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Abstract: This paper describes an approach for pre-service or in-service mathematics education that teaches sophisticated mathematics using only the tools of high school mathematics. The idea is to start with a standard problem from high school mathematics and let the solution to this problem serve as a platform for asking good mathematical questions and searching for deeper mathematical structure beyond the obvious "answer" to the question. Surprisingly, it turns out that high school mathematics is remarkably open to this sort of analysis, and the results are more interesting mathematically than one might initially think.

Response to Reviewers: We have tried to respond to all reviewers, along the lines that Anne Watson suggested (several original reviewers did not know our paper was in conjunction with Irene Bloom's).

Extended Analyses: Finding Deep Structure in Standard High School Mathematics

Dick Stanley and Manya Raman

Abstract

This paper describes an approach for pre-service or in-service mathematics education that teaches sophisticated mathematics using only the tools of high school mathematics. The idea is to start with a standard problem from high school mathematics and let the solution to this problem serve as a platform for asking good mathematical questions and searching for deeper mathematical structure beyond the obvious “answer” to the question. Surprisingly, it turns out that high school mathematics is remarkably open to this sort of analysis, and the results are more interesting mathematically than one might initially think.

1. Introduction

Recently in the U.S. there have been increasing calls for pre-service programs that both embody strong mathematical content and ground this content firmly in the actual mathematics that the teachers will be teaching. For example, Recommendation 1 in *The Mathematical Education of Teachers (2001)* states: "Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach."

While many educators seem to be in general agreement with this recommendation, there are large differences in what is meant by “deep understanding”. One view has been that deep understanding comes from exposing students to abstract mathematics above the level the prospective teacher would teach (Wu 1999). Unfortunately teachers rarely make the relevant connections between high school and university level mathematics (Cuoco, 2001). But what are the alternatives? Should one develop deeper theorems about standard topics like congruence and functions? (Usiskin, et al. 2003) Should one provide historical motivation? (Williamson, 1979) Should one look for abstract mathematics embedded in the high school mathematics? (Cuoco, 2001) All of these approaches have mathematical and educational merit. However, most of them draw in some way upon mathematics outside the confines of the high school curriculum, and this adds a cognitive burden to pre- and in- service teachers.

In this paper, we suggest an alternative approach that develops mathematics in a deep and meaningful way *wholly within* the territory of high school mathematics. This approach builds on a tradition that goes as far back as Isaac Newton (see Whiteside 1972, p. 129-159), and continues with explorations in geometry, algebra, and other high school topics (Brousseau 2006, Watson and Mason 2007). The idea behind this approach, which we call “extended analysis,” is to look beyond the initial features of a mathematics task and try to reveal a deeper mathematical structure underlying the task. Without being technically difficult, the approach builds important mathematical behaviors that are rarely addressed in typical high school curricula, such as generalizing a problem, coaxing algebraic representations into forms that reveal the structure of the problem situation, interpreting algebraic representations geometrically, and generating related problems.

We will illustrate these behaviors in a problem below. As typical with extended analysis, we start with a fairly standard word problem from the high school curriculum. Still, the reader should not assume that our approach falls into the category of the many existing approaches for helping high school students better understand and solve such problems. Rather, our extended analysis takes the initial problem *and its solution* as a starting point, and goes on to bring out deeper features of the mathematics of these problems, features that we have never seen discussed in this way elsewhere. This is precisely the type of work we feel teachers should be exposed to as part of their preparation for teaching courses in which these problems occur. It is work very much in the spirit of the *Taken as Shared* editorial in this volume.

The exploration of this particular problem goes through the following phases:

0. Using a question about the “solution” as a starting point for exploration
1. Looking at extreme cases in a qualitative analysis of the problem situation
2. Generalizing the problem and finding meaning in the algebraic structure of the solution
3. Using this structure to compare this problem with other tasks

For more tasks together with a discussion of possible extended analyses of them, see Usiskin et al., 2003, and Stanley and Callahan, 2002. The present paper focuses on the mathematical ideas contained in an extended analysis. The following paper, by Irene Bloom, discusses how an extended analysis approach could be used in a course for teacher education.

