda \neq 0$. The basin of attraction $A = A(z_{fix}) \subset S$ consists of all $z \in S$ for which $\lim_{n \to \infty} f^o_n(z)$ exists and is equal to $z_{fix}$. The immediate basin of attraction is defined to be the connected component of $A$ which contains $z_{fix}$. [Mi1]

The definitions of a critical point and a critical value looks as follows.
2.2.2 (Critical point) Suppose that $f : S \to S$ is a holomorphic map from a Riemann surface $S$ into itself. The set of critical points of $f$ is defined as

$$\text{Crit}(f) = \left\{ z \in S : \frac{d(f(z))}{dz} = 0 \right\}.$$  

The image of a critical point $z_{cp}(f)$ is called a critical value of $f$ and it is in this thesis written $z_{cv}(f)$.

Again the notations $z_{cp}$ and $z_{cv}$ are used when there is no doubt of which map they belong to.

Remark: Proof of how fixed points, critical points and critical values of polynomials of the form $f_c = z^2 + c$ can be obtained is given in the appendix.

2.3 External and internal rays

Another important conjunction which is as a backbone in the theory of mating Julia sets is the one discovered by Lucjan Böttcher in 1904.

Theorem 2.3.1 (Böttcher’s Theorem) Suppose $f$ is a holomorphic polynomial map of degree $n \geq 2$ with a superattracting fixed point $z_{\text{fix}}$. Then there exists a holomorphic map $w = \phi(z)$ which in a neighborhood of $z_{\text{fix}}$ conjugates $f$ to the $n$th power map $w \mapsto w^n$. Near $z_{\text{fix}}$, $f$ is conjugate to the map of the form

$$\phi \circ f \circ \phi^{-1} : w \mapsto w^n.$$

Proof: See [Mi1] page 90.

The map $\phi(z)$ in Theorem 2.3.1 is known as the Böttcher isomorphism or simply the Böttcher map.

When referring to the concept of mating it is meant that the exterior of the boundaries $\partial K_1$ and $\partial K_2$ of two filled Julia sets are attached to one another in reverse order. To understand how this is done we first need to elaborate on something called external rays. Let $f$ be a polynomial map of degree $n \geq 2$. Quoting [Mi1].

... the complement $\mathbb{C} \setminus K$ to any connected filled Julia set $K$ is conformally isomorphic to $\mathbb{C} \setminus \overline{D}$ under the Böttcher isomorphism

$$\phi : \mathbb{C} \setminus K \xrightarrow{\phi} \mathbb{C} \setminus \overline{D}$$

which conjugates the map $f$ outside $K$ to the $n$th power map $w \mapsto w^n$ outside the closed unit disk, with $\phi(z)$ asymptotic to the identity map at infinity.
2.3 External and internal rays

A continuous function $G : \mathbb{C} \to \mathbb{R}$ known as Green's function for $K$ is then defined by

$$G(z) = \begin{cases} \log |\phi(z)| > 0 & z \in \mathbb{C} \setminus K \\ 0 & z \in K \end{cases}$$

and each curve

$$G^{-1}(c) = \{ z ; G(z) = c \} \quad c > 0$$

is called an equipotential curve around $K$. The trajectories

$$\{ z : \arg(\phi(z)) = \text{constant} \}$$

orthogonal to the equipotential curves are the definition of the external rays for $K$. An external ray with angle $\theta$ will be denoted $ER(\theta)$ where $0 \leq \theta < 1$. We say that the angle $\theta$ is a landing angle for $z$ if the external ray $ER(\theta)$ lands at $z$. With $\phi^{-1}$ being the inverse Böttcher map, $ER(\theta)$ is said to land at $\gamma(\theta)$ if the limit

$$\lim_{r \to 1^+} \phi^{-1}(r \cdot e^{2\pi i \theta})$$

exists. From this we are ready to state a theorem which is highly important for the thesis. It will be given as written by J. Milnor.

**Theorem 2.3.2 (The Landing criterion Theorem)**  For any given map $f$ with connected Julia set, the following four conditions are equivalent.

- Every external ray $ER(\theta)$ lands at a point $\gamma(\theta)$ which depends continuously on the angle $\theta$.
- The Julia set $J$ is locally connected.
- The filled Julia set $K$ is locally connected.
- The inverse Böttcher map $\phi^{-1} : \mathbb{C} \setminus \overline{D} \to \mathbb{C} \setminus K$ extends continuously over the boundary $\partial \overline{D}$, and this extended map carries each $e^{2\pi i \theta} \in \partial \overline{D}$ to $\gamma(\theta) \in J(f)$.

Furthermore, whenever these conditions are satisfied, the resulting map $\gamma : \mathbb{R} \setminus \mathbb{Z} \to J(f)$ satisfies the semiconjugacy identity

$$\gamma(n\theta) = f(\gamma(\theta))$$

and maps the circle $\mathbb{R} \setminus \mathbb{Z}$ onto the Julia set $J(f)$.

**Proof:** See [Mi1] page 191. □

What this theorem reveals is that as long as one is dealing with a locally connected filled Julia set $K$, there exists an inverse Böttcher map $\phi^{-1}(\theta)$ such that one point and one point only of $\partial K$ corresponds to each $0 \leq \theta < 1$. Also the final addendum of the landing criterion can be simplified to the needs in question for this thesis and be restated as The double angle Theorem.
Theorem 2.3.3 (The double angle Theorem) Suppose \( f \) is a polynomial map of degree \( n = 2 \) with locally connected filled Julia set \( K \). Further suppose that the points \( z, w \in \partial K \) relate according to \( w = f(z) \) and that the external ray \( ER(\theta) \) land at \( z \). Then \( ER(2\theta \mod Z) \) land at \( w \).

**Proof:** See Theorem 2.3.2. \( \Box \)

The points on the boundary of \( K \) can be categorized as being **uni-, double- or multi-accessible**. The definitions are as follows.

**Definition 2.3.1 (Uni-, double- and multi-accessible)** Suppose one and only one external ray land at a point \( z \in \partial K \) of some filled Julia set \( K \). Then \( z \) is said to be **uni-accessible** on \( \partial K \). In similar manner \( z \) is **double-accessible** on \( \partial K \) if exactly two external rays land at it and so on. \( \Box \)

Another ray similar to the external one is the **internal ray**. Again let \( f \) be a polynomial map of degree \( n \geq 2 \). A simple connected Fatou component \( F \) of \( f \) is conformally isomorphic to the unit disk \( \mathbb{D} \) under the Böttcher map

\[
\phi : F \cong \mathbb{D}
\]

and it conjugates \( f \) inside \( F \) to the \( n \)th power map \( w \mapsto w^n \) inside the unit disk.

An internal ray with angle \( \theta \) will be denoted \( IR(\theta) \) and for the inverse Böttcher map \( \phi^{-1} \), \( IR(\theta) \) is said to land at \( \gamma(\theta) \) if the limit

\[
\lim_{r \to 1^-} \phi^{-1}(r \cdot e^{2\pi i \theta})
\]

exists.

Notable is that The double angle Theorem (Theorem 2.3.3) also applies for internal rays.

### 2.4 Properties of the quadratic map

The maps being mated in this thesis are all of the form \( f_c = z^2 + c \).

For \( f_c \) we write the fixed points, critical points and critical values as \( z_{fix}(f_c) \), \( z_{cp}(f_c) \) and \( z_{cv}(f_c) \) respectively. They are as follows,

\[
\begin{align*}
z_{fix}(f_c) & \in \left\{ \frac{1 \pm \sqrt{1 - 4c}}{2}, \infty \right\} \\
z_{cp}(f_c) & \in \text{Crit}(f_c) = \{0, \infty\} \\
z_{cv}(f_c) & \in \{c, \infty\}.
\end{align*}
\]
2.4 Properties of the quadratic map

The fixed point $z_{\text{fix}}(f_c) = \infty$ is superattracting (see appendix for proof). Some other properties of $f_c$ are not surprisingly related to the Mandelbrot set $M$. The Mandelbrot set, dened as

$$M = \{ c : |f_c^n(0)| < \infty \forall n \in \mathbb{Z} \},$$

and is seen illustrated in figure 2.4.1. As will be given shortly in Lemma 2.4.1, one can in J. Milnor’s paper “Pasting Together Julia Sets: A Worked Out Example of Mating (2003)” ([Mi2]) find a relation between the $\alpha$-fixed point (defined in Lemma 2.4.1) of a Julia set $J(f_c)$ and the limbs of $M$ containing $c$. A limb of $M$ will here be defined as it is in “Periodic orbits, external rays and the Mandelbrot set: An expository account (1999)” by J. Milnor ([Mi3]), which in short terms can be explained as a closed subset of $M$ only connected to the main cardioid at a single point. We define the main cardioid as the set of complex numbers $c$ such that the iteration of $f_c(0)$ converges to a bounded fixed point. The main cardioid of $M$ is the largest cardioid visible in figure 2.4.1. The approach next is to exploit the fact that for each complex number $\lambda$ there is only one $f_c$ having a fixed point $f_c(z) = z$ with multiplier $f'_c(z) = \lambda$. The boundary of the main cardioid can be parameterized to the unit circle such that when $c$ varies over the boundary of the main cardioid the set of solutions
for $\lambda(c)$ will correspond to the unit circle, i.e. $\lambda = e^{2\pi i \theta}$ for some $0 \leq \theta < 1$. A connected subset of $\mathcal{M}$, which is connected to the main cardioid only at $c(\lambda)$, is then referred to as the $(p/q)$-limb, where $p/q \in \mathbb{Q}$ and $\lambda = e^{2\pi i p/q}$. Stated in [Mi2] is the following.

**Lemma 2.4.1** Suppose $c \in (p/q)$-limb of $\mathcal{M}$. Then there exists a fixed point $z_\alpha \in K(f_c)$ such that there are exactly $q$ external rays

$$\{ER(\theta_1), ER(\theta_2), \ldots, ER(\theta_q)\},$$

where $0 < \theta_1 < \cdots < 2 \cdot \theta_m = \theta_{m+1} < \cdots < \theta_q < 1$, landing at $z_\alpha$. The point $z_\alpha$ is also known as the $\alpha$-fixed point of $f_c$. □

The $\beta$-fixed point of $f_c$ is defined as the point at which the external $ER(0)$ and only $ER(0)$ lands.

In the next chapter we are going to take a closer look on some specific $f_c$ which for the goal in question is of great interest.
3 The Basilica, the dendrites and the $R_a$ family

Chapter 3 discusses properties and relations about the different maps mated in examples presented in latter chapters. Also necessary definitions and lemmas are introduced.

Out of the family of maps given by the varieties of $f_c = z^2 + c$, only $c \in \{-1, i, -i\}$ are utilized in matings in this thesis. In this chapter each of these three maps and their Julia sets will thoroughly be analyzed. Also in order to do this in a pedagogic manner, some definitions and lemmas need to be introduced.

3.1 The Basilica

The filled Julia set $K_{f_{-1}}$ is commonly referred to as the basilica and defined as

$$K_{f_{-1}} = \left\{ z : \left| f_{-1}^n(z) \right| < \infty \forall n \in \mathbb{Z} \right\}.$$

It is found illustrated in figure 3.1.1, which is a modification of an illustration presented in [AsYa].

![Figure 3.1.1](image)

Figure 3.1.1. The basilica is in the illustration represented by the grey area. It coincides with the filled Julia set of the map $f_{-1}(z) = z^2 - 1$. Word of warning, the external rays illustrated in this figure are merely of topologically value and do not occupy the exact same set of points as the true external rays of the basilica, except for the landing points. Original illustration is from [AsYa].

The basilica possesses a super attracting periodic orbit of period $p = 2$ between the points

$$-1 \mapsto 0$$
The basilica, the dendrites and the \( R_a \) family

which both are in \( K_{f^{-1}} \cap (\mathbb{C} \setminus J_{f^{-1}}) \).

The map \( f^{-1}(z) \) however also has

\[
z_{\text{fix}}(f^{-1}) = \infty
\]
as a superattracting fixed point (or orbit of period \( p = 1 \)).

Also \( f^{-1}(z) \) has critical points

\[
z_{\text{cp}}(f^{-1}) \in \{0, \infty\}
\]
and critical values

\[
z_{\text{cv}}(f^{-1}) \in \{-1, \infty\}.
\]

Only the bounded points are included in \( K_{f^{-1}} \).

Proposed by Aspengren and Yampolsky in [AsYa], is to let \( B_0 \) and \( B_1 \) be the immediate basins of attraction of 0 and \(-1\) respectively for \( f_{-1} \) and let \( B_\infty \) be the basin of attraction at infinity. They then write the following three lemmas, which all also are proven in [AsYa].

**Lemma 3.1.1** For any two Fatou components \( A \) and \( B \) of \( f_{-1} \), neither of which is the attracting basin of infinity, exactly one of the following holds:

1. \( \overline{A} \cap \overline{B} = \emptyset \).
2. \( \overline{A} \cap \overline{B} \) is only one point, which is a pre-fixed point of \( f_{-1} \).
3. \( A = B \). \( \square \)

Since \( f_{-1} \) is hyperbolic we have that \( J_{f_{-1}} \) is locally connected ([CaGa]), thus according to the landing criterion every external ray \( ER(\theta) \) lands at a point \( \gamma(\theta) \in J_{f_{-1}} \), which depends continuously on the angle \( \theta \).

**Lemma 3.1.2** Let \( F_i \) be an arbitrary infinite sequence of distinct Fatou components of \( f_{-1} \). Then \( \text{diam } F_n \to 0. \) \( \square \)

**Lemma 3.1.3** A point \( z \in J_{f_{-1}} \) is a landing point of precisely two external rays if and only if \( z \) is a preimage of the \( \alpha \)-fixed point. No other points \( z \in J_{f_{-1}} \) are biaccessible. \( \square \)

These lemmas help revealing the behavior of the basilica. For further analysis of it some new conceptions are required. Next is therefore the definition, as it is given in [AsYa], of a so called bubble.
3.1 The Basilica

Definition 3.1.1 (Bubble of the basilica) A bubble of $K_{f_{-1}}$ is a Fatou component $F \subset K_{f_{-1}}$. The generation of a bubble $F$ is the smallest non-negative $n = \text{Gen}(F)$ for which $f_{-1}^n(F) = B_0$. The center of a bubble $F$ is the preimage $f_{-1}^{-\text{Gen}(F)}(0) \cap F$ and is denoted by $\text{Cent}(F)$. [AsYa] □

Recognizing the bubbles of the basilica and iterating their points under $f_{-1}$, one will find that all bubbles are preimages of $B_0$ and $B_1$. In figure 3.1.1 some bubbles have been tagged. The relations between these six bubbles are as follows,

\[
\begin{align*}
 f_{-1}(B_0) &= B_1 \\
 f_{-1}(B_1) &= f_{-1}(B_2) = B_0 \\
 f_{-1}(B_2) &= f_{-1}(B_6) = B_2 \\
 f_{-1}(B_3) &= f_{-1}(B_4) = B_5.
\end{align*}
\]

As supposedly already understood from its definition but underlined for clarification, in fact any bubble of the basilica will if iterated under $f_{-1}$ a sufficient number of times equal $B_0$.

