Master of Science Thesis in Physics

Cyclotron Damping in Magnetized Plasmas

ROLANDO AYLLON SALINAS

Supervisor: Gert Brodin, Department of Physics, Umeå University.
Examiner: Jens Zamanian, Department of Physics, Umeå University.

Department of Physics
Umeå University
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Abstract

The aim of this thesis was to study the cyclotron damping in magnetized plasmas using a different approach to the dielectric tensor that is the stardard way to study this case. In this approach given we deduce a set of coupled differential equations that give us the evolution of the electric field and the distribution function. The system of coupled equations can not be solved analitically, that is why we have found numerical solutions. The algorithm we used to obtain the numerical solutions is the staggered leap-frog method that common used in problems involving electromagnetic fields.

We have studied two cases where we consider two different initial conditions for our distribution function in the velocity space. In the first example we used $\tilde{g}(t = 0, v_n) = 0$. In this case we found that the electric field decays exponentially and there is phase mixing in the evolution of the distribution function. As second example we used as initial condition the expression $\tilde{g}(t = 0, v_n) = E_n/(iv_n - \gamma)$. In this case the phase mixing is less pronounced compared to the first example, and the electric field start growing until the oscillations of the distribution function start to become important, then the electric field start to decay slowly.

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Contents

Chapter 1

Introduction

With the progress made in astrophysics and theoretical physics during the last century, it was realized that most of the matter in the known universe exist in a plasma state, almost 99% of it. On the earth's surface the presence of plasmas occuring naturally are limited to a few examples: The flash of a lightning bolt, the glow of the Aurora Borealis, the conducting gas inside a neon sign. It would seem that we live in the 1% where the plasma state does not appear naturally. Even though there are few examples of plasmas in nature, it is possible to create plasmas in the laboratory, and the study of this has lead to important applications in different fields of research, industry and technology [1],[2],[3]. For such reasons the study of plasmas has grown exponentially in the last fifty years.

One of the plasma phenomena widely studied is Landau damping. In 1946 Lev Davidovich Landau was studying the propagation of waves in warm plasmas and he found in his calculations an unexpected result. He noticed that there was a decay of the amplitude of the waves; this was a surprise due to the fact that there was no external factor producing this effect.

In the original paper [4] Landau exposed this result in a completely mathematical framework without any physical explanation for this behaviour. For this reason his result was not believed for almost twenty years. The first evidence of damped waves in collisionless plasmas was in a experiment performed by J.H. Malmberg and C.B. Wharton [5].

After its experimental confirmation, the Landau problem has become of fundamental interest in plasma physics and in other research areas were this phenomenon can be encountered. Due to the mathematical complexities in Landau's calculation, many people has been and still trying to develop new approaches to make this subject easier to understand. In 1961 J.M Dawson [6], made a proposal where the mathematics behind the problem is simpler compared to Landau's approach. Additionally it provides a more deep understanding of the physics involved in the phenomenon. The inconvenience with this approach is that is long and tedious.

These two ways to deal with the Landau problem are present in most books in plasma physics. With this idea in mind in 1995 G. Brodin made a new proposal to handle the Landau problem. This new approach gives a set of differential equations that in some special cases can be solved analytically, but in the majority of times must be solved numerically. This approach present several advantages compared to the traditional ways [7].

All the approaches mentioned before consider the case were the plasma is not embedded in an external magnetic field. In the case we study electromagnetic waves propagating along magnetic field we have what is called the cyclotron damping problem. The general way to deal with this problem is related to the calculation of the dielectric tensor. In this work we will apply the approach presented by Brodin to analyse the cyclotron damping. Furthermore, we will solve numerically the set of equations obtained with this approach for different initial conditions.

The structure of the thesis is as follows. Chapter 2 deals with the theoretical background and introduces all the terminology needed to understand the Landau problem. Chapter 3 details the different approaches of the Landau damping, to compare the different subtleties in the theories; and finally Chapter 4 is devoted to the study of the Landau damping in magnetized plasmas.

Chapter 2

Review of Basic Concepts in Plasma Physics

The essential part of the work we have done in this thesis was the study of wave-particle interaction in a magnetized plasma. For that reason it is important to set some basic notions in plasma physics. The aim of this chapter is to introduce these basic concepts in order to have a clear picture of the plasma state and the physics presented in this work.

2.1 Definition of Plasma

The word plasma was introduced by the physiologist Jon Evangelista Purkinje to denote the clear substance remaining after the removal of all the corpuscular material of the blood. In 1922 the physicist Irving Langmuir proposed that the electron, neutrons and ions inside an ionized gas could be considered as corpuscular material inside some fluid medium, and in a similar way called this medium plasma.

The plasma is an ionized gas. It can be generated by heating the gas enough such that the atoms colliding with each other knock the electrons off, which means that we have different species in the gas; ions and electrons. But not all the ionized gases are plasmas. A useful definition is found in [8], which states: "*A plasma is a quasineutral gas of charged and neutral particles which exhibits collective behavior*". The collective behavior in plasmas is the result of the long range of the electromagnetic forces, since each charge particle interact simultaneously with a considerable number of other charged particles.