2. An example of extended analysis: A round trip with and against a wind

The task we have chosen to illustrate an extended analysis comes from high school algebra. The reader might best appreciate the analysis by following along (and forging ahead!) with pencil and paper.

Round trip with wind task:

An airplane makes a round trip where the one-way distance is 1,000 km. On the out-leg the plane faces a head wind of 50 km/hr, while on the return it is helped by a tail wind of the same speed. The speed of the plane in still air is 400 km/hr. What is the total time for the trip?

This task is often given in an algebra I¹ class as an application of solving linear equations.² It can be solved by finding the time out and the time back, each as a distance divided by a speed. The result is that the total trip takes about 5.079 hours.

This is where the typical high school task usually ends, but for us it is where the real analysis begins. As a first step we can ask if there is anything to arouse our curiosity about the answer

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¹ In the US, Algebra 1 is typically taught in 7th, 8th, or 9th grades (12, 13, or 14 year olds)

² Sometimes students are taught rote ways of solving stereotyped problems of this sort, and do not learn very much. As a result, tasks of this sort are sometimes rejected wholesale as being artificial. One purpose of this paper is to revive the mathematical respectability of such tasks. In doing so we will go on to bring out more mathematics in them than is typically done even when they are taught well.

itself. And indeed there is. An initial conjecture might well be that the effect of the wind on the way out should *cancel* its effect on the return. This would make the total time with wind equal to the round trip time with no wind: $2,000 \text{ km}/400 \text{ km/hr} = 5 \text{ hr}$. But the answer is greater than 5 hours.

Why is that so? This simple question can lead to various explanations, and pursuing these gives much more mathematical insight than does a mere computation of the answer. (We invite you to try to come up with some such explanations before reading on.)

A. Qualitative approaches

One approach is to analyze the situation qualitatively, using an extreme case. Suppose the wind speed is very large. The net speed of the plane on the out-leg would then be very small, so the time on the out-leg would be large, asymptotically approaching infinity as the wind speed approaches the plane's speed. We can represent this behavior graphically.

This graph gives the time on the out-leg only; the total time will be even longer. For example, if the wind speed is 200 km/hr, the net speed of the plane is only 200 km/hr, so that the time for just the out-leg is 5 hours, which is the time required to go out *and* back with no wind. Since the wind factor does not cancel out in this case, we would be very surprised if it did for slower wind speeds.

Another approach is to represent distance as the area under a speed vs. time graph. A comparison of the areas under the graphs in the with-wind and no-wind situations can show, with no algebra, that the total time will be larger. (We invite the reader to carry out such an analysis.)

It may come as a surprise to some teachers that a straightforward algebraic task can be elucidated with the help of this sort of qualitative or geometrical analysis. This idea of stepping back from a problem and viewing it in a non-algebraic way is one of the key features of many extended analyses. Still, developing a sophisticated algebraic approach, one that goes beyond the role of algebra in finding a initial numerical answer, is one of the most powerful features of an extended analysis. We illustrate such a development in the next section.

B: Algebraic/Structural approaches

Sophisticated use of algebra can show that quite ordinary problems of the high school curriculum, such as this one, have what can be called a mathematical "deep structure". This structure may not be apparent to students, even those who solve the problems correctly, and often may not be apparent even to the teachers who assign these problems. This section will present in some detail some of the algebraic features of an extended analysis of the round trip problem as a way of getting at this mathematical deep structure.

The initial round-trip problem is given with specific numerical input, but if we assign variables, we can express the relationships between time, distance, and speed algebraically in a more meaningful way, a way that will reveal some of the deep structure.