Next is the fixed points of $K_{f_{-1}}$. It has two and they are,

\[
z_{\pm_{\text{fix}}} = \frac{1 \pm \sqrt{5}}{2}.
\]

From Lemma 3.1.1, Lemma 3.1.3 and the fact that $J_{f_{-1}}$ is connected it can, since $z_{\pm_{\text{fix}}} = (1-\sqrt{5})/2 \in \mathbb{R}$ lies between two Fatou components, be concluded that it has two external rays landing at it. Thus $z_{\pm_{\text{fix}}}$ must be the $\alpha$-fixed point $(z_{\alpha})$ of $f_{-1}$. This also concurs with the prediction made by Lemma 2.4.1 and the fact that $c = -1 \in 1/2$-limb of $M$. These two external rays are $\{ER(1/3), ER(2/3)\}$. The role of immediate preimage of $z_{\alpha}$ is apart from $z_{\alpha}$ itself also held by the point in figure 3.1.1 denoted $z_0$. Lemma 3.1.3 says that exactly two external rays land at $z_0$. Due to The double angle Theorem (Theorem 2.3.3) they are determined to be $\{ER(1/6), ER(5/6)\}$. In the same manner the immediate preimages of $z_0$, denoted $z_1$ and $z_2$, has external rays $\{ER(5/12), ER(7/12)\}$ and $\{ER(1/12), ER(1/12)\}$ respectively landing at them. Evidently for these four points, is that preimages of intersection points between bubbles are themselves intersection points between bubbles. This can be proven to be valid throughout the whole of the basilica.

Lemma 3.1.4 All preimages of an intersection point between two bubbles of the basilica are also intersection points between two bubbles of the basilica.

Proof: Note that $f_{-1}(z) = z^2 - 1$ is a holomorphic map with $f_{-1}'(z) \neq 0 \forall z \in \mathbb{C} \setminus \{0\}$, thus it is conformal on $\mathbb{C} \setminus \{0\}$. □
Lemma 3.1.4 will later prove itself to be of great value analyzing matings partly originating from the basilica.

Now an interesting thing about the basilica worth noting before leaving its external ray theory for a moment is that it is symmetric over both the $x$- and $y$-axis. This insight will later in the thesis prove valuable.

### 3.1.1 Bubble rays, axes and crossing curves

In order to describe actions undertaken by the basilica we need to distinguish between the conception of an intersection and a crossing.

**Definition 3.1.1.1 (Crossing curves)** Suppose $\Gamma_A(t)$ and $\Gamma_B(t)$ are continuous parameterized curves which for $0 \leq t \leq 1$ propagates from $\Gamma_A(0)$ to $\Gamma_A(1)$ and from $\Gamma_B(0)$ to $\Gamma_B(1)$ respectively.

Further suppose that $\ell_A$ and $\ell_B$ are continuous curves that propagates from $z_{A,0} = \Gamma_A(0)$ to $z_{A,1} = \Gamma_A(1)$ and $z_{B,0} = \Gamma_B(0)$ to $z_{B,1} = \Gamma_B(1)$ respectively.

Suppose also that $\Gamma_A$ and $\Gamma_B$ do not intersect one another at any of their endpoints and that:

\[
\Gamma_A \cap \ell_A = \{z_{A,0}, z_{A,1}\} \\
\Gamma_B \cap \ell_B = \{z_{B,0}, z_{B,1}\} \\
\Gamma_A \cap \ell_B = \Gamma_B \cap \ell_A = \emptyset.
\]

If there exists $\ell_A$ and $\ell_B$ satisfying the above conditions and $\ell_A \cap \ell_B = \emptyset$, then $\Gamma_A$ and $\Gamma_B$ are defined as **non-crossing curves**. Otherwise $\Gamma_A$ and $\Gamma_B$ are said to be **crossing curves** or that $\Gamma_A$ cross $\Gamma_B$ and vice versa. □

Consider a bubble $F \neq B_0$ as given in the definition above. In the set of bubbles whose closures intersect with $\overline{F}$ there exists one that is of lowest generation found among the bubbles in the set. Denote it $G$ and refer to it as the **predecessor** of $F$. The point $x \equiv \overline{F} \cap \overline{G}$ is then known as the **root** of $F$. The next step is to put bubbles together and form a chain. This chain will be known as a **bubble ray**. The definition is from [AsYa].

**Definition 3.1.1.2 (Bubble ray)** A **bubble ray** $B$ is a collection of bubbles $\cup_{0}^{m \leq \infty} F_k$ such that for each $k$ the intersection $\overline{F_k} \cap \overline{F_{k+1}} = \{x_k\}$ is a single point, and $\text{Gen}(F_k) < \text{Gen}(F_{k+1})$. □

Remembering Lemma 3.1.1 and Lemma 3.1.4 it is safe to claim that each of the points $x_k$ in the Bubble ray definition are preimages of the $\alpha$-fixed point of $f_{-1}$.

Proposed in [AsYa] is also that for an infinite bubble ray $B = \cup_{0}^{\infty} F_k$ there exists a unique point $x \in J_{f_{-1}}$ called the **landing point** of $B$, such that $F_k \to x$ in Hausdorff sense. The landing points of the infinite bubble rays in the basilica
may also be referred to as the horns of the basilica. The next definition is as given in [AsYa].

**Definition 3.1.1.3 (Axis of a Bubble ray)**  The axis of a bubble ray $B = \{F_k\}_{m \leq \infty}$ is the closed union

$$\gamma(B) \equiv \bigcup_{0}^{\infty} \gamma_k,$$

where $\gamma_k$ for $k \geq 1$ is the union of two internal rays of $F_k$ connecting its center to the points $x_{k-1}$ and $x_k$, and $\gamma_0$ is the internal ray of $F_0$ terminating at $x_0$. □

The spine of the basilica is defined as the axis of the the bubble ray initiated at the horn where $ER(1/2)$ lands and terminating at the horn where $ER(0)$ lands. Note that the spine is essentially made up by $K_{f-1} \cap \mathbb{R}$. We are now ready to define the meaning of saying that two bubble rays cross each other.

**Definition 3.1.1.4 (Crossing bubble rays)**  Given two bubble rays $B_A$ and $B_B$, suppose their respective axes $\Gamma_A$ and $\Gamma_B$ are non-crossing curves. Then $B_A$ and $B_B$ are defined as non-crossing bubble rays, otherwise they are said to be crossing bubble rays. □

In figure 3.1.2 an example of two intersecting non-crossing bubble rays is given.

As one studies a bubble ray carefully and perhaps observes that its axis does not coincide with a straight line, it is apparent that further tools for describing its properties are needed.

**Definition 3.1.1.5 (Bubble ray and axis orientation)**  Suppose $\Gamma(t)$ for $0 \leq t \leq 1$ is a parameterized curve coinciding with the axis of a bubble ray $B = \{F_k\}_{m \leq \infty}$, where $\Gamma(0)$ is placed at the center of $F_0$ and $\Gamma(1)$ at the center of $F_m$. Propagating along $\Gamma$ from $t = 0$ to $t = 1$, a right and left side of $\Gamma$ in $B$ under this direction can be identified. This orientation is also defined to be the orientation of the axis of $B$ and the orientation of $B$ is defined to be that of its axis. □
Figure 3.1.2. An example of two bubble rays that intersect but are non-crossing. Seen is that the axes $\Gamma_A$ and $\Gamma_B$ intersect but the curves $\ell_A$ and $\ell_B$ do not.

**Definition 3.1.1.6 (Bubble ray and axis diversion)** Suppose

$\left( \bigcup_{k=1}^{m-1} \left\{ IR_k^{(1)}, IR_k^{(2)} \right\} \right) \cup \left\{ IR_0^{(2)}, IR_m^{(1)} \right\}$

are the internal rays constituting the axis of the bubble ray $B = \{ F_k \}_{0 \leq k < \infty}$, where $IR_k^{(1)}$ and $IR_k^{(2)}$ are the internal rays from the center of bubble $F_k$ to the bubble intersection points $x_{k-1}$ and $x_k$ respectively for all $k \in [0, m]$. The bubble intersection points are as previously defined by $\{ x_k \} = F_k \cap F_{k+1}$. Suppose further that $\Theta_k(r)$ is a circle centered at $x_k$ with radius $r$ such that

$$r < \min \left( |\text{Cent}(F_k) - x_k|, |x_k - \text{Cent}(F_{k+1})| \right).$$

As $\Theta_k(r)$ cross $IR_{k+1}^{(1)}$ and $IR_k^{(2)}$ it can be divided into two arcs which together form the complete circle. One of these arcs, call it $\Theta_k^{(left)}(r)$, is placed entirely on the left side of the axis of $B$ and the other, $\Theta_k^{(right)}(r)$, is placed entirely on the right side of the axis of $B$. If the length of $\Theta_k^{(right)}(r)$ exceeds that of $\Theta_k^{(left)}(r)$ as $r \to 0$, the bubble ray and its axis are, given that the limit exists, said to divert to the left at $x_k$ with **diversion angle** (given in units of a full rotation)

$$\theta_k = \lim_{r \to 0} \frac{\text{length} \left( \Theta_k^{(left)}(r) \right)}{2\pi r}.$$  

Vice versa of course applies for $\text{length} \left( \Theta_k^{(right)}(r) \right) < \text{length} \left( \Theta_k^{(left)}(r) \right)$ as $r \to 0$. □
3.2 The Dendrites

The definition of the word **dendrite** is a “branching treelike structure”. For mathematical contexts a suitable description found in [Mi1] is

\[ \text{Dendrite} - \text{A compact, connected set without interior which does not separate the plane.} \]

In this investigation only \( f_{\pm i}(z) \) will be considered. The Julia set of \( f_i(z) \) is in [Mi1] classified as a dendrite. The lack of interior points provides \( K_{f_i} = J_{f_i} \), which is illustrated in figure 3.2.1. Its complex conjugate is \( K_{f_{-i}} = \overline{K_{f_i}} \) (see appendix for proof), hence \( K_{f_{-i}} \) is also a dendrite.

3.2.1 The map \( f_i \)

The critical points and critical values of the map

\[ f_i = z^2 + i \]

are

\[ z_{cp}(f_i) \in \{0, \infty\} \]

and

\[ z_{cv}(f_i) \in \{i, \infty\}. \]

Due to the definition of a filled Julia set (definition 2.1.4) only the bounded of these points are included in \( K_{f_i} \). Apart from the attracting periodic point at
infinity there also exists a repelling periodic orbit (hence it lies in the Julia set) between the points
\[-1 + i \leftrightarrow -i.\]

An iterated preimage of this repelling periodic orbit is \(z_{cp}\).

**Lemma 3.2.1.1** Suppose
\[X_{\pm} \equiv \bigcup_{n \geq 0} \left(f_{\pm}^{0(-n)}(0)\right),\]
that is, the set consisting of all preimages to the critical point of \(J_{f_{\pm}}\). Then \(X_{\pm}\) is everywhere dense in \(J_{f_{\pm}}\).

**Proof:** See corollary 4.13 in [Mi1] page 49. \(\Box\)

Lemma 3.2.1.1 will in upcoming chapters play a significant role.

There are two fixed points belonging to \(J_{f_{\pm}}\) and they are
\[z_{\pm}^{\pm} = \frac{1 \pm \sqrt{1 - 4i}}{2} \in J_{f_{\pm}}.\]

Turning to the external rays of \(J_{f_{\pm}}\), since \(ER(1/6)\) land at the critical value of \(J_{f_{\pm}}\) (see [CaGa] p. 149) the external rays \(ER(1/3)\) and \(ER(2/3)\) land at the repelling periodic orbit points \((-1 + i, -i)\) respectively. Through backwards iteration of The double angle Theorem (Theorem 2.3.3) it can also be deduced that both \(ER(1/12)\) and \(ER(7/12)\) land at the critical point of \(J_{f_{\pm}}\). Now \(c = i\) belongs to the \(1/3\)-limb of the Mandelbrot set \(M\), so according to Lemma 2.4.1 there are then exactly three distinct external rays landing at the \(c\)-fixed point \(J_{f_{\pm}}\). They are \(ER(1/7), ER(2/7)\) and \(ER(4/7)\).

### 3.2.2 The map \(f_{-i}\)

For \(f_{-i}\) the story is almost the same as just explained above. The critical points and critical values are
\[z_{cp}(f_{-i}) \in \{0, \infty\}\]
and
\[z_{cv}(f_{-i}) \in \{-i, \infty\}.\]

The attracting periodic point at infinity is again present and the repelling periodic orbit now oscillate between the points (which lie in the Julia set of \(f_{-i}\))
\[-1 - i \leftrightarrow i.\]

Fixed points of \(J_{f_{-i}}\) are
\[z_{f_{-i}}^{\pm} = \frac{1 \pm \sqrt{1 + 4i}}{2} \in J_{f_{-i}}.\]
3 THE BASILICA, THE DENDRITES
AND THE $R_a$ FAMILY

3.3 The $R_a$ family

The same external rays as for $f_i$ land at the repelling periodic orbit of $f_{-i}$. But now $ER(2/3)$ land at $-1 - i$ and $ER(1/3)$ land at $i$, since $ER(5/6)$ (see [CaGa] p. 149) land at the critical value of $J_{f_{-i}}$. The external rays $ER(5/12)$ and $ER(11/12)$ land at the critical point of $J_{f_{-i}}$. The value $c = -i$ belongs to the $2/3$-lobe of $M$, so the three external rays landing at the $\alpha$-fixed point $J_{f_{-i}}$ are $ER(3/7)$, $ER(5/7)$ and $ER(6/7)$.

3.3 The $R_a$ family

The rational map $R_a$ is defined as

$$R_a(z) \equiv \frac{a}{z^2 + 2z} \quad a \in \mathbb{C}. $$

It is conformal on $\mathbb{C} \setminus \{-1, \infty\}$ and continuous on the whole complex plane. Critical points of $R_a$ are $z_{cp}(R_a) \in \{-1, \infty\}$, and their corresponding critical values are $z_{cv}(R_a) \in \{-a, 0\}$. Since it contains a critical point the periodic orbit

$$0 \leftrightarrow \infty$$

of $R_a$ is superattracting, hence it lies in the Fatou set of $R_a$. The Fatou component containing unbounded points is denoted $A_{\infty}$.

The fixed points of $R_a$ are given by the roots to the polynomial $z^3 + 2z^2 - a$. At the moment finding these roots can be a bit cumbersome and will not contribute information of significant value. Thus they are left for later chapters.