Let:

v_p = speed of plane (in still air)	(v_p = 400 km/hr in the problem statement)
v_w = speed of wind	(v_w = 50 km/hr in the problem statement)
d = one-way distance	(d = 1,000 km in the problem statement)
t = total time for the trip	(t is the unknown to be found in the problem)

Using exactly the steps we would use to solve this problem numerically for the total time (5.079 hours), we arrive at the following expression as a *general* solution for the total time t :

$$(1) \quad t = \frac{d}{v_p - v_w} + \frac{d}{v_p + v_w}$$

This representation is general, but not (yet) particularly revealing. In the extended analysis below we will algebraically manipulate this representation for the purpose of expressing the relationship in a way that better illuminates the situation.³

To begin with, since formula (1) is a sum, we haven't yet algebraically combined the times for the two legs. Finding a common denominator does this:

$$(2) \quad t = 2d \frac{v_p}{v_p^2 - v_w^2}$$

Recall that we had earlier asked why the total trip time with wind was *not* the same as the time for a round trip with *no* wind. We now observe that the time for a round trip with *no* wind is $2d/v_p$, and that this expression *almost* occurs in (2). If we multiply both numerator and denominator by v_p , it *will* occur:

$$(3) \quad t = \frac{2d}{v_p} \cdot \frac{v_p^2}{v_p^2 - v_w^2}$$

This form has more symbols than the previous step, but its structure is more revealing. Moreover, we can coalesce the two occurrences of v_p^2 . If at the same time we use the symbol t_0 for the factor $2d/v_p$ on the left (the time for a round trip with no wind), we get, finally:

$$(4) \quad t = t_0 \cdot \frac{1}{1 - \left(\frac{v_w}{v_p}\right)^2}$$

This form of the formula for the round trip time is particularly useful. Note that it does not involve the ultimately irrelevant parameter for the distance d . Moreover, it shows that the time does not depend on the individual speeds of the plane and the wind, but only on their *ratio*.

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³ In a typical high school algebra class, students are asked to condense or expand expressions like this one, but they never see examples such as this one that show why it is useful to do so.

In fact, the "structure" of formula (4) gives us a lot of information. It shows that the time for the round trip *with* the wind is the time t_0 with *no* wind multiplied by a *dimensionless factor*. (The factor to the right of t_0 is dimensionless since v_p and v_w occur in a ratio with the speed dimension canceling out.) Moreover, it is clear structurally that the dimensionless factor is greater than 1. (Why?) Thus it is also clear why $t > t_0$. So we not only see *that* the round trip time in a wind situation takes longer, but we also see *why* it does.

The usefulness of the expression in (4) is not limited to this particular problem. In fact, it comes up in important applications outside of school mathematics. One is special relativity (for example, see Lorentz Transformations at <http://hyperphysics.phy-astr.gsu.edu/hbase/relativ/ltrans.html>).

A typical extended analysis involves "coaxing" algebraic representations into useful forms in just the way we have illustrated here in progressing from (1) to (4). You don't always know where you are going at the outset, but with some experience you learn to take advantage of features (such as ratios and dimensionless factors) to help reveal the deeper mathematical structure behind a seemingly simple task.

3. Some related and unrelated motion problems

If, in the way we have illustrated, you work at bringing out deep mathematical structure in a variety of other high school problems, you are sure to notice two surprising things. One is that problems that seem on the surface to be quite *different* may actually have *similar* underlying mathematical structure. Irene Bloom discusses these as "isomorphic" tasks in the following article in this volume. Another example appears in Stanley and Walukiewicz, 2004, p 252-253.

The other surprise is that two problems that seem on the surface to be quite *similar*, might have very different "deep structures". Take for instance this task from Bloom's article: "Suppose that on a round trip you go at 30 mph on the way out and at 60 mph on the way back. What is your average speed?" This question and the one we have discussed above both involve round trips with a fixed distance where the speed going out and coming back are not the same. However, an extended analysis of one of these problems is rather different in character from an extended analysis of the other.