**Definition 3.3.1 (Bubble of $R_a$)** A bubble of $R_a$ is a Fatou component $F \subset \hat{\mathbb{C}} \setminus J(R_a)$. The *generation* of a bubble $F$ is the smallest non-negative $n = \text{Gen}(F)$ for which $R_a^{\text{Gen}(F)}(F) = A_{\infty}$. The center of a bubble $F$ is the preimage $R_a^{-\text{Gen}(F)}(\infty) \cap F$ and is denoted by $\text{Cent}(F)$. $\square$

Bubble rays and bubble ray axes of $R_a$ may now be defined in similar ways as for the ones of the basilica. Proven further into the thesis is that the Fatou set of $R_a$ can be observed as merely being a stretching and bending of the basilica on the Riemann sphere. Hence a reasonable description of the structure of $R_a$ is a web of bubbles. The Julia and Fatou sets of $R_3$ are illustrated in figure 3.3.1. The illustration is given as presented by V. Timorin ([Ti1] p. 18).

Studying figure 3.3.1 a repeating pattern of two blue and two red bubbles, with only one point in common and ordered every second color reveals itself. This will be known as a *clover* and is defined later in the thesis.
Figure 3.3.1. An illustration of the Fatou and Julia sets for the rational map $R_a$, where $a = 3$. The union of the blue and red components is the Fatou set. The Julia set is the infinitely small space connecting the components. The critical point $z_{cp} = -1$ is found at the very center of the figure.
4 Theory of mating

The conception of mating and its basic theory is in its short version presented in this chapter.

To perform a mating of two maps simply mean that these maps are joined by connecting the exterior of their corresponding filled Julia sets to one another in reverse order. Through this procedure the filled Julia sets roll up, tangle and give rise to a new set on which the map called the mating map onto itself.

For a mating to be performable the maps out of which it is conceived must be mateable. To understand what the term mateable means we must first introduce the idea of ray equivalence.

4.1 Ray equivalence

Consider two complex maps $f$ and $g$ with locally connected filled Julia sets $K_f$ and $K_g$. As given by The landing criterion Theorem (Theorem 2.3.2) in chapter 2 there exists semiconjugacies

$$
\gamma_f(\theta) : \mathbb{R}/\mathbb{Z} \rightarrow \partial K_f
$$

and

$$
\gamma_g(\theta) : \mathbb{R}/\mathbb{Z} \rightarrow \partial K_g
$$

continuously depending on $\theta$, which map the landing points of the external rays $ER_{K_f}(\theta)$ and $ER_{K_g}(\theta)$ respectively. A space $K_f \sqcup K_g$ is constructed by identifying $\gamma_f(\theta)$ with $\gamma_g(-\theta)$ for all $\theta \in \mathbb{R}/\mathbb{Z}$ writing $\gamma_f(\theta) \sim \gamma_g(-\theta)$. For the case of $\gamma_f(\theta) \sim \gamma_g(-\theta)$ corresponding to a single point $\gamma_f(\theta) = \gamma_g(-\theta)$ for all $\theta \in \mathbb{R}/\mathbb{Z}$ we define the set

$$
X = (K_f \sqcup K_g) / (\gamma_f(\theta) \sim \gamma_g(-\theta))
$$

Definition 4.1.1 (Ray equivalent) With definitions as given above in this section, suppose

$$
z \in \partial K_f \quad w \in \partial K_g$$

and

$$
\theta_z \in \gamma_f^{-1}(z) \quad \theta_w \in \gamma_g^{-1}(w)
$$

If $\theta_z = -\theta_w$ we then say that $z$ and $w$ are ray equivalent and write

$$
z \sim w
$$

Suppose further that there exists sequences $\{z_k\} \subset \partial K_f$ and $\{w_k\} \subset \partial K_g$ such that $z_k \sim w_k \sim z_{k+1}$ for all $1 \leq k \leq n$. We then write $z_1 \sim w_n$ and say that $z_1$ and $w_n$ are ray equivalent. For all $1 \leq k < j \leq n$, $z_k$ and $z_j$ are also said to be ray equivalent, that is $z_k \sim z_j$. A point is also defined to be ray equivalent with itself. □
4.2 The mating and mateability

Up on reading papers on the subject, various types of matings are encountered. A topological mating is here defined as presented in [AsYa].

Definition 4.2.1 (Topological mating)  Suppose that \( f \) and \( g \) are quadratic polynomial maps. Suppose further that the topological space \( X \) as given above is homeomorphic to \( S^2 \). Then \( f \) and \( g \) are said to be topologically mateable and the continuous map

\[
F = f \sqcup \tau g : X \to X
\]

is called the topological mating of \( f \) and \( g \). □

Another type of mating is the conformal mating. It is defined as in [AsYa].

Definition 4.2.2 (Conformal mating)  Again suppose \( f, g, F \) and \( X \) are as in the above definitions, i.e. \( f \) and \( g \) are topologically mateable. Assume the existence of the homeomorphic change of coordinate

\[
\psi : X \to \hat{\mathbb{C}}
\]

conformal on \( \hat{K}_f \cup \hat{K}_g \) such that

\[
R = \psi \circ F \circ \psi^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}
\]

is a rational map. This relation can also be written as

\[
\psi \circ F = R \circ \psi : X \to \hat{\mathbb{C}}.
\]

\( R = f \sqcup g \) is then said to be a conformal mating of \( f \) and \( g \), which under these circumstances are referred to as conformally mateable. □

So the ability for two maps \( f_{c_1} \) and \( f_{c_2} \) to be conformally mateable is depending on the existence of the change of coordinate \( \psi \) just described. \( \psi \) can be divided into two parts, one map for each filled Julia set. For \( i = 1, 2 \) we write

\[
\psi(z) : X \to \hat{\mathbb{C}}
\]

which corresponds to

\[
\psi_{c_i}(z) : K_{f_{c_i}} \to \hat{\mathbb{C}} \quad z \in K_{f_{c_i}}
\]

where

\[
\psi_{c_i} \circ f_{c_i} = R \circ \psi_{c_i} : K_{f_{c_i}} \to \hat{\mathbb{C}}
\]
must hold. Also \( \psi_{c_i} \) is conformal on \( \hat{K}_{f_{c_i}} \) but unlike \( \psi \) it is not invertible. We have \( z \sim w \) if and only if \( \psi_{c_i}(z) = \psi_{c_j}(w) \).

The worked out examples given in chapter 5 and 6 are conformal matchings and we will therefore throughout the rest of the thesis settle for referring to them just as matchings. When mating maps of the form \( f_c \) a satisfying property can be found proven in [Mi2].

**Lemma 4.2.1** Suppose \( f_{c_1} \) and \( f_{c_2} \) are mateable with the mating \( F = f_{c_1} \sqcup f_{c_2} \). Then the two critical points of \( F \) are given by the set

\[
\{ \cup_{i=1,2} \psi_{c_i}(z_{\text{cp}}(f_{c_i})) \}
\]

for

\[ z_{\text{cp}}(f_{c_i}) \in K_{f_{c_i}}. \]

**Proof:** See [Mi2]. \( \square \)

### 4.3 Shared matings

There exist cases when matchings are equal up to Möbius transformation, although the original maps from which the matchings were build are not. Such matchings are said to be **shared matchings** and we write, for matchings mated from maps of the form \( f_c \),

\[ f_{c_1} \sqcup f_{c_2} \cong f_{c_3} \sqcup f_{c_4}. \]

Using Thurston equivalence and its applications, D. Dudko explores and provides a proof of existence of these shared matchings in *Matings with Laminations* from 2011 ([Du]).

In the next chapter a description of the two shared matchings

\[ f_{-1} \sqcup f_i = f_{-1} \sqcup f_{-i} = R_3 \]

will be given and we will argue that they both equal the same rational function. Notable is that the supplement of an allowed Möbius transformation is thus here redundant.
5 Mating the basilica with the dendrites

We have now arrived at the first of the two main chapters of this thesis. Here the maps, from the majority of the basilica onto the Fatou and Julia sets of the rational map $R_a$, will be defined and investigated in depth. New tools suited for the objective are introduced and one will by the end of the chapter understand how the basilica under these matings deforms into $R_a$.

Provided $f_c$ is a dendrite and $c \notin \mathcal{I}$-limb of $\mathcal{M}$, it is in [AsYa] proven that $f_{-1}$ and $f_c$ are conformally mateable and that their mating is unique up to Möbius transformation.

However the approach in this thesis is by defining the changes of coordinates $\psi^{-1}_{\pm}$ from the basilica onto to the Julia and Fatou set of $R_a$ on the Riemann sphere, also prove that $\psi^{-1}_{\pm}$ exists. If one were to add the existence of changes of coordinates from the dendrites onto the Julia set of $R_a$, it would show that $f_{-1}$ and $f_{\pm i}$ are conformally mateable. However the later proposition is not in the scope of this thesis.

5.1 Connecting the Julia sets

According to Böttcher’s Theorem the Böttcher maps

$$\phi_{-1} : \mathbb{C} \setminus K_{f_{-1}} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$$

$$\phi_{\pm i} : \mathbb{C} \setminus K_{f_{\pm i}} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$$

do exist. By the Landing criterion, $\phi_{-1}^{-1}$ and $\phi_{\pm i}^{-1}$ can be extended continuously to $\partial \mathbb{D}$. External rays land at the points $\gamma_{-1}(\theta)$ and $\gamma_{\pm i}(\theta)$ which continuously depend on $\theta$ where

$$\gamma_{-1}(\theta) = \lim_{r \rightarrow 1^+} \phi_{-1}^{-1}(e^{2\pi i\theta}) : \mathbb{R} \setminus \mathbb{Z} \rightarrow J(f_{-1})$$

$$\gamma_{\pm i}(\theta) = \lim_{r \rightarrow 1^+} \phi_{\pm i}^{-1}(e^{2\pi i\theta}) : \mathbb{R} \setminus \mathbb{Z} \rightarrow J(f_{\pm i})$$

Hence the topological spaces

$$X_{\pm i} = (K_{f_{-1}} \sqcup K_{f_{\pm i}}) / (\gamma_{-1}(\theta) \sim \gamma_{\pm i}(-\theta))$$

do exist and they are created by letting every point $\gamma_{-1}(\theta) \in J(f_{-1})$ also correspond to the point $\gamma_{\pm i}(-\theta) \in J(f_{\pm i})$.

A definition of homeomorphic maps from $X_{\pm i}$ onto $\hat{\mathbb{C}}$ is now the next step needed to be done, where the submaps are denoted as

$$\psi^{-1}_{-1} : K_{f_{-1}} \rightarrow \hat{\mathbb{C}}$$

$$\psi_{\pm i} : K_{f_{\pm i}} \rightarrow J(R_a)$$
5.2 Specifying the parameter $a$ in $R_a$

for the conformal mating $f_{-1} \sqcup f_i$, and

$$\psi_{-1} : K_{f_{-1}} \to \hat{\mathbb{C}}$$
$$\psi_{-i} : K_{f_{-1}} \to J(R_a)$$

for the conformal mating $f_{-1} \sqcup f_{-i}$. According to the previous chapter it is necessary to have

$$\psi_{-1} \circ f_{-1} = R_a \circ \psi_{-1} : K_{f_{-1}} \to \hat{\mathbb{C}}$$
$$\psi_{\pm i} \circ f_{\pm i} = R_a \circ \psi_{\pm i} : K_{f_{\pm i}} \to \hat{\mathbb{C}}$$

and the existence of $\psi_{-1}$ and $\psi_{\pm i}$ is now the missing puzzle piece essential for the conformal matings $f_{-1} \sqcup f_{\pm i}$. The definition of $\psi_{-1}$ on the set

$$L = \hat{K}_{f_{-1}} \cup \left( \bigcup_{n=0}^{\infty} f_{-1}^{o(-n)}(z_\alpha) \right)$$

will in this chapter be derived. The definitions of $\psi_{-1}$ on $K_{f_{-1}} \setminus L$ and $\psi_{\pm i}$ on $K_{f_{\pm i}}$ will not be considered in this investigation. But first we specify $R_a$.

5.2 Specifying the parameter $a$ in $R_a$

Suppose for a brief moment that the conformal matings $f_{-1} \sqcup f_i$ and $f_{-1} \sqcup f_{-i}$ do exists. Lemma 4.2.1 then states

$$\text{Crit}(f_{-1} \sqcup f_{\pm i}) = \{ \psi_{-1}(z_{cp}(f_{-1})), \psi_{\pm i}(z_{cp}(f_{\pm i})) \}$$

for

$$z_{cp}(f_{-1}) \in K_{f_{-1}}, \quad z_{cp}(f_{\pm i}) \in K_{f_{\pm i}}.$$ 

thus

$$\psi_{-1}(0) \in \{-1, \infty\}, \quad \psi_{(\pm i)}(0) \in \{-1, \infty\}.$$ 

The superattracting periodic orbit

$$0 \mapsto \infty$$

is contained in $R_a$, and $f_{-1}$ contains the superattracting periodic orbit

$$-1 \mapsto 0.$$

The dendrite polynomials $f_{\pm i}$ do not contain any attracting periodic orbits (since $K_{f_{\pm i}} = J_{f_{\pm i}}$). Remembering $\psi_{-1} \circ f_{-1} = R_a \circ \psi_{-1} : K_{f_{-1}} \to \hat{\mathbb{C}}$, clearly the image of an attracting periodic orbit of $f_{-1}$ under $\psi_{-1}$ must also represent an attracting periodic orbit, hence $\psi_{-1}(0) = \infty$, $\psi_{-1}(-1) = 0$ and $\psi_{\pm i}(0) = -1$. From the forward iterations of $f_{\pm i}$ from their bounded critical point the following mapping scheme is made,
5.3 Defining $\psi^+_{-1}$ and $\psi^-_{-1}$

The definitions of the change of coordinates $\psi^\pm_{-1}$ mapping part of the basilica onto the Fatou set and part of the Julia set of $R_3$ will in this section be given.