The reason is that in Bloom's problem, extending and generalizing the initial answer brings one face to face with the mathematics of the harmonic mean of the two given vehicle speeds, and with how it is different from the arithmetic mean. But in our problem, the speed of the wind plays a role that has no parallel in the other problem, and the harmonic and arithmetic means do not figure at all.⁴ In short, these two round trip problems lead in a natural way to rather different mathematical deep structures.

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⁴ The reader is invited to verify this by expressing the average speed of the plane in its round trip in terms of its speed in still air and of the wind speed. This average has nothing to do with harmonic or arithmetic means. Rather, it is the plane's speed multiplied by the very same dimensionless factor that is the denominator of (4) above.

Another round trip motion problem that brings out an important feature of extended analyses was given as a “puzzler” on National Public Radio’s Car Talk:

You start at a dock and row upstream for a mile, at which point your hat blows off into the river. You decide not to worry, and keep rowing. But 10 minutes later you realize your concert tickets are in the hatband. You turn around quickly, and row downstream. You get to the hat just as it is passing the dock. Question: What is the speed of the current?

This problem involves precisely the same situation as the round-trip problem that we have been discussing: a vehicle making a round trip both with and against a current. In spite of the whimsical formulation, this puzzle problem is mathematically interesting because of this fact: *The problem has a simple solution that can be arrived at through common sense, with **no algebra**.* In this respect it is similar to the solutions discussed in section 2A above. (We encourage the curious reader to try to find a no-algebra solution before reading further.)

Here is such a solution. Suppose you look at the problem from the point of view of the hat. You have just been dropped into the river, and are floating with the current. You see the boat travel away from you up the river for 10 minutes, and turn around. Then you see the boat come back down the river at the same speed and catch up to you. This must also take 10 minutes. (Why?) Since you’ve drifted the 1 mile back to the dock in that 20 minute time period, the speed of the current is 1 mile divided by 20 minutes, or 3 mph. Finding this solution has resulted from simply being able to picture the relevant features of the situation, and has shown that no algebra is necessary for a solution.⁵

It is important to say why this puzzle problem is important for teachers. Often there will be a few students in a class who come up with this sort of non-algebraic solution on their own to problems posed for them that are meant to illustrate algebraic techniques. A teacher needs to be able to understand what such a student is saying, and be prepared to give appropriate validation for the underlying insight the student had. This is a healthy opposite to what is unfortunately a more typical situation: a student uses a learned mathematical procedure to come up with a correct numerical solution to a problem, but has no idea of the meaning of the problem situation.

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⁵ It is also instructive to attempt an algebraic solution to this problem. You will be able to find the current speed (though perhaps not easily). But you will **not** be able to find the speed of the boat, since it turns out that the data is compatible with **any** boat speed. Hence you cannot find the total round trip time or distance. So if your algebraic method focused on finding any of these three quantities first, you may conclude (wrongly) that the problem has no solution! This elusive feature is characteristic of many puzzle problems, and indeed is the sort of thing that makes many puzzle problems different from school problems.

4. Discussion

We have followed a task from its initial formulation as a typical textbook exercise into what we hope the reader will agree is fairly interesting and sophisticated mathematics. In particular we began by raising a question about the solution, and used a fairly simple graphical analysis to understand why the answer to this question is not simply 5 hours. From there we generalized the task, and took care to express the solution algebraically in a way that revealed something about the problem situation. This allowed us to appreciate that not all round-trip tasks are similar. Finally in the Car Talk puzzler we saw that relying on algebra does not always lead to a result, and that common sense about the problem situation may be required instead.