Shown in the previous section was that $\psi^\pm_{-1}(0) = \infty$ must hold, so lets make the periodic point at infinity the definition of $\psi^\pm_{-1}$ for the periodic point at the origin. We write

$$\psi^\pm_{-1} : 0 \rightarrow \infty.$$ 

Remember that the immediate basin of attraction for $f_{-1}(0)$ is $B_0$ and denote the immediate basin of attraction for $R_3$ at $\infty$ as $A_\infty$. Both the Fatou components $B_0$ and $A_\infty$ are simply connected subsets of $K_{f_{-1}}$ and $\hat{\mathbb{C}}$ respectively each containing a periodic point of order 2. Hence there exists invertible Böttcher maps such that

$$\phi_{B_0} : B_0 \rightarrow \mathbb{D}$$
$$\phi_{A_\infty} : A_\infty \rightarrow \mathbb{D}$$
$$\phi_{B_0}(0) = \phi_{A_\infty}(\infty) = 0.$$ 

At every point of $B_0$ we then define $\psi^\pm_{-1}$ as

$$\psi^\pm_{-1} = \phi_{A_\infty}^{-1} \circ \phi_{B_0} : B_0 \rightarrow A_\infty.$$ 

Next $\psi^\pm_{-1}$ will be defined for the Fatou components constituting all immediate preimages of points included in $B_0$. They are denoted as $\{B_1, B_2\} = f_{-1}^{(1)}(B_0)$ and can be seen illustrated in fig 3.1.1 in section 3.1. Every point $z \in A_\infty$ has exactly two immediate preimages $R_3^{(1)}(z) = \{w_1, w_2\}$. Now a useful lemma.

$$0 \rightarrow \pm i \rightarrow -1 \pm i \rightarrow \mp i$$

$$\psi_{\pm i}(0) \rightarrow \psi_{\pm i}(\pm i) \rightarrow \psi_{\pm i}(-1 \pm i) \rightarrow \psi_{\pm i}(\mp i)$$

$$-1 \rightarrow R_a(-1) \rightarrow R_a^3(-1) = R_a^3(-1).$$

Remember from chapter 3 that external rays $ER(1/3)$ and $ER(2/3)$ land at $z_\alpha \in \partial K_{f_{-1}}$ and external rays $ER(1/3) = ER(2/3)$ and $ER(2/3) = ER(1/3)$ land at $-1 \pm i$ and $\mp i$ respectively. This means that the equality $\psi^\pm_{-1}(z_\alpha) = \psi_{\pm i}(-1 \pm i) = \psi_{\pm i}(\mp i)$ is valid. The points $-1 \pm i$ and $\mp i$ did here under $\psi_{\pm i}$ provide the same image. This phenomenon will throughout the thesis be known as fusing and will later in chapter 5 be further discussed.

The requirement that the equality $R_a^{(2)}(-1) = R_a^{(3)}(-1)$ must hold can now put a condition on $a$. $R_a^{(2)}(-1) = R_a^{(3)}(-1)$ has only $a \in \{1, 3\}$ as possible solutions. However $a = 1$ is a trivial non useful case since the only point mapping onto $z_{cp}(R_1) = -1$ under $R_1$ is $z_{cp}(R_1) = -1$ itself. Therefore we get

$$f_{-1} \cup f_{\pm i} = R_3.$$
Lemma 5.3.1 \hspace{1em} There exists exactly two Fatou components $A_1$ and $A_2$, such that $A_1 \cap A_2 = \emptyset$ and $R_3(A_1) = R_3(A_2) = A_\infty$.

**Proof:** Let $z \in A_\infty$. Then $z$ has exactly two immediate preimages under $R_3$. Suppose there exist more than two disjoint sets which under $R_3$ map onto $A_\infty$. This would create a contradiction because every $z \in A_\infty$ then has more than two immediate preimages. Suppose now instead that there exist only one connected set $A_1$ which under $R_3$ map onto $A_\infty$. The Euler characteristic for a simply connected proper subset of $\hat{\mathbb{C}}$, denoted $T$, is $\chi(T) = 1$. Lemma 2.3.1 in section 2.1 here states that the number of critical points in $A_1$ is equal to $\chi(A_\infty) \cdot d - \chi(A_1)$. $A_1$ is a simply connected set not containing any critical points thus saying that $\chi(A_1) = 2$. But this is the Euler characteristic of the whole Riemann sphere and therefore implies the trivial case of $\hat{\mathbb{C}}$ only consisting of one single Fatou component. □

Each of the Fatou components $A_1$ and $A_2$ described in Lemma 5.3.1 contains exactly one immediate preimage to every point of $A_\infty$. Let $A_1$ be the Fatou component containing $z = 0$ and $A_2$ the Fatou component containing $z = -2$. We define two branches of $R_3$ as

$$g_1 = R_3 : A_1 \to A_\infty$$
$$g_2 = R_3 : A_2 \to A_\infty.$$ 

Since $\psi_{\pm}^1$ is already well defined on $B_0$, it is possible to, via a detour through $B_0$ and $A_\infty$, also define $\psi_{\pm}^1$ on $B_1$ and $B_2$,

$$\psi_{\pm}^1 = g_1^{-1} \circ \psi_{\pm}^1 \circ f_{-1} : B_1 \to A_1$$
$$\psi_{\pm}^2 = g_2^{-1} \circ \psi_{\pm}^2 \circ f_{-1} : B_2 \to A_2.$$ 

The fixed point $z_\alpha \in K_{f_{-1}}$ is under $\psi_{\pm}^1$ mapped onto the only fixed point of $R_3$ included in $\overline{A_\infty}$, that is

$$\psi_{\pm}^1 : z_\alpha \to 1.$$ 

Since they are mapping fixed points both $f_{-1}^\pm(z_\alpha)$ and $R_3^{\alpha(-1)}(1)$ represents, apart from the fixed points themselves, only one point each. Straight forward this provides

$$\psi_{\pm}^1 : z_0 \to -3.$$ 

Let us expand Lemma 3.1.1 to what is a version of Lemma 2.7 in [AsYa].

Lemma 5.3.2 \hspace{1em} For any two Fatou components $A$ and $B$ of $R_3$ exactly one of the following holds:

1. $\overline{A} \cap \overline{B} = \emptyset$.
2. $\overline{A} \cap \overline{B}$ is only one point, which is a pre-fixed point of $R_3$.
3. $A = B$. □
Lemma 5.3.1 and Lemma 5.3.2 shows that $\overline{A_1} \cap \overline{A_\infty}$ and $\overline{A_2} \cap \overline{A_\infty}$ each contain a maximum of one point. Because $R_3(A_\infty) = \overline{A_1}$ and the fixed point $\psi_{-1}^\pm (z_\alpha) \in \overline{A_\infty}$ we derive that $\psi_{-1}^\pm (z_\alpha) \in \overline{A_1}$ must also be true. And so

$$\overline{A_\infty} \cap \overline{A_1} = \psi_{-1}^\pm (z_\alpha) = 1.$$ Using the landing criterion, the inverse Böttcher map $\phi_{A_\infty}^{-1}$ can be extended continuously over the boundary $\partial \mathbb{D}$ and provide

$$\gamma_{A_\infty} (\theta) : \mathbb{R} \setminus \mathbb{Z} \to \partial \overline{A_\infty}$$

which satisfies the semiconjugacy identity

$$\gamma_{A_\infty} (2\theta) = R_3 (\gamma_{A_\infty} (\theta)).$$

Since $1 \in \partial \overline{A_\infty}$ is a fixed point, the external ray $ER(0 \text{ mod } \mathbb{Z})$ lands at $\phi_{A_\infty}(1)$. For the immediate preimage of this fixed point we have the relation

$$\gamma_{A_\infty} (2\theta) = R_3 (\gamma_{A_\infty} (\theta)) = R_3 (-3) = \gamma_{A_\infty} (0 \text{ mod } \mathbb{Z}).$$

So $\theta = 1/2$ and thereby $-3 \in \partial \overline{A_\infty}$. Since it is a critical value there only exists one immediate preimage to $-3$ when mapped under $R_3$, namely $-1$. Since both $\overline{A_1}$ and $\overline{A_2}$ map onto $\overline{A_\infty}$ under $R_3$ we have $-1 \in \partial \overline{A_1}$ and $-1 \in \partial \overline{A_2}$ and hence

$$\overline{A_1} \cap \overline{A_2} = -1.$$ There also must exist a point $w \in \partial \overline{A_2}$ such that $R_3 (w) = 1 \in \partial \overline{A_\infty}$. The solutions are $w \in \{-3, 1\}$. However if $w = 1$ the set $\overline{A_1} \cap \overline{A_2}$ consists of two points which is a contradiction to Lemma 5.3.2. So

$$\overline{A_2} \cap \overline{A_\infty} = -3.$$ And so the topological properties of $A_1$ and $A_2$ in $R_3$ are decided.

The immediate preimages to $z_0$ under $f_{-1}$ are $z_1$ and $z_2$ (shown in fig. 3.1.1). However $R_3^{2(-1)} (\psi_{-1}^\pm (z_0))$ only represents a single point, namely $z_{cp} = -1$. But the semiconjugacy $\psi_{-1}^\pm \circ f_{-1} = R_3 \circ \psi_{-1}^\pm$ must hold for all values of $K_{f_{-1}}$, and so we define

$$\psi_{-1}^\pm : \{z_1, z_2\} \to -1.$$ Since every bubble is an iterated preimage of $A_\infty$ and $\overline{A_\infty} \cap \overline{A_1} = 1$ we get the following lemma.

**Lemma 5.3.3** Suppose $A$ and $B$ are bubbles of $R_3$. If $\overline{A} \cap \overline{B} = \{z\}$ then $z$ is an iterated preimage of 1 and at most four bubbles intersect at $z$.

**Proof:** Recall $\overline{A_\infty} \cap \overline{A_1} = 1$. Every point $v_1 \in \hat{C} \setminus J(R_3)$ belonging to the Fatou set of $R_3$ is a preimage to the super attracting periodic cycle of period 2 lying in $A_\infty \cup A_1$. Thus every bubble $A$ of $R_3$ is a preimage to $A_\infty$ and $A_1$.
Hence if \( v_2 = \overline{A} \cap \overline{B} \) for two bubbles \( A \) and \( B \) of \( R_3 \) then \( v_2 \) must also be a preimage of \( z = 1 \).

Since \( -1 \) is critical (of order 2) and \( R_3^2(-1) = 1 \) we have four bubbles intersecting at \( z = -1 \). Every other preimage \( w \) of \( z = 1 \) thus have at most four bubbles intersecting at \( w \), since \( R_3 \) is conformal on \( \hat{\mathbb{C}} \setminus \{-1, \infty\} \).

The closures of the four bubbles meeting at the critical point \( -1 \) of \( R_3 \) will be denoted \( \overline{A_1}, \overline{A_2}, \overline{A_5} \) and \( \overline{A_6} \). The bubbles \( A_5 \) and \( A_6 \) are disjoint and map onto \( A_2 \) under \( R_3 \). How the four closures are connected will further in this chapter be described.

**Definition 5.3.1 (Clover)** Suppose a point \( z = F_1 \cap F_2 \cap F_3 \cap F_4 \) is the only intersection point of the closures of four disjoint Fatou components. The set

\[
\bigcup_{m=1}^4 \overline{F_m}
\]

is then called a clover and \( z \) the core of the clover. □

In figure 3.3.1 illustration of clovers can be seen.

**Corollary 5.3.1** All cores of the clovers in \( R_3 \) are iterated preimages to \( z_{cp}(R_3) = -1 \). □

For the continued definition of \( \psi_{-1} \) an introduction of new concepts is required.

### 5.3.1 Fusion points

As briefly just encountered, as well as in section 5.2, the subject of points fusing is now about to unravel more profoundly. Initially the definition of a fusion point is given.

**Definition 5.3.1.1 (Fusion point)** If for a map \( f : X \to Y \) the equality \( f(z) = f(w) \) is true the points \( \{z, w\} \in X \) are said to be fusion points of \( X \) in \( Y \) for \( f \). □

When performing a mating one is likely to encounter the phenomenon of several points fusing into one. As an example consider a conformal mating

\[
F = f \cup g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}
\]

with filled Julia sets \( K_f \) and \( K_g \). Also

\[
\psi_f \circ f = F \circ \psi_f : K_f \to \hat{\mathbb{C}}
\]

and

\[
\psi_g \circ f_g = F \circ \psi_g : K_g \to \hat{\mathbb{C}}.
\]
5.3 Defining $\psi_{-1}^+ \text{ and } \psi_{-1}^-$ WITH THE DENDRITES

The maps $\phi_f$ and $\phi_g$ are the Böttcher maps from $\mathbb{C} \setminus K_f$ and $\mathbb{C} \setminus K_g$ onto $\mathbb{C} \setminus \mathbb{D}$. Assume that two external rays $ER(\theta_1)$ and $ER(\theta_2)$ land at a point $v_1 \in K_f$. If $ER(-\theta_1)$ and $ER(-\theta_2)$ land at points $\{v_2 \neq v_3\} \subset K_g$ then

$$\lim_{r \to 1^+} \phi_g^{-1}(r \cdot e^{2\pi i(-\theta_1)}) \neq \lim_{r \to 1^+} \phi_g^{-1}(r \cdot e^{2\pi i(-\theta_2)})$$

and because $v_2 \sim v_1 \sim v_3$ are ray equivalent we have $\psi_g(v_2) = \psi_g(v_3) = \psi_f(v_1)$. This means that when describing $\psi_g(K_g)$ in the Julia and Fatou sets of $F$ one will find that there is one and only one point $w$ such that $w = \psi_g(v_2) = \psi_g(v_3)$. In this thesis the points $v_2$ and $v_3$ will then be said to have fused in $\mathbb{C}$ for $\psi_g$. This technique of finding fusion points, in order to be able to topologically understand what the filled Julia sets being mated undergoes, will in a moment prove itself highly valuable.

In the above example no concern has been given to whether there are more than one external ray landing at each of $v_2$ and $v_3$. If this is the case there are several ways of fusing $v_2$ and $v_3$. The next three definitions provide notifications useful to specify how these points are fused, where one is a special case only applicable on the basilica.

**Definition 5.3.1.2 (Fusion point connection angle)** Suppose that $f \sqcup g$ exists and that external rays $ER(\theta_1)$ and $ER(\theta_2)$ land at the points $w_1 \in K_f$ and $w_2 \in K_f$ respectively. Further suppose that $w_1$ and $w_2$ fuse in $Y$ for $\psi_f : K_f \to Y$ and that there exists a point $w_3 \in K_g$ at which the external rays $ER(-\theta_1)$ and $ER(-\theta_2)$ land. For the fusion of $w_1$ and $w_2$ of $K_f$ in $Y$ for $\psi_f$, the landing angles $\theta_1$ and $\theta_2$ are then defined as **fusion point connection angles** (or simply connection angles) for $w_1$ and $w_2$ respectively, and the landing angle pair $\{\theta_1, \theta_2\}$ is defined as a fusion point connection angle pair (or simply connection angle pair) for $w_1$ and $w_2$. □

Let $\ell_{-1}$ be an equipotential curve surrounding $K_{f-1}$, given by the Böttcher map

$$\ell_{-1} = \phi_{-1}^{-1}\left(\{r_0 \cdot e^{2\pi i(\theta)} : 0 \leq \theta < 1\}\right) \text{ for some fixed } r_0 > 1.$$ 

**Definition 5.3.1.3 (Fusion point connection ray, FCR)** Suppose that $\gamma(d_1, d_2) \subset \mathbb{D}$ is a hyperbolic geodesic curve on the closed unit disc connecting the points $d_1 \in \partial \mathbb{D}$ and $d_2 \in \partial \mathbb{D}$. Define $M = \{z : |z| \geq r_0 > 1\}$ as the exterior of a disc with radius $r_0$ centered at the origin. Suppose further that $h$ is a homeomorphic map of $\mathbb{D}$ onto $M$ and that $h(\gamma(d_1, d_2))$ is a hyperbolic geodesic in $M$ between $h(d_1)$ and $h(d_2)$.