We have illustrated above what *doing mathematics* can look like at the high school level. We have not used any sophisticated mathematical tools in the analysis, nor any reasoning that is beyond the high school level. However, we have done a few things that are typical of real research in mathematics and need sufficient space in pre-service or in-service teacher education curricula if teachers are to know how to generate further questions and reasoning than textbooks provide. These are:

1. Rather than stopping with the answer, using it is the starting point of a deeper analysis
2. Using algebra and graphing as a tool for reasoning and not as an end goal
3. Searching for a general formulation of the solution that illuminates the problem situation

In summary, we feel, above all, that it is important that teachers be aware of the sort of mathematical richness that exists when one simply pushes traditional tasks a little further than usual. We have tried to illuminate some of this richness here. The task of getting teachers to seek out these riches on their own is the topic of the following paper by Bloom.

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Format: Mathematics Teaching around the World

Extended Analyses: There is more to high school math than you might think!

Abstract

This paper describes an approach for pre-service or in-service mathematics education that teaches fairly sophisticated mathematics using only the tools of high school mathematics. The idea is to start with a fairly standard problem from high school mathematics, and let the solution to this problem serve as a platform for asking good mathematical questions and searching for deeper mathematical structure beyond the obvious "answer" to the problem. Surprisingly, it turns out that high school mathematics is remarkably open to this sort of analysis, and the results are more interesting than one might expect.

Key Words: Extended analyses, open-ended problem solving, high school mathematics, habits of mind, curiosity

1. Introduction

Recently in the U.S. there have been increasing calls for pre-service programs that both embody strong mathematical content and ground this content firmly in the actual mathematics that the teachers will be teaching. For example, Recommendation 1 in *The Mathematical Education of Teachers (2001)* states: "Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach."

While many educators seem to be in general agreement with this recommendation, it seems that there are large differences in what is meant by "deep understanding". One view has been that deep understanding comes from exposing students to abstract mathematics above the level the prospective teacher would teach (Wu 1999). Unfortunately teachers rarely make the relevant connections between high school and university level mathematics (Cuoco, 2001). But what are the alternatives? Should one develop deeper theorems about standard topics like congruence and functions? (Usiskin, et al. 2003) Should one provide historical motivation? (Williamson, 1979) Should one look for abstract mathematics embedded in the high school mathematics? (Cuoco, 2001) All of these approaches have mathematical and educational merit. However, most of them draw in some way upon mathematics outside the confines of the high school curriculum.

In this paper, we suggest an alternative approach that develops mathematics in a deep and meaningful way wholly within the territory of high school mathematics. Without being technically difficult, our approach takes seemingly straightforward tasks and turns them into an arena for doing sophisticated mathematics that often leads to deep and surprising insights. This approach, called "extended analysis", helps teachers (and, ultimately, their students) develop good mathematical habits of mind. It also reveals high school mathematics to be a much more interesting subject than we might have thought.

In the next three sections we illustrate some of the features of an extended analysis using, as a basis, a typical "round trip" problem from high school algebra.

2. An example of extended analysis: A round trip with and against a wind

Among the most important habits of mind that we hope to develop with extended analyses is a sense of curiosity about mathematical situations. In that spirit we invite the curious reader to follow along with paper and pencil—exploring the tasks on your own. You may very well discover a direction we did not address. If so, you are really getting the experience of an extended analysis.

The round trip task is as follows:

Round trip with wind task:

An airplane makes a round trip where the one-way distance is 1,000 km. On the out-leg the plane faces a head wind of 50 km/hr, while on the return it is helped by a tail wind of the same speed. The speed of the plane in still air is 400 km/hr. What is the total time for the trip?

This task is often given in an algebra I class as an application of solving linear equations.ⁱ It can be solved by finding the time out and the time back, each as a distance divided by a speed. The result is that the total trip takes about 5.079 hours.

This is where the typical high school task usually ends, but for us it is where the real analysis begins. As a first step we can ask if there is anything to arouse our curiosity about the answer itself. And indeed there is. An initial analysis might well argue that the effect of the wind on the way out should *cancel* its effect on the return. This would make the total time with wind equal to the round trip time with no wind: $2,000 \text{ km} / 400 \text{ km/hr} = 5 \text{ hr}$. But the answer is greater than 5 hours.

Why is that so? This simple question can give rise to several explanations that give much more mathematical insight than a mere computation of the answer provides. (We invite the curious reader to try to come up with some such explanations before reading on.)

A. Qualitative approaches

One approach is to analyze the situation qualitatively, using an extreme case. Suppose the wind speed were very large. The net speed of the plane on the out-leg would then be very small, so the time on the out-leg would be large, asymptotically approaching infinity as the wind speed approaches the plane's speed. We can represent this behavior graphically.

This graph gives the time on the out-leg only; the total time will be even longer. For example, if the wind speed is 200 km/hr, the net speed of the plane is only 200 km/hr, so that the time for just the out-leg is 5 hours, which is the time required to go out *and* back with no wind. Since the wind factor does not cancel out in this case, we would be very surprised if it did for slower wind speeds.

Another approach is to represent distance as the area under a speed vs. time graph. A comparison of the areas under the graphs in the with-wind and no-wind situations can show, with no algebra, that the total time will be larger. (We invite the reader to carry out this analysis.)

It may come as a surprise to some teachers that a straightforward algebraic task can be elucidated with the help of this sort of qualitative or geometrical analysis. This idea of stepping back from a problem and viewing it in a non-algebraic way is one of the key features of many extended analyses. Still, developing a sophisticated algebraic approach, one that goes beyond the role of algebra in finding a numerical answer, is one of the most powerful features of an extended analysis. We explore such an approach in the next section.

B: *Algebraic/Structural approaches*

A sophisticated use of algebra can show that quite ordinary problems of the high school curriculum have what can be called a mathematical "deep structure". This structure may not be apparent to students, even those who solve the problems correctly, and often may not be apparent even to the teachers who assign these problems. This section will present in some detail some of the algebraic features of an extended analysis of the round trip problem as a way of getting at this mathematical deep structure.

The round-trip problem is given with specific numbers, but if we assign variables, we can express the relationships between time, distance, and speed algebraically in a more meaningful way, a way that will reveal some of the deep structure.

Let:

v_p = speed of plane (in still air)	(v_p = 400 km/hr in the problem statement)
v_w = speed of wind	(v_w = 50 km/hr in the problem statement)
d = one-way distance	(d = 1,000 km in the problem statement)
t = total time for the trip	(t is the unknown to be found in the problem)

Using exactly the steps we would use to solve this problem numerically for the total time, we arrive at the following expression as the *general* solution for the total time t :

$$(1) \quad t = \frac{d}{v_p - v_w} + \frac{d}{v_p + v_w}$$

In the extended analysis below, we will algebraically manipulate this representation for a particular purpose, that of trying to express the relationship in a way that illuminates the situation.ⁱⁱ

To begin with, since formula (1) is a sum, we haven't yet algebraically combined the times for the two legs. Finding a common denominator does this:

$$(2) \quad t = 2d \frac{v_p}{v_p^2 - v_w^2}$$

Note that the time for a round trip with *no* wind is $2d/v_p$, and that this expression *almost* occurs in (2). If we multiply both numerator and denominator by v_p , it *will* occur:

$$(3) \quad t = \frac{2d}{v_p} \cdot \frac{v_p^2}{v_p^2 - v_w^2}$$

This form has more symbols than the previous step, but note that we can coalesce the two occurrences of v_p^2 . If we do this, and if at the same time we use the symbol t_0 , for the factor $2d/v_p$ on the left (the time for a round trip with no wind), we get, finally:

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This form of the formula for the round trip time is particularly useful. Note that it does not involve the ultimately irrelevant parameter for the distance d . Moreover, it shows that the time does not depend on the individual speeds of the plane and the wind, but only on their *ratio*.