Suppose also that $\Gamma = \phi_{-1}^{-1}(h(\gamma(d_1, d_2)))$ is a curve in $\mathbb{C} \setminus K_{f-1}$ that intersects $\ell_{-1}$ (as defined above) at the points $v_1 = \phi_{-1}^{-1}(h(d_1))$ and $v_2 = \phi_{-1}^{-1}(h(d_2))$. Define

$$ER(\theta, r) = \left\{\phi_{-1}^{-1}\left(t \cdot e^{2\pi i(\theta)}\right) : 1 < t < r\right\}.$$
Suppose that \( \{w_1, w_2\} \subset K_{f-1} \) for \( R_3 \) are fusion points with connection angle pair \( \{\theta_1, \theta_2\} \) such that \( v_1 \in ER(\theta_1, r_0) \) and \( v_2 \in ER(\theta_2, r_0) \). The **fusion point connection ray** of \( w_1 \) and \( w_2 \), or simply \( FCR(\theta_1, \theta_2) \), is then defined as
\[
FCR(\theta_1, \theta_2) = \Gamma \cup ER(\theta_1, r_0) \cup ER(\theta_2, r_0).
\]

\[\square\]

For the definition to be useful another lemma is needed.

**Lemma 5.3.1.1** Suppose the connection angles of \( FCR(\theta_{11}, \theta_{12}) \) and \( FCR(\theta_{21}, \theta_{22}) \) are such that \( \theta_{11} < \theta_{21} < \theta_{22} < \theta_{12} \). Then \( FCR(\theta_{11}, \theta_{12}) \) and \( FCR(\theta_{21}, \theta_{22}) \) do not intersect.

**Proof:** Suppose \( \gamma(d_1, d_2) \subset \mathbb{D} \) is a hyperbolic geodesic curve on the open unit disc between the points \( \{d_1, d_2\} \subset \mathbb{D} \). Suppose further that \( g \) is a homeomorphic map of \( \mathbb{D} \) onto the upper half of the complex plane \( S = \{z : \text{Im}(z) \geq 0\} \) and that \( g(\gamma(d_1, d_2)) \) is a hyperbolic geodesic in \( S \) between \( g(d_1) \) and \( g(d_2) \). Theorem 5.3.3 in [JoSi] proves that \( g(\gamma(d_1, d_2)) \) for all \( \{d_1, d_2\} \subset \mathbb{D} \) is an arc of a semi-circle in \( S \) between two points lying on the real axis. Let
\[
\Gamma_1 = FCR(\theta_{11}, \theta_{12}) \setminus (ER(\theta_{11}, r_0) \cup ER(\theta_{12}, r_0))
\]
and
\[
\Gamma_2 = FCR(\theta_{21}, \theta_{22}) \setminus (ER(\theta_{21}, r_0) \cup ER(\theta_{22}, r_0)).
\]

With \( h \) given as in the definition 5.3.1.3, \( g^{-1} \circ h^{-1} \circ \phi_{-1}(\Gamma_1) \) and \( g^{-1} \circ h^{-1} \circ \phi_{-1}(\Gamma_2) \) do not intersect, and thus neither do \( \Gamma_1 \) and \( \Gamma_2 \). From the definition of \( l_{-1} \) we have that \( ER(\theta_{11}, r_0), ER(\theta_{12}, r_0), ER(\theta_{21}, r_0) \) and \( ER(\theta_{22}, r_0) \) do not intersect, hence \( FCR(\theta_{11}, \theta_{12}) \) and \( FCR(\theta_{21}, \theta_{22}) \) do not intersect. \( \square \)

The definition of \( \psi^\pm_{-1} \) can now be extended. For every fusion point pair \( \{w_1, w_2\} \subset J(f_{-1}) \) with connection angles \( \theta_1 \) and \( \theta_2 \) we define
\[
\psi^\pm_{-1} : FCR(\theta_1, \theta_2) \to \psi^\pm_{-1}(w_1) = \psi^\pm_{-1}(w_2).
\]

Analyzing fusion points a fusion point connection ray can when added to an image give a clearer understanding of how a set is deformed when mapped into another set.

Derived, from what already in part has been shown in section 5.2, is the following schematic illustration (fig. 5.3.1.1) showing coincidings of repelling periodic orbits in \( f_{-1} \cup f_{\pm 1} \).
Denoting $\psi^+\pm 1$ and $\psi^-\pm 1$ and the Basilica with the Dendrites

\[
\begin{align*}
0 & \xrightarrow{f_{\pm i}} \pm i & \xrightarrow{f_{\pm i}} & -1 \pm i \xrightarrow{f_{\pm i}} \mp i \\
\psi^+ & \downarrow & \psi^+ & \downarrow & \psi^+ & \downarrow \\
-1 & \xrightarrow{R_3} -3 & R_3 & \xrightarrow{R_3} & 1 & \text{\circ} \\
\psi^- & \uparrow & \psi^- & \uparrow & \psi^- & \uparrow \\
z_1 & \xrightarrow{f_{-1}} z_0 & f_{-1} & \xrightarrow{f_{-1}} & z_\alpha & \text{\circ} \\
z_2 & \text{\circ} & \\
\end{align*}
\]

**Figure 5.3.1.1.** A schematic illustration showing images of the origin under $f_{\pm i}$ and images of $z_1$ and $z_2$ under $f_{-1}$. Also seen is how these points under $\psi_{\pm i}$ and $\psi_{\pm -1}$ map onto $-1$ and its images under $R_3$.

Definitions of the elements in the set $\{z_\alpha, z_0, z_1, z_2\}$ are as described in section 3.1. Since $f_{\pm i}$ are second degree polynomials, it follows that each point of $K_{f_{\pm i}}$, apart from the immediate preimage of the critical value, has two preimages. But as found proven in the appendix, $K_{f_{\pm i}}$ only includes one critical value, $z_{cv}(K_{f_{\pm i}}) = \pm i$, and one preimage of this critical value, namely the critical point $z_{cp}(K_{f_{\pm i}}) = 0$. Due to this result not only the identification

$$
\psi_{\pm -1}(z_\alpha) = \psi_{\pm i}(-1 \pm i) = \psi_{\pm i}(\mp i)
$$

but also

$$
\psi_{\pm -1}(z_1) = \psi_{\pm -1}(z_2) = \psi_{\pm i}(0)
$$

can be made. This once again reveals that $\{z_1, z_2\} \subset K_{f_{-1}}$ are fusing points in $\hat{C}$ for $\psi_{\pm -1}$. The external rays landing on $z_1$ and $z_2$ are as observed in figure 3.1.1 $\{ER(5/12), ER(7/12)\}$ and $\{ER(1/12), ER(11/12)\}$ respectively. The question needed an answer now is naturally, which of the four possibilities for the two connection angles, denote them $\theta_z$ and $\theta_{z_2}$, of the fusion point connection ray between $z_1$ and $z_2$ that should be applied. The next lemma will help answer this question.

**Lemma 5.3.1.2** For the maps $\psi_{\pm -1}$, fusion point connection rays connecting the fusion points in the set $f_{-1}^{(\alpha-n)}(z_\alpha) \in K_{f_{-1}}$, are never intersected by other fusion point connection rays.

**Proof:** If a fusion point connection ray (not being $FCR(\theta_{z_1}, \theta_{z_2})$ itself) were to intersect $FCR(\theta_z, \theta_{z_2})$ it would, due to Lemma 5.3.1.1 and the definition of $\psi_{\pm -1}$ on fusion point connection rays, imply that there exists a point $z_3 \in J(f_{-1})$.  

lying at one of the ends of the intersecting fusion point connection ray, satisfying the equality
\[ \psi_{-1}^\pm(z_1) = \psi_{-1}^\pm(z_2) = \psi_{-1}^\pm(z_3) = -1 \in J(R_3). \]
Recall \( \overline{B}_1 \cap \overline{B}_5 = z_1 \) and \( \overline{B}_2 \cap \overline{B}_6 = z_2 \). Due to continuity
\[ \psi_{-1}^\pm(\overline{B}_1) \cap \psi_{-1}^\pm(\overline{B}_5) = \psi_{-1}^\pm(\overline{B}_2) \cap \psi_{-1}^\pm(\overline{B}_6) = -1. \]
Since \( z_3 \in J(f_{-1}) \) it must belong to the boundary of some bubble \( B_{23} \subset K_{f_{-1}} \). Adding \( \psi_{-1}^\pm(\overline{B}_{23}) \) there are then at least five closures of bubbles of \( R_3 \) intersecting at \(-1\). This contradicts Lemma 5.3.3. Hence there are no other fusion point connection ray intersecting \( FCR(\theta_{z_1}, \theta_{z_2}) \). Remember that \( R_3 \) is conformal on \( \hat{\mathbb{C}} \setminus \{-1, 0, \infty\} \) and so exactly four closures of bubbles of \( R_3 \) will intersect at all preimages of \(-1\), thus fusion point connection rays do not intersect. \( \square \)

With some theory on fusion points we are now ready to continue defining \( \psi_{-1}^\pm \) for the remaining Fatou components and prefixed points of \( K_{f_{-1}} \). As presented in section 3.1, \( f_{-1}(B_3) = f_{-1}(B_6) = B_2 \). Every point \( z \in A_2 \) has exactly two immediate preimages under \( R_3 \). One in each of two different Fatou components denoted \( A_5 \) and \( A_6 \). Define the branches of \( R_3 \)
\[ g_5 = R_3 : A_5 \to A_2 \]
\[ g_6 = R_3 : A_6 \to A_2 \]
where \( A_5 \) and \( A_6 \) are yet to be determined. (In fact they will depend on \( \psi_{-1}^\pm \) and \( \psi_{+1} \)).

Define \( \psi_{-1}^\pm \) for \( B_5 \) and \( B_6 \)
\[ \psi_{-1}^\pm = g_6^{-1} \circ \psi_{-1}^\pm \circ f_{-1} : B_5 \to A_5 \]
\[ \psi_{+1}^\pm = g_6^{-1} \circ \psi_{+1}^\pm \circ f_{-1} : B_6 \to A_6. \]
Since \( z_1 \in \overline{B}_5 \) and \( z_2 \in \overline{B}_6 \) we know from Lemma 5.3.2 that
\[ \overline{A}_5 \cap \overline{A}_6 = -1. \]
But \( z_3 = -1 \in \overline{A}_2 \) and its immediate preimages are \( \{-1 + i\sqrt{2}, -1 - i\sqrt{2}\} \), which since \( R_3 \) is continuous lie one in each of \( \overline{A}_5 \) and \( \overline{A}_6 \). Remember that \( R_3(\overline{A}_\infty) = \overline{A}_1 \). Further \( z_3 = -1 \in \overline{A}_1 \), hence with assistance of Lemma 5.3.2
\[ (\overline{A}_5 \cup \overline{A}_6) \cap \overline{A}_\infty = \{-1 + i\sqrt{2}, -1 - i\sqrt{2}\} \subset \overline{A}_\infty. \]
A topological understanding of \( \overline{A}_5 \) and \( \overline{A}_6 \) has now been made except for the problem that we do not know which of them contains which of the points \( \{-1 + i\sqrt{2}, -1 - i\sqrt{2}\} \). Denote the immediate preimages to \( z_1 \) and \( z_2 \) under \( f_{-1} \) as \( f_{-1}(z_{11}) = f_{-1}(z_{12}) = z_1 \) and \( f_{-1}(z_{21}) = f_{-1}(z_{22}) = z_2 \), where \( z_{11} \in B_0 \cap B_3, z_{12} = B_0 \cap B_4, z_{21} \in B_5 \) and \( z_{22} \in B_6 \). As stated earlier
\[ \mathcal{A}_5 \cap \mathcal{A}_6 = \psi_{-1}^\pm (z_2) = \psi_{-1}^\pm (z_1) \text{ thus } \psi_{-1}^\pm (z_{21}) \neq \psi_{-1}^\pm (z_{22}) \text{ since otherwise } \mathcal{A}_5 \cap \mathcal{A}_6 \text{ would consist of more than one point. Since } \psi_{-1}^\pm \text{ is already well defined on } \mathcal{B}_0 \text{ we also get } \psi_{-1}^\pm (z_{11}) \neq \psi_{-1}^\pm (z_{12}). \]

This means that exactly one element from each of the sets \{ \psi_{-1}^\pm (z_{11}), \psi_{-1}^\pm (z_{12}) \} and \{ \psi_{-1}^\pm (z_{21}), \psi_{-1}^\pm (z_{22}) \} is defined as equal to \(-1 + i\sqrt{2}\) and the other as equal to \(-1 - i\sqrt{2}\). A choice as of which element is defined to which value has to be made and it is this very choice that will separate \(\psi_{-1}^+\) and \(\psi_{-1}^-.\)

Since \(f_{\pm_1}\) has external rays \(ER(\pm 1/7), ER(\pm 2/7)\) and \(ER(\pm 4/7)\) landing at its \(\alpha\)-fixed point, the points

\[ \{ z_{\mathcal{E}R(\pm 1/7)}; z_{\mathcal{E}R(\pm 2/7)}; z_{\mathcal{E}R(\pm 4/7)} \} \subset K_{f,-}. \]

with external rays \(ER(\mp 1/7) = ER(\pm 6/7), ER(\mp 2/7) = ER(\pm 5/7)\) and \(ER(\mp 4/7) = ER(\pm 3/7)\) landing at them respectively, need to fuse in \(\mathcal{C}\) for \(\psi_{-1}^\pm\). Let \(\ell\) be the part of the real axis enclosed on the interval \([z_1, z_2]\). The curve \(\Gamma = (\ell \cup FCR(\theta_{z_1}, \theta_{z_2}))\) then forms a closed loop. \(\Gamma\) separates two connected subsets of \(\mathbb{C}\), where one is bounded and the other is not. The bounded subset will be known as the inside of \(\Gamma\) and the unbounded as the outside of \(\Gamma\).

With Lemma 5.3.1.2 proven it is safe to claim that the three points \(\{ z_{\mathcal{E}R(\pm 1/7)}; z_{\mathcal{E}R(\pm 2/7)}; z_{\mathcal{E}R(\pm 4/7)} \} \subset K_{f,-}\), all need to lie on the same side of \(\Gamma\). Now consider the choice

\[ -1 - i\sqrt{2} \in \mathcal{A}_5 \quad -1 + i\sqrt{2} \in \mathcal{A}_6. \]

Denote \(w(r, \theta) = z_{\mathcal{C}P} + r \cdot e^{2\pi i \theta}\). If \(r > 0\) is sufficiently small and \(\theta\) is varied on \(0 \leq \theta < 1\) there will exist a specific order of how \(w\) will appear in the bubbles \(A_1, A_2, A_5\) and \(A_6\). Given \(A_1\) and \(A_2\) this order must be

\[ \begin{array}{c}
\mathcal{A}_6 \swarrow \quad A_6 \\
[A_2 \quad [z_{\mathcal{C}P} = -1] \quad A_1 \\
\searrow \quad \mathcal{A}_5
\end{array} \]

The fusion point connection ray providing this orientation is \(FCR(5/12, 11/12)\). With help from figure 3.1.1 it is now clear that this is the fusion point connection ray connecting \(z_1\) and \(z_2\) for the mating \(f_{-1} \cup f_1\). Hence we define

\[
\psi_{-1}^+: \{ z_{11}, z_{21} \} \rightarrow -1 - i\sqrt{2} \\
\psi_{-1}^+: \{ z_{12}, z_{22} \} \rightarrow -1 + i\sqrt{2}.
\]

The landing point on \(K_{f_i}\) for both the external rays \(ER(-5/12)\) and \(ER(-11/12)\) is \(z_{\mathcal{C}P}(K_{f_{\pm_1}}) = 0\).

If we instead make the choice

\[ -1 + i\sqrt{2} \in \mathcal{A}_5 \quad -1 - i\sqrt{2} \in \mathcal{A}_6. \]

The fusion point connection ray \(FCR(1/12, 7/12)\) connects \(z_1\) and \(z_2\) for the mating \(f_{-1} \cup f_{-1}\). We define

\[
\psi_{-1}^+: \{ z_{11}, z_{22} \} \rightarrow -1 - i\sqrt{2} \\
\psi_{-1}^+: \{ z_{12}, z_{21} \} \rightarrow -1 + i\sqrt{2}.
\]
The landing point on $K_{f_-}$ for both the external rays $ER(-1/12)$ and $ER(-7/12)$ is also $z_{cp}(K_{f_-}) = 0$.

In the basilica the preimage relation $f_-^{-1}(B_3) = f_-^{-1}(B_4) = B_3$ holds. Every point $z \in A_5$ has exactly two immediate preimages under $R_3$. One in each of the Fatou components denoted $A_3$ and $A_4$. We define the branches of $R_3$

$$g_3 = R_3 : A_3 \rightarrow A_5$$
$$g_4 = R_3 : A_4 \rightarrow A_5$$

where $A_3$ and $A_4$ are yet to be determined.

Define $\psi_{-1}$ for $B_3$ and $B_4$

$$\psi^\pm_{-1} = g_3^{-1} \circ \psi^\pm \circ f_-^{-1} : B_3 \rightarrow A_3$$
$$\psi_{-1}^\pm = g_4^{-1} \circ \psi^\pm \circ f_-^{-1} : B_4 \rightarrow A_4.$$ 

Since $z_{11} = B_3 \cap B_0$ and $z_{12} = B_4 \cap B_0$ we have

$$\psi_{-1}^+(z_{11}) = \psi_{-1}^+(z_{21}) = \overline{A_3} \cap \overline{A_5} = \overline{A_3} \cap \overline{A_5} = -1 - i\sqrt{2}$$
$$\psi_{-1}^-(z_{22}) = \overline{A_4} \cap \overline{A_6} = \overline{A_4} \cap \overline{A_6} = -1 + i\sqrt{2}$$

5.3.2 Bubble rays do not cross

Upcoming are some crucial lemmas concerning bubble rays.

**Lemma 5.3.2.1** Suppose $h(z) = a(z - z_0)^2 + O((z - z_0)^3)$ is analytic. Suppose further that $\ell_{cv} \subset S$ is a line such that the point $z_{cv}(h) \in \ell_{cv}$. The preimages $\{\ell_{cp,1}, \ell_{cp,2}\} = h^{\circ(-1)}(\ell_{cv})$ of $\ell_{cv}$ are then two curves only intersecting at $z_{cp}(h)$, at which they both divert in the same direction with diversion angle $\theta_{div} = 1/4$.

**Proof:** On $S$, $h$ behaves the same way as $g(z) = az^2 + O(z^3)$ does on $S_0 = \{z - z_0 : z \in S\}$. (At the origin $g$ behaves as $f(z) = az^2$). Define two subsets of $\ell_{cv}$ as

$$\ell_{cv} = \ell_1 \cup \ell_2 \quad z_{cv} = \ell_1 \cap \ell_2.$$ 

Define four subsets of $\ell_{cp,1}$ and $\ell_{cp,2}$ as

$$\ell_{cp,1} = \ell_{11} \cup \ell_{12} \quad \ell_{cp,2} = \ell_{21} \cup \ell_{22}$$
$$z_{cp} = \ell_{11} \cap \ell_{12} = \ell_{21} \cap \ell_{22}$$

$$f : \left\{ \begin{array}{ll}
\{\ell_{11}, \ell_{21}\} \rightarrow \ell_1 \\
\{\ell_{12}, \ell_{22}\} \rightarrow \ell_2.
\end{array} \right.$$
From standard theory of complex analysis it follows that if the diversion angle between $\ell_1$ and $\ell_2$ is $1/2$, then the diversion angle between $\ell_{11}$ and $\ell_{12}$ and the diversion angle between $\ell_{21}$ and $\ell_{22}$ is in both cases equal to $1/4$. Since $z_{cp}(f) \in \ell_{cp,1} \cap \ell_{cp,2}$ and $z_{cv}(f) \in \ell_{cv}$, we conclude that $z_{cp}(f) = \ell_{cp,1} \cap \ell_{cp,2}$.

For the case of $\ell_{cv}$ being the horizontal line segment with positive direction in the same direction as the real axis, we are left with the two possible solutions shown in fig 5.3.2.1. □

As the next lemma will show, Lemma 5.3.2.1 applied on $R_3$ reveals some of the actions performed on the basilica as it is mapped under $\psi_{\pm 1}$.

**Lemma 5.3.2.2** The images $\psi_{\pm 1}(B_1)$ and $\psi_{\pm 1}(B_2)$ of two arbitrary bubble rays, $B_1 \in K_{f_{-1}}$ and $B_2 \in K_{f_{-1}}$, are themselves bubble rays which are always non-crossing. The axis of $\psi_{\pm 1}(B_1)$ divert with diversion angle $\theta_{div} = 1/4$ at each intersection point between its bubbles in a unified direction (left for $\psi_{\pm 1}$ and right for $\psi_{-1}$) throughout $\hat{\mathbb{C}}$. An exception is for the images of the critical point of $R_3$ where the axis of $\psi_{\pm 1}(B_1)$ belongs to the real axis.

**Proof:** Notations of bubbles and points are in this proof inherited from figure 3.1.1 in chapter 3.
By definition this is a line such that \( \psi \) is a real axis. This brings us, as seen in figure 5.3.2.1, two solutions of how the axes \( \{ \psi \} \) are arranged. The solutions are as seen in figure 5.3.2.2 where for pedagogic purposes a direction has been given the axes. In both cases the bubble rays are non-crossing and their axes divert with diversion angle \( \theta_{\text{div}} = \frac{1}{4} \) either to the left or right exclusively. Next note that \( R_3(z) \) is conformal on \( \mathbb{C} \setminus [-1,0] \). From here it can be concluded that all preimages of \( z_{\text{cp}}(R_3) \) and their neighborhood will behave the same way as shown at \( z_{\text{cp}}(R_3) \) in figure 5.3.2.2. Earlier in section 5.3 a specific orientation of the bubbles \( A_1, A_2, A_5 \) and \( A_6 \) around \( z_{\text{cp}}(R_3) \) was presented. It looked as

\[
\begin{array}{cccc}
A_2 & [z_{\text{cp}} = -1] & A_1 & A_2 & [z_{\text{cp}} = -1] & A_1 \\
\nearrow & A_6 & \searrow & A_5 & \nearrow & A_6 \\
& \text{for } \psi_{-1}^+ & & \text{for } \psi_{-1}^- & & .
\end{array}
\]

Hence the bubble rays turn to the left when mapped under \( \psi_{-1}^+ \) and to the right when mapped under \( \psi_{-1}^- \). \( \Box \)
5.3 Defining $\psi^+_1$ and $\psi^-_1$

5 MATING THE BASILICA WITH THE DENDRITES

Figure 5.3.2.2. Illustration of the polynomial mapping $z \rightarrow -3 \cdot (1 + (1 + z)^2)$. The axes of bubble pairs $\{\psi^+_1(B_1), \psi^+_1(B_5)\}$ and $\{\psi^+_1(B_2), \psi^+_1(B_6)\}$ divert to the left and the axes of bubble pairs $\{\psi^-_1(B_1), \psi^-_1(B_5)\}$ and $\{\psi^-_1(B_2), \psi^-_1(B_6)\}$ divert to the right. The images of all bubble pair axes equal the same desired axis of the bubble pair $\{\psi^+_1(B_0), \psi^+_1(B_2)\}$ lying along the real axis.

A conclusion that $\psi^+_1$ is merely a stretching and bending of the basilica is here at hand due to Lemma 5.3.2.2. Now note for the fusion connection ray $FCR(\theta_{12}, 11/12)$ that it joins the basilicas bubble rays containing $z_1$ and $z_2$, on their right side, whereas for $FCR(\theta_{12}, 7/12)$ the bubble rays are joined on their left side. This is pictured in illustrations A and B of figure 5.3.2.3. As the next lemma will show there exists a generalization to be explored here.

Lemma 5.3.2.3 For the map $\psi^+_1$ every intersection between bubbles in the basilica is joined on the right side, and on the right side only, by a fusion point connection ray. The same relation applies for the map $\psi^-_1$ except that the fusion point connection rays now join the intersections on the left side.

(For definition of right/left see definition 3.1.1.5 in section 3.1.)

Proof: Remember that $R_3(z)$ is conformal on $\hat{\mathbb{C}} \setminus \{-1, 0, \infty\}$. As derived earlier, $FCR(\theta_{12}, \theta_{22})$ joins $\{z_1, z_2\} \subset K^+_{1/2}$ on their right or left side exclusively, depending on which of the maps $\psi^+_1$ is considered. Now recall corollary 5.3.1. Since $R_3$ is conformal all bubble rays of the basilica, when mapped under $\psi^+_1$, will at each intersection point between two bubbles of $R_3$, given that the
intersection point is a preimage of the bounded critical point of $R_3$, divert in
a unified direction with a fixed diversion angle ($\theta_{\text{div}} = 1/4$ proved in Lemma
5.3.2.2) throughout $R_3$. Thus the fusion point connection rays on the basilica
for the maps $\psi^+_1$ and $\psi^-_1$ can only join the intersection points of the bubbles
on the right and left side respectively. □

Instructive is also pointing out, as now will be proven in Lemma 5.3.2.4, that
another relationship similar to Lemma 5.3.2.3 exists.

Lemma 5.3.2.4 Suppose $z \in \ell_z$ is an intersection point between two
bubbles of the basilica lying on a bubble ray axis $\ell_z$. A preimage $f_{-1}^{\circ(-n)}(ER(\theta))$
of an external ray $ER(\theta)$ landing at $z$ on the right/left side of $\ell_z$, does itself
land on the right/left side of the axis $\ell_w$ at a point $w = f_{-1}^{\circ(-n)}(z) \in \ell_w$.

Proof: Note that $f_{-1}$ is conformal on $\hat{\mathbb{C}} \setminus \{0, \infty\}$ which include the space
where the external rays lie. Hence, $f_{-1}$ preserves orientation and the side and
angle from which an external ray approaches and lands at intersection points of
bubbles in the basilica will be preserved for its preimages. □

For convenience let us for a moment devote the attention only to the map
$\psi_{-1}$. Recognizing $FCR(5/12, 11/12)$, The double angle Theorem (Theorem 2.3.3)
and Lemma 5.3.2.4 allows us to identify more fusion point connection rays.
The preimages to the external ray $ER(5/12)$ are $ER(5/24)$ and $ER(17/24)$. They
land on the right sides of the preimages to the point $z_1 \in K_{f_{-1}}$. In the same
way the preimages to $ER(11/12)$ are $ER(11/24)$ and $ER(13/24)$ which land on
the right side of the preimages to the point $z_2 \in K_{f_{-1}}$. Note that both sets
$\{ER(5/24), ER(17/24)\}$ and $\{ER(11/24), ER(13/24)\}$ have one element on each
side of $FCR(5/12, 11/12)$. Result due to Lemma 5.3.1.2 is that the preimages
to $z_1$ and $z_2$ are forced to fuse under $FCR(5/24, 17/24)$ and $FCR(11/24, 13/24)$.
Next step will be finding the preimages to $ER(5/24)$, $ER(11/24)$, $ER(17/24)$ and
$ER(23/24)$ which, as a reaction to $z_1$ and $z_2$ fusing and the fact that each point of
$R_3$ has a maximum of two preimages, also must fuse in sets of two. From here on
the identification of further fusion point connection rays is purely mechanical.
The derivation method of fusion point connection rays for the map $\psi_{-1}$ is similar
and its detail are therefore left out.

Note that solutions to preimages of $\theta$ under double angle mapping are

$$2\nu = \theta \mod \mathbb{Z}$$

$$\nu = \left\{ \begin{array}{ll}
\frac{\theta}{2} & \text{ mod } \mathbb{Z} \\
\frac{\theta}{2} + \frac{1}{2} & \text{ mod } \mathbb{Z}.
\end{array} \right.$$
5.3 Defining $\psi_{-1}^+$ and $\psi_{-1}^-$

With the Dendrites

Figure 5.3.2.3. Fusion point connection rays on the basilica for the maps $\psi_{-1}^+$ (illustration A) and $\psi_{-1}^-$ (illustration B). Consideration has been given points $\{z_1, z_2\} \subset K_{f_{-1}}$ and their preimages up to 3rd generation. Fusion point connection rays illustrated in this figure do not occupy the exact same set of points as the true fusion point connection rays of $\psi_{-1}^\pm$, except for the landing points.
Lemma 5.3.2.5 Suppose the external ray $ER(\theta)$ lands at the point $z \in \partial K_{f_{-1}}$. Then $ER(\theta + \frac{1}{2})$ lands at $-z \in \partial K_{f_{-1}}$.

Proof: Suppose $ER(\theta)$ lands at $z \in \partial K_{f_{-1}}$, and $ER(\theta + \frac{1}{2})$ lands at $w \in \partial K_{f_{-1}}$. Shown above is that $z$ and $w$ are then preimages to the same point. We get the equation $f_{-1}(z) = f_{-1}(w)$ with the solutions $z = \pm w$. The solutions $z = w$ implies that $z = 0$ which is impossible since $0 \notin J(f_{-1})$. Hence $w = -z$, has to be the landing point of $ER(\theta + \frac{1}{2})$. □

Lemma 5.3.2.5 says that the landing points of $ER(\theta/2)$ and $ER(\theta/2 + \frac{1}{2})$ only differ on a minus sign. The effect this lemma has on the basilica is that it coils up in the same way on each side of $FCR(\frac{5}{12},1,1/12)$ (or $FCR(1,12,7/12)$) as the fusion point connection rays collapses. This serves a perception of how the basilica under $\psi_{\pm 1}$ transforms into $R_3$. The identification of some of the fusion point connection rays for both the maps $\psi_{\pm 1}$ are found illustrated in figure 5.3.2.3. It is advised that the reader take a moment to observe figure 5.3.2.3 in detail and picture the progress from basilica to $R_3$ as the fusion point connection rays shorten.