In fact, the whole "structure" of formula (4) is transparent. It shows that the time for the round trip *with* the wind is the time t_0 with no wind multiplied by a *dimensionless factor*. (The factor to the right of t_0 is dimensionless since v_p and v_w occur in a ratio with the speed dimension canceling out.) Moreover, it is clear structurally that the dimensionless factor is greater than 1. (Why?) Thus it is also clear why $t > t_0$. So we not only see *that* the round trip time in a wind situation takes longer, but we also see *why* it does.

The usefulness of the expression in (4) can be appreciated by seeing how well it communicates information in the context of this problem. Its usefulness can be affirmed when we see that it comes up in important applications outside of school mathematics. One is special relativity. For example, look under Lorentz Transformations at <http://hyperphysics.phy-astr.gsu.edu/hbase/relativ/ltrans.html>.

A typical extended analysis involves "coaxing" algebraic representations into useful forms in just the way we have illustrated here. One doesn't always know where one is going at the outset, but with some experience one learns to take advantage of features such as ratios and dimensionless factors to help see the deep mathematical structure behind a seemingly simple task.

3. Some related and unrelated motion problems

If you work at bringing out deep mathematical structure in a variety of other high school problems in the way we have just illustrated, you are sure to notice two surprising things. One is that problems that seem on the surface to be quite *different* may actually have *similar* underlying mathematical structure. Irene Bloom discusses these so-called "isomorphic" tasks in an article in this volume. Another example appears in Stanley and Walukiewicz, 2004, p 252-253.

The other surprise is that two problems that seem on the surface to be quite *similar* might have very different "deep structures". Take for instance this task from Bloom's article: "Suppose that on a round trip you go at 30 mph on the way out and at 60 mph on the way back. What is your average speed?" This question and the one we have discussed above both involve round trips with a fixed distance where the speed differs going out and coming back. However, an extended analysis of one of these problems will likely be rather different in character from an extended analysis of the other.

The reason is that in Bloom's problem, extending and generalizing the initial answer brings one face to face with the mathematics of harmonic and arithmetic means of the two given vehicle speeds. But in our problem, the speed of the wind plays a role that has no parallel in the other

problem, and these two means do not figure at all.ⁱⁱⁱ In short, these two round trip problems lead in a natural way to different mathematical deep structures.

Another round trip motion problem that brings out an important feature of extended analyses was given as a “puzzler” on National Public Radio’s Car Talk:

You start at a dock and row upstream for a mile, at which point your hat blows off into the river. You decide not to worry, and keep rowing. But 10 minutes later you realize your concert tickets are in the hatband. You turn around quickly, and row downstream. You get to the hat just as it is passing the dock. Question: What is the speed of the current?

This problem involves precisely the same situation as the round-trip problem that we have been discussing: a vehicle making a round trip both with and against a current. In spite of the whimsical formulation, this puzzle problem is mathematically interesting because of this fact: *The problem has a simple solution that can be arrived at through common sense, with **no algebra**.* In this respect it is similar to the solutions discussed in section 2A above. (We encourage the curious reader to try to find a no-algebra solution before reading further. It may not be easy!)

Here is such a solution. Suppose you look at the problem from the point of view of the hat. You have just been dropped into the river, and are floating with the current. You see the boat travel away from you up the river for 10 minutes, and turn around. Then you see the boat come back down the river at the same speed and catch up to you. This also takes 10 minutes. (Why?) Since you've drifted the 1 mile back to the dock in that 20 minute time period, the current speed is 1 mile divided by 20 minutes, or 3 mph. Simply being able to picture the relevant features of the situation has meant that only arithmetic, and no algebra, is necessary for a solution.^{iv}

It is important to say why this puzzle problem is important for teachers. Often there will be a few students in a class who come up with this sort of non-algebraic solution on their own to problems posed for them. A teacher needs to be able to understand what such a student is saying, and be prepared to give appropriate validation for the underlying insight the student had. After all, this is a healthy opposite to what is unfortunately a more typical situation: a student uses a mathematical procedure to come up with a correct numerical solution to a problem, but then has no idea of the meaning of the problem situation.