In this section a certain method of how to define $\psi_{\pm 1}$ for the bubbles $B_0$ through $B_6$ of the basilica and their connection points was derived. Also lemmas and methods of how to find fusion point connection rays were provided and demonstrated. Continuing utilizing these methods one is able to define $\psi_{\pm 1}$ on the whole of

$$ L = \hat{K}_{f_{-1}} \cup \left( \bigcup_{n=0}^{\infty} f_{-1}^{\pm n}(z_0) \right) $$

such that

$$ \psi_{\pm 1} \circ f_{-1} = R_3 \circ \psi_{\pm 1} : L \to \hat{\mathbb{C}}. $$

Hence $\psi_{\pm 1}$ exists on $L$.

5.3.3 Lamination

An informative way to describe the map $\psi_{\pm 1} : L \to \hat{\mathbb{C}}$ is with schematic diagrams as the ones seen in figure 5.3.3.1. The diagrams illustrate the relation between the landing angles of the external rays landing at some of the fusion points of the basilica and how fusion point connection rays joins them. This technique is known as lamination. For the case of the maps $\psi_{\pm 1}$ turn to the laminations $A$ and $B$ in figure 5.3.3.1.

As seen in lamination $A$, denote the landing points of $ER(\frac{5}{24})$ and $ER(23/24)$ as $z_{5/24}$ and $z_{23/24}$. The two immediate preimages of $z_{5/24}$ have external rays $ER(\frac{5}{48})$ and $ER(\frac{29}{48})$ landing at them. We therefore denote them

$$ f_{-1}^{0(-1)}(z_{5/24}) = \{ z_{5/48}, z_{29/48} \}. $$

Observed is that $|\frac{5}{24} - 0| > |\frac{5}{48} - 0|$ and $|\frac{5}{24} - 1/2| > |\frac{29}{48} - 1/2|$. The two immediate preimages of $z_{23/24}$ have external rays $ER(\frac{47}{48})$ and $ER(\frac{23}{48})$ landing at them.
5.3 Defining $\psi_1^+$ and $\psi_1^-$

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Figure 5.3.3.1. Lamination giving a schematic illustration of the relation between landing angles of some of the fusion points of the basilica and how their fusion point connection rays joins them. Laminations A and B are for the map $\psi_1^+$ and C and D are for $\psi_1^-$. As seen, the connected angles narrows in on $\theta = 0$ or $\theta = \frac{1}{2}$ as more fusion point connection rays are derived.

Again denote

$$f^{5(-1)}_{-1} \left(\frac{23}{24}\right) = \left\{\frac{23}{48}, \frac{247}{48}\right\}$$

and observe that $|\frac{23}{24} - 0| > |\frac{47}{48} - 0|$ and $|\frac{23}{24} - \frac{1}{2}| > |\frac{23}{48} - \frac{1}{2}|$. Thus the landing point of $ER(\frac{5}{24})$ has one immediate preimage with landing angle closer to 0 than the landing angle of $ER(\frac{5}{24})$, and one immediate preimage with landing angle closer to $\frac{1}{2}$ than the landing angle of $ER(\frac{5}{24})$. The same relation between the landing point of $ER(\frac{23}{24})$ and its preimages is also true. For a moment focus on the angles lying closer to the zero angle. That is the landing angles of $ER(\frac{5}{48})$ and $ER(\frac{47}{48})$. The set of immediate preimages under the double angle map

$$T(\theta) = 2\theta \mod \mathbb{Z}$$

for $\frac{5}{48}$ and $\frac{47}{48}$ also contains two angles which are closer to the zero angle than their image under $T$. The immediate preimages of $\frac{5}{48}$ and $\frac{47}{48}$ also contains
two angles closer to the half angle than their image under $T$. Continuing finding the immediate preimages under $T$ for all angles which are closer to the zero angle than their images and illustrating their fusion point connection rays, the behavior in lamination $B$ of figure 5.3.3.1 arises (see appendix for proof). As the number of iterations, of the process described above, grows large the points of the basilica that are picked in this manner and fuses have decreasing angular distance (shortest possible) between their connection angles. Also every fusion encloses either the zero or the half angle. However, generally angle distance between fusion connection angles do not necessarily decrease to zero.

The corresponding results from the above method applied on the map $\psi_{-1}$ are illustrated in laminations $C$ and $D$ of figure 5.3.3.1.

Interesting is now of course how the patterns of the fusion point connection rays in a schematic illustration of the maps $\psi_{\pm 1}$ will evolve if all preimages are included. For fusion point connection rays up to fourth generation ($n = 4$) this is found illustrated in figure 5.3.3.2.

Studying figure 5.3.3.2 and figure 5.3.2.3 one obtains a revealing description of what is going on in the transformation $\psi_{\pm 1} : L \to \hat{C}$. Why it is that $\psi_{\pm 1}$ map $L$ to the same set of points, is a question confronted in chapter 6.
5.3 Defining $\psi_{-1}^+$ and $\psi_{-1}^-$ with the Dendrites

Figure 5.3.3.2. Lamination giving a schematic illustration of the relation between the landing angles of fusion points up to fourth generation of the basilica and how their fusion point connection rays joins them. Lamination A is for the map $\psi_{-1}^+$ and B is for $\psi_{-1}^-$. 
6 Analyzing the shared matings

Chapter 6 is the final of the primary chapters and here a comparison of \( \psi^+_{-1}(L) \) and \( \psi^-_{-1}(L) \) is studied in pursuit of understanding why they are identical. The whereabouts of \( \psi^\pm_{-1}(L) \) in \( \hat{\mathbb{C}} \) is also illustrated.

Being the main problem of the thesis, explanations to why \( \psi^+_1(L) = \psi^-_1(L) \) will in this chapter be presented. In the first section an analytical approach is applied and in the second we will simply bend and stretch the basilica into the Fatou and Julia sets of \( R_3 \).

6.1 Complex conjugate matings

In order to avoid mix up with the the closure of a set, the notation \( \text{cct} (S) \) is used to denote the complex conjugate transformation of \( S \).

Observing figure 5.3.2.3 one might suspect that the two illustrations are each others complex conjugate, i.e. the complex conjugate transform applied on the set of points covered in the union of the basilica and the fusion point connection rays in illustration A of figure 5.3.2.3 equals the set of points covered in the union of the basilica and the fusion point connection rays in illustration B of figure 5.3.2.3. If this is continued true when all fusion point connection rays are present, then so must also the equality \( \text{cct} (\psi^+_1(L)) = \psi^-_1(L) \) be. This is the case but an even stronger relation between the two sets will shortly be proven. First a lemma about the Fatou and Julia sets of \( R_3 \) is needed.

**Lemma 6.1.1** Suppose \( z \in (\mathbb{C} \setminus J(R_3)) \) and \( w \in (J(R_3)) \). Then \( \text{cct}(z) \in (\mathbb{C} \setminus J(R_3)) \) and \( \text{cct}(w) \in (J(R_3)) \).

**Proof:** Define \( \Lambda = \bigcup_{n=0}^{\infty} R_3^{\circ(-n)}(1) \). The immediate preimages of \( z_0(R_3) = 1 \) are \( R_3^{\circ(-1)}(1) = \{-3, 1\} \). The immediate preimage of \(-3\) is \( R_3^{\circ(-1)}(-3) = -1 \). The immediate preimages of \(-1\) are \( R_3^{\circ(-1)}(-1) = \{-1 \pm i\sqrt{2}\} \). Suppose the points \( v \) and \( \text{cct}(v) \) both lie in \( \Lambda \) and denote

\[
\begin{align*}
R_3^{\circ(-1)}(v) &= -1 \pm \sqrt{1 + \frac{3}{v}} = \{v_1^+, v_1^-= \} \\
R_3^{\circ(-1)}(\text{cct}(v)) &= -1 \pm \sqrt{1 + \frac{4}{\text{cct}(v)}} = \{v_2^+, v_2^-\}. 
\end{align*}
\]

Note that with the principal branch as our branch cut choice we have \( v_1^+ = \text{cct}(v_2^-) \) and \( v_1^- = \text{cct}(v_2^+) \), which all belong to \( \Lambda \). This means that if \( R_3^{\circ(-n)}(v) \in \Lambda \) then so does \( \text{cct} \left( R_3^{\circ(-n)}(v) \right) \in \Lambda \). Choosing \( v = -1 + i\sqrt{2} \) we have \( \Lambda = \text{cct}(\Lambda) \). Since all the points of \( \Lambda \) are preimages to \( z_0(R_3) = 1 \), it is dense in \( J(R_3) \).

Suppose \( w \in J(R_3) \). Then there exists a sequence \( w_k \in \Lambda \subset J(R_3) \) such that \( w_k \to w \) as \( k \to \infty \). Because \( \Lambda = \text{cct}(\Lambda) \) there also exists a sequence \( \text{cct}(w_k) \in \Lambda \subset J(R_3) \). Due to continuity \( \text{cct}(w_k) \to \text{cct}(w) \) as \( k \to \infty \), hence
$cct(w) \in J(R_3)$. Since the Fatou set by definition is the complement of the Julia set and no point can belong to $J(R_3)$ (which is closed) without its complex conjugate also doing so, the statement of the lemma immediately follows for the Fatou set of $R_3$ as well. \qed

Immediately from (1) follows that
\[ R_3(cct(w)) = cct(R_3(w)) \]
and thus also
\[ R_3^{-1}(cct(w)) = cct(R_3^{-1}(w)). \]

**Lemma 6.1.2** The equality $\psi^+_1(L) = \psi^-_1(L)$ is true.

**Proof:** Denote the bubbles of the basilica as they are denoted in figure 3.1.1.

First we want to show that $\psi^+_1(B_2) = cct(\psi^{-}_1(B_2))$. The real interval $(-3,-1) \subset \psi^{\pm}_1(B_2)$. $\psi^{-}_1(B_2)$ is from the definition of Fatou component a connected set. Suppose $z \in \psi^{\pm}_1(B_2)$. Then there exists a curve $\gamma \subset \psi^{\pm}_1(B_2)$ such that it connects $z$ and the real valued point $-3 < x < -1$. Lemma 6.1.1 now says that $cct(\gamma) \in \left( \hat{\mathbb{C}} \setminus J(R_3) \right)$. Due to continuity requirement on $cct(\gamma)$ it lies completely in $\psi^{\pm}_1(B_2)$ and thereby so does $cct(z) \in \psi^{\pm}_1(B_2)$. Hence
\[ \psi^{\pm}_1(B_2) = cct(\psi^{\pm}_1(B_2)). \]

Next we want to show that $\psi^+_1(B_5) = cct(\psi^{-}_1(B_6))$.

Since
\[ R_3^{-1}(\psi^+_1(B_2)) = \psi^{\pm}_1(B_5) \cup \psi^{\pm}_1(B_6) \]
we have
\[ R_3(\psi^-_1(B_5)) = \psi^+_1(B_2), \]
thus
\[ cct(R_3(\psi^+_1(B_5))) = cct(\psi^+_1(B_2)) = \psi^-_1(B_2). \]
From (1) we have
\[ cct(R_3(\psi^+_1(B_5))) = R_3(cct(\psi^+_1(B_5))) = \psi^+_1(B_2) \]
hence
\[ R_3(cct(\psi^+_1(B_5))) = R_3(\psi^+_1(B_5)) = \psi^+_1(B_2). \]

For $\psi^+_1$ it was in chapter 5 derived that
\[ -1 - i\sqrt{2} \in \overline{\psi^+_1(B_5)} -1 + i\sqrt{2} \notin \overline{\psi^-_1(B_5)}. \]

To keep the continuity for $\psi^+_1$ it is conclude that
\[ \psi^+_1(B_5) \neq cct(\psi^+_1(B_5)). \]
and thus
\[ \text{cct}(\psi^+_1(B_5)) = \psi^+_1(B_6). \]

The same result is also obtained for \( \psi^-_1 \). It was here derived that
\[ -1 - i\sqrt{2} \notin \psi^-_1(B_5) \quad -1 + i\sqrt{2} \in \psi^-_1(B_5) \]
and thus
\[ \text{cct}(\psi^-_1(B_5)) = \psi^-_1(B_6). \]

Combining \( \psi^+_1(B_2) = \psi^-_1(B_2) \) with \( \psi^+_1(B_5) = \text{cct}(\psi^-_1(B_6)) \) it is concluded that
\[ \psi^+_1(B_5) = \psi^-_1(B_6) \]
\[ \psi^-_1(B_5) = \psi^-_1(B_5). \]

Since these two pair of sets all map under \( R_3 \) onto \( \psi^+_1(B_2) \) the following equalities for their preimages are valid
\[ R_3^{\alpha(-n)}(\psi^+_1(B_5)) = R_3^{\alpha(-n)}(\psi^-_1(B_6)) \]
\[ R_3^{\alpha(-n)}(\psi^-_1(B_6)) = R_3^{\alpha(-n)}(\psi^-_1(B_5)). \]

By the same reasoning so is also the equality
\[ R_3^{\alpha(n)}(\psi^+_1(B_2)) = R_3^{\alpha(n)}(\psi^-_1(B_2)). \]

Hence \( \psi^+_1(L) = \psi^-_1(L) \).

6.2 Identifying the basilica in \( R_3 \)

Along the chapters rules on how the basilica may be treated as it deforms into the Fatou and Julia sets of the rational map \( R_3 \) have been described. It is now time that these findings come together when we try to illustrate \( \psi^\pm_1(L) \) in \( \hat{\mathbb{C}} \).

Figure 6.2.1 with its marked points, bubbles and rays poses as a reference map of the basilica. The spine of the basilica is represented by a black filled line with forward direction to the right and the line
\[ \ell_{fm} = \{ z = it : -\infty < t < \infty, \; t \in \mathbb{R} \} \cap K_{f^{-1}} \]
by an orange filled line with forward direction upwards. The dotted lines propagate inside the bubble rays. They all start at the origin and they land at points which has an element from the set \{1/7, 2/7, 3/7, 4/7, 5/7, 6/7\} as landing angle for some external ray. The images under \( \psi^\pm_1 \) for all of these reference objects are illustrated in figure 6.2.2.
6.2 Identifying the basilica in $R_3$

Figure 6.2.1. The basilica with a selection of points, bubbles and rays marked for use in identification in figure 6.2.2. External and fusion point connection rays illustrated in this figure do not occupy the exact same set of points as the true external and fusion point connection rays of the basilica and $\psi_{-1}^\pm$.

Figure 6.2.2 below, originates from the illustration of the Fatou and Julia set of $R_3$ presented by Timorin ([Ti1], p. 18). In this illustration Timorin has used two different colors, blue and red. Both colors represent Fatou components. The remaining derivations of this section are best read parallel along examining figure 6.2.2.