4. Characteristic features of an extended analysis

The extended analysis we have presented in this paper illustrates good mathematical behaviors that can be brought into play in the analysis of other problems. Here are some of these behaviors:

* Qualitative solutions

Giving an initial “qualitative” analysis of a problem, often by looking at extreme cases, can serve as a guide for a more detailed quantitative analysis,

* Generalization of numerical problems

A general solution found by replacing numbers with parameters shows structurally how different quantities are related to each other, and illustrates the particular numerical problem in the context of a more general solution. It is an important way to get at the mathematical deep structure.

* Units and dimensional analysis

Paying attention to units through dimensional analysis and appreciating the role of dimensionless factors can help give insight into the structure of the relationships being derived,

The round trip tasks discussed here highlight only some of the possible features of an extended analysis approach. Other problems might illustrate other features. Briefly stated, here are a few:

* Changing the conditions of a problem and seeing how this affects the solution.

* Providing both algebraic and geometric analyses of the same problem

* Looking for different interpretations of the solution of a problem.

* Being on the lookout for interesting variants and related problems.

For a collection of more extended analysis tasks and some of the ways they can be analyzed, see Usiskin et al 2003, and Stanley and Callahan, 2002.

5. A few words about implementation

The approach in this chapter is based on having teachers work with actual high school problems. However, the mathematical depth and generality at which they analyze these problems goes beyond what would be expected of their students in a typical high school classroom. This is an essential feature of the approach. It is based on the belief that teacher professional development should be a place where teachers can explore mathematics, take risks, and learn to make connections without the burden of feeling they need to replicate this precise experience directly in their own classrooms.

Extended analyses are less about particular answers to problems and more about developing tools, both mathematical and affective, to find interesting questions and search for paths through sometimes uncharted domains. This is not an ability that can be developed overnight. A professional development program based on extended analysis must have a number of features, such as scaffolded questions and non-graded sharing sessions, to help teachers move beyond their role as ship captain towards a role as a real mathematical discoverer. We must take into account the risks for teachers involved in such a shift. It is often difficult enough to solve a mathematical task when a simple answer is all that is asked. It is another thing to learn to pull structure and meaning out of tasks without any guidance except one's own mathematical curiosity.

Still, this is both the danger and the reward of doing the real mathematical inquiry required here. And we argue that such inquiry should be a part of every high school mathematics teacher's mathematical education. Few people will disagree with that. But what we have hoped to demonstrate here is that a significant depth of mathematical understanding is possible without going outside the domain of high school mathematics. After one has played around with enough problems in the spirit of an extended analysis, one reaches the perhaps surprising conclusion that high school mathematics is not as boring as one might think!

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ⁱ Sometimes students are taught rote ways of solving stereotyped problems of this sort, and do not learn very much. As a result, tasks of this sort are sometimes rejected wholesale as being artificial. We will attempt below to revive the mathematical respectability of such tasks, and will go on to bring out more mathematics in them than is typically done even when they are taught well.

ⁱⁱ Note that this is doing algebra for a purpose, not just simplifying or expanding expressions as an empty exercise as is often seen in traditional algebra classes.

ⁱⁱⁱ The reader is invited to verify this by expressing the average speed of the plane in its round trip in terms of its speed in still air and of the wind speed. It is the plane's speed multiplied by the very same dimensionless factor that is the denominator of (4) above.

^{iv} It is also instructive to attempt an algebraic solution to this problem. You will be able to find the current speed (though perhaps not easily). But you will **not** be able to find the speed of the boat, since the data is compatible with **any** boat speed. Hence you cannot find the total round trip time or distance. So if your algebraic method focused on finding any of these three quantities first, you may conclude (wrongly) that the problem has no solution!