Derived in section 5.2 was $\psi_{-1}^\pm(0) = \infty$. There is only one Fatou component in each of mating cases $A$ and $B$ in figure 6.2.2 that contains unbounded points. It is in both cases the surrounding blue area, sometimes denoted $A_{\infty}$, and here marked as $\psi_{-1}^\pm(B_0)$. Remember the result

$$
\begin{align*}
\psi_{-1}^\pm(z_1) &= -1 \\
\psi_{-1}^\pm(z_2) &= -1 \\
\psi_{-1}^\pm(z_0) &= -3 \\
\psi_{-1}^\pm(z_\alpha) &= 1
\end{align*}
$$

from section chapter 5. The Fatou set of $R_3$ is a stretching and bending of the basilica where bubble rays may not cross, the thus points $\psi_{-1}^\pm(z_0)$, $\psi_{-1}^\pm(z_1) = \psi_{-1}^\pm(z_2) = z_{cp}$ and $\psi_{-1}^\pm(z_\alpha)$ are identified in Timorins illustration. One could now naturally consider it a problem to stretch and bend the bubble $B_0 \subset K_{f,-1}$ onto $\psi_{-1}^\pm(B_0) = A_{\infty}$ and still place the rest of the bubbles in a bounded
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Figure 6.2.2. The Fatou set of $R_3$ labeled for identification of the basileicas image under $\psi_{-1}$. Label significance is found in figure 6.2.1.
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part of the complex plane. The answer to the dilemma is by remembering that $R_3$ operates on the Riemann sphere. $B_0$ is as the matter of fact stretched enough for it to cover the larger part of the Riemann sphere. The bubble rays $\{\psi^\pm_1(B_1), \psi^\pm_1(B_3)\}$ and $\{\psi^\pm_1(B_2), \psi^\pm_1(B_6)\}$ then, like a pair of arms, grasp around the Riemann sphere in order to intersect at the critical point $\psi^\pm_1(z_1) = \psi^\pm_1(z_2)$ (see orientation in figure 6.2.2). This also explains the surprising left/right orientation of $\psi^\pm_1(z_0)$ and $\psi^\pm_1(z_\alpha)$.

As proven in Lemma 5.3.2.2, a bubble ray $B$ of the basilica mapped under $\psi^\pm_1$ is itself a bubble ray. At the intersection points between the bubbles in $\psi^\pm_1(B)$ (provided they are preimages to $z_{cp}$ seen in figure 6.2.2), $\psi^\pm_1(B)$ will divert either to the left or to the right depending on the choice of sign. The diversion angle is always $\theta_{div} = \pi/4$. Interesting now is to follow a bubble ray in illustration A of figure 6.2.2, starting at either $\psi^+_1(z_0)$ or $\psi^-_1(z_\alpha)$, moving towards $z_{cp}$ and turning left in a right angle at every intersection with a new bubble. It might not seem the case at first glance, but progressing in this manner actually allows us to reach every single bubble of $R_3$. The exact same procedure is of course applicable in illustration B of figure 6.2.2, only the diversion is then done to the right.

Using this method it is not a particular cumbersome task identifying the remaining Fatou components of figure 6.2.1 under $\psi^\pm_1$. Although it at this stage has been thoroughly analyzed and even proven, particularly interesting is still to observe how the spine of the basilica and $\ell_{Im}$, when mapped under $\psi^+_1$, are so different from when mapped under $\psi^-_1$. And yet the equality $\psi^+_1(L) = \psi^-_1(L)$ hold.

The dotted lines in figure 6.2.2 now reveals the location of the points

$$\psi^\pm_1(z_{ER(\pm \gamma/7)}) = \psi^\pm_1(z_{ER(\pm \gamma/7)}) = \psi^\pm_1(z_{ER(\pm \gamma/7)}) = \psi^\pm_1(z_{\alpha (f_{\pm \gamma})}).$$

Due to insufficient resolution of figure 6.2.2, $\chi_\pm$ are printed as areas in which these points lie. Rounding up, one last instructive illustration, seen in figure 6.2.3, of how the basilica coils to become is given. It portrays the basilica in a variety of colors and how the bubbles painted in a specific color, under $\psi^\pm_1$, are mapped onto Fatou components on the Riemann sphere having the same color.
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**Figure 6.2.3.**
The basilica divided into six differently coloured subsets (illustration A) and how these subsets are mapped onto the Riemann sphere under $\psi_{(-1)}$ (plus sign for illustration B and minus sign for illustration C). Be aware that these illustrations are only given as for the reader to obtain a perception of the matings and may contain colouring inaccuracies on small scale bubbles.
7 References


8 Appendix

Proof that $R_a : z \to -a(1 + z^2)$ at its critical point $z_{cp} = -1$. The critical point of $R_a$ in focus here is $z_{cp} = -1$. With some algebra one can obtain the following

$$R_a(z) = \frac{a}{z^2 + 2z} = -\frac{a}{2} \left( \frac{1}{1 - (z + 1)} + \frac{1}{1 + (z + 1)} \right).$$

For all $|z + 1| < 1$, which includes $z_{cp}$, $R_a$ can be written as the series below

$$R_a(z) = -\frac{a}{2} \left( \sum_{n=0}^{\infty} (z + 1)^n + \sum_{n=0}^{\infty} (z + 1)^n(-1)^n \right) = -a \sum_{n=0}^{\infty} (z + 1)^{2n} = -a(1 + (z + 1)^2) + O((z + 1)^4)$$

which as the critical point is approached behaves like the second degree polynomial

$$P(z) = -a(1 + (z + 1)^2).$$

□

All polynomials have an attracting point at infinity. Consider

$$P_n(z) = a_0 + a_1z + \cdots + a_n z^n.$$

On $\hat{\mathbb{C}}$, $P_n \left( \frac{1}{z} \right)$ behaves in a neighborhood of $z = 0$ as $P_n(z)$ does around the infinity point. So we have

$$\lim_{z \to \infty} P_n(z) = \lim_{z \to 0} P_n \left( \frac{1}{z} \right) = a_0 + a_1 \frac{1}{z} + \cdots + a_n \left( \frac{1}{z} \right)^n$$

where the right hand side of the equation clearly is unbounded as $z \to 0$ if $n \geq 1$. Hence $z = \infty$ is a fixed point of $P_n(z)$. In the section below it will be proven that this point is also a critical point of $P_n(z)$. Final conclusion is that $\infty$ is a superattracting periodic orbit of period $p = 1$. □

Finding critical points of polynomial $P_n(z)$. Consider

$$P_n(z) = a_0 + a_1z + \cdots + a_n z^n.$$

The critical point $z_{cp}$ is defined by

$$\left. \frac{d(P_n(z))}{dz} \right|_{z = z_{cp}} = 0$$

so

$$P'(z_{cp}) = a_1 + 2a_2z + \cdots + na_n z^{n-1} = 0.$$
If \( n \geq 2 \) and \( a_1 = 0 \), one solution is \( z_{cp} = 0 \). But there exists one more. For a function \( g(z) \) which is holomorphic in a neighborhood of infinity consider the following

\[
\lim_{z \to 0} \left[ \frac{d}{dz} \left( \frac{1}{g(z)} \right) \right] = \lim_{z \to 0} \left[ -\frac{1}{(g(z))^2} \cdot \frac{d}{dz} \left( g \left( \frac{1}{z} \right) \right) \right] = 0 \Rightarrow
\]

\[
\lim_{z \to 0} \left[ \frac{d}{dz} \left( g \left( \frac{1}{z} \right) \right) \right] = 0. \quad (2)
\]

Since \( \frac{d}{dz} \left( g \left( \frac{1}{z} \right) \right) \) behaves the same way around origin as \( \frac{d}{dz} \left( g \left( \frac{z}{2} \right) \right) \) behaves around infinity on the Riemann sphere, \( g(z) \) will have a critical point at infinity if (2) is satisfied.

\[
\lim_{z \to 0} \left[ \frac{d}{dz} \left( \frac{1}{P_n \left( \frac{1}{z} \right)} \right) \right] = \left( (-1) \cdot a_1 z^{-2} + (-2) \cdot a_2 z^{-3} + \cdots + (-n) \cdot a_n z^{-n-1} \right) \frac{1}{(a_0 + a_1 z^{-1} + \cdots + a_n z^{-n})^2}
\]

\[
\lim_{z \to 0} \frac{a_1 z^{-2} + 2a_2 z^{-3} + \cdots + na_n z^{-n-1}}{a_0^2 + 2a_0a_1 z^{-1} + \cdots + a_n^2 z^{-2n}} = \frac{z^{2n}}{z^{2n}} = 0.
\]

Hence \( z_{cp} = \infty \) is also a critical point of \( P_n(z) \). \( \square \)

**Proof that** \( K_{f_{-i}} = \overline{K_{f_i}} \). The critical point \( z_{cp}(K_{f_i}) = z_{cp}(K_{f_{-i}}) = 0 \) in common for \( f_i \) and \( f_{-i} \) is a preimage to the repelling periodic orbits \( -1+i \xlongequal{-i} -i \) and \( -1-i \xlongequal{i} i \) respectively, so \( z_{cp} \in J_{f_i} \cap J_{f_{-i}} \). Note that

\[
\overline{z_{cp}} = z_{cp}
\]

\[
z^2 + i = \overline{z^2 - i}
\]

\[
\pm \sqrt{z^2 - i} = \pm \sqrt{z^2 + i}.
\]

Thus the set \( X_i \) of pre- and postimages of \( z_{cp} \) for \( f_i \) is a complex conjugate of the set \( X_{-i} \) of pre- and postimages of \( z_{cp} \) for \( f_{-i} \). \( X_i \) and \( X_{-i} \) are everywhere dense in \( J_{f_i} \) and \( J_{f_{-i}} \) respectively, hence \( J_{f_i} = J_{f_{-i}} \). \( \square \)
Proof of laminations \(B\) and \(D\) in figure 5.3.3.1. Consider a number \(0 < \theta_0 < 1\) (The case of \(\theta_0 = 0\) is trivial). Find two new numbers \(\theta_1\) and \(\theta_2\) which under doubling is equal to \(\theta_0 \mod \mathbb{Z}\). That is

\[ \theta_1 = \theta_0 \cdot \frac{1}{2} \]

\[ \theta_2 = (\theta_0 + 1) \cdot \frac{1}{2} = \theta_0 \cdot \frac{1}{2} + \frac{1}{2}. \]

Call \(\theta_1\) and \(\theta_2\) the preimages of \(\theta_0\). The preimages of \(\theta_1\) are

\[ \theta_{11} = \theta_1 \cdot \frac{1}{2} = \theta_0 \cdot \left(\frac{1}{2}\right)^2 \]

and

\[ \theta_{12} = \theta_1 \cdot \frac{1}{2} + \frac{1}{2} = \theta_0 \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} \]

and the preimages of \(\theta_2\) are

\[ \theta_{21} = \theta_2 \cdot \frac{1}{2} = \theta_0 \cdot \left(\frac{1}{2}\right)^2 \]

and

\[ \theta_{22} = \theta_2 \cdot \frac{1}{2} + \frac{1}{2} = \theta_0 \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} \]

Observe that

\[ \theta_{11} < \theta_1 < \theta_{21} < \theta_{12} < \theta_2 < \theta_{22}. \]

The preimages of \(\theta_{11}\) are

\[ \theta_{111} = \theta_{11} \cdot \left(\frac{1}{2}\right)^3 \]

and

\[ \theta_{112} = \theta_{11} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{2} \]

and the preimages of \(\theta_{22}\) are

\[ \theta_{221} = \theta_{22} \cdot \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 \]

and

\[ \theta_{222} = \theta_{22} \cdot \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 + \frac{1}{2} \]

Once again observe the magnitude order of the third generation preimages

\[ \theta_{111} < \theta_{11} < \theta_{221} < \theta_{112} < \theta_{22} < \theta_{222}. \]
Continuing picking out the smallest and the largest number of \((n-1)\) generation preimages in the preproceeded manner, the \(n\) generation preimages will be of the forms

\[
\begin{align*}
\theta_{1\ldots111}^n &= \theta_0 \cdot \left(\frac{1}{2}\right)^n \\
\theta_{1\ldots112}^n &= \theta_0 \cdot \left(\frac{1}{2}\right)^n + \frac{1}{2} \\
\theta_{2\ldots221}^n &= \theta_0 \cdot \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \cdots + \left(\frac{1}{2}\right)^2 \\
\theta_{2\ldots222}^n &= \theta_0 \cdot \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \cdots + \left(\frac{1}{2}\right)^2 + \frac{1}{2}.
\end{align*}
\]

That

\[
0 < \theta_{1\ldots111}^n < \theta_{1\ldots112}^n < \theta_{2\ldots221}^n < \theta_{2\ldots222}^n
\]

is trivial.

Remembering

\[
\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2 \rightarrow \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \rightarrow \sum_{n=2}^{N} \left(\frac{1}{2}\right)^n < \frac{1}{2} \ \forall N < \infty
\]

settles the magnitude

\[
\theta_{2\ldots221}^n < \theta_{1\ldots112}^n.
\]

Clearly

\[
\theta_{1\ldots112}^n < \theta_{1\ldots112}^{n-1} < \theta_{2\ldots222}^{n-1} < \theta_{2\ldots222}^n
\]

\[
\theta_{1\ldots111}^n < \theta_{1\ldots111}^{n-1} < \theta_{2\ldots221}^{n-1} < \theta_{2\ldots221}^n
\]

but what about

\[
\theta_{2\ldots221}^{n-1} < \theta_{2\ldots221}^n
\]

which if true immediately would lead to

\[
\theta_{2\ldots222}^{n-1} < \theta_{2\ldots222}^n.
\]
Consider

$$\theta^{2}_{n-1} < \theta^{2}_{n} \rightarrow \theta_{0} \cdot \left(\frac{1}{2}\right)^{n-1} < \theta_{0} \cdot \left(\frac{1}{2}\right)^{n} + \left(\frac{1}{2}\right)^{n} \rightarrow 2\theta_{0} < \theta_{0} + 1$$

which is true for $\theta_{0} < 1$.

Also note that

$$\lim_{n \to \infty} \theta^{1 \cdots 111}_{n} = 0^{+}$$

$$\lim_{n \to \infty} \theta^{2 \cdots 221}_{n} = \frac{1}{2}$$

$$\lim_{n \to \infty} \theta^{1 \cdots 112}_{n} = \frac{1}{2}$$

$$\lim_{n \to \infty} \theta^{2 \cdots 222}_{n} = 1^{-}.$$ 

Hence the following is valid for all $n$

$$0 < \theta^{1 \cdots 111}_{n} < \theta^{1 \cdots 111}_{n} < \theta^{2 \cdots 221}_{n} < \theta^{2 \cdots 221}_{n} < \frac{1}{2} < \theta^{1 \cdots 112}_{n} < \theta^{1 \cdots 112}_{n} < \theta^{2 \cdots 222}_{n} < \theta^{2 \cdots 222}_{n} < 1.$$ 

Setting $\theta_{0} = \frac{5}{6}$, as is the angle of the external ray landing on the right side of the critical value point of the Basilica, in the above given derivation then $\theta_{1} = \frac{5}{12}$, $\theta_{2} = \frac{11}{12}$ and the desired consistency with laminations B and D in figure 5.3.3.1 are acquired. □