Furthermore there are three parameters that characterize a plasma:

1. The particle density n.
2. The temperature of each species.
3. The strenght of the magnetic field B

2.1.1 Thermal Distribution of Velocities in a Plasma

A plasma in thermal equilibrium has particles moving with all velocities, and the Maxwellian distribution is the most probable distribution for these velocities.

$$f(v) = Ae^{-\frac{mv^2}{2k_BT}}; \tag{2.1}$$

where k_B is the Boltzmann's constant $k_B = 1.38 \times 10^{-23} JK^{-1}$, T is the absolute temperature in Kelvin and A is the normalization constant defined in this case as

$$A = n \left(\frac{m}{2\pi k_B T} \right)^{3/2}.$$

From equation (2.1) we can relate the number density n, as the three dimensional integral of $f(v)$ over all the velocities $\mathbf{v}$, i.e.,

$$n(\mathbf{x}, t) = \iiint f(v) d\mathbf{v}. \tag{2.2}$$

It will become useful to define the thermal velocity v_{th}, which is a measure of the width of the peak of our distribution function. This velocity is defined as

$$v_{th} = \sqrt{\frac{2k_B T}{m}}. \tag{2.3}$$

From (2.3) we observe that the temperature is related to the kinetic energy of the particles. In order to have a better picture of the meaning of temperature, we compute the average kinetic energy in this distribution.

Without losing generality we will use (2.1) in one dimension, which is given

$$f(u) = A_1 e^{-\frac{mu^2}{2k_B T}} = A_1 e^{-\frac{u^2}{v_{th}^2}}, \tag{2.4}$$

with

$$A_1 = n \left(\frac{m}{2\pi k_B T} \right)^{1/2} = \frac{n}{\sqrt{\pi} v_{th}}.$$

The average kinetic energy is

$$< E >= \frac{\int_{-\infty}^{\infty} \frac{mu^2}{2} f(u) du}{\int_{-\infty}^{\infty} f(u) du}, \tag{2.5}$$

replacing (2.4) and (2.2) into (2.5), then performing the integral, we get the result

$$< E >= \frac{1}{2} k_B T. \tag{2.6}$$

It is not difficult to extend this result to the three dimensions [8], the result is

$$< E >= \frac{3}{2} k_B T. \tag{2.7}$$

So we can conclude that $< E >= (1/2) k_B T$ per degree of freedom. Since the average kinetic energy is related to the temperature, it is customary in plasma physics to represent the temperatures in energy. For definition we use $k_B T = 1\text{eV}$, knowing that $1\text{eV} = 1.6 \times 10^{-19}\text{J}$ solving for the temperature we obtain $T = 11600\text{K}$.

Thus the conversion factor is

$$1\text{eV} \leftrightarrow 11600\text{K}. \tag{2.8}$$

2.1.2 Debye Shielding

The charged particles inside a plasma have the ability to shield out electric potentials. This behavior is a consequence of the collective effects of the plasma particles.

To compute the thickness of this shielding distance we start with the Poisson equation

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}, \tag{2.9}$$

where ρ is the charge density and ϕ is the electric potential.

Without losing generality we can use the Poisson's equation in one dimension, and the charge density defined as the charge by the total density n. Since we have inside the plasma ions and electrons the total charge density is

$$\rho = e\left(n_i - n_e\right), \tag{2.10}$$

where we have set $q = -e$ for the charge of the electrons and $q = e$ the charge of the ions.

The Poisson's equation in one dimension and introducing the charge density, we get

$$\frac{d^2\phi(x)}{dx^2} = -\frac{e}{\varepsilon_0}\left(n_i - n_e\right). \tag{2.11}$$

Due to the fact that the electrons are less massive than the ions, the electrons respond faster to the change of the electric potential. For such a reason the charge density of electrons n_e is allowed to vary, while the ions are assumed that they do not move but form a background of positive charge. The density far away is uniform n_0, now the density of the ions is uniform in any point thus $n_i = n_0$.

To find how n_e changes we have to use (2.2) and to introduce the potential energy in our distribution,

$$f(u) = A_1 e^{-\left(\frac{1}{2}mu^2 - e\phi\right)/k_BT}; \tag{2.12}$$

which is the genralization of (2.1) when $\phi \neq 0$. Using (2.2) and integrating over the velocity u, we get the density of electrons

$$n_e = n_0 e^{e\phi/k_BT}. \tag{2.13}$$

Replacing these densities into (2.11) our equation take the form

$$\frac{d^2\phi(x)}{dx^2} = -\frac{en_0}{\varepsilon_0}\left(1 - e^{e\phi/k_BT}\right). \tag{2.14}$$

In the region where $\left|\frac{e\phi}{k_BT}\right| \ll 1$, we can expand the right side of (2.14). Taking only the linear term we obtain the following equation

$$\frac{d^2\phi(x)}{dx^2} = \frac{e^2n_0}{\varepsilon_0 k_BT}\phi. \tag{2.15}$$

The solution of (2.15) is $\phi(x) = \phi_0 \exp\left\{-\left|x\right|/\lambda_D\right\}$ where we define the Debye length as

$$\lambda_D = \left(\frac{\varepsilon_0 k_BT}{e^2n_0}\right)^{1/2}, \tag{2.16}$$

which is a measure of the shielding thickness.

2.1.3 Plasma Frequency

The Debye length λ_D is an important parameter that characterizes a plasma. As we have mentioned before this is a consequence of the collective behavior of the plasma. However, there is another parameter that gives us the typical time of response to the electric fields. This time is given by the plasma frequency ω_p, i.e. $t_p = 2\pi/\omega_p$.

If we perturb a quasineutral plasma moving the electrons slightly from their initial positions, the unbalance in the charge distribution will create an electric field in attemp to restore the charge neutrality. But the inertia of the electrons will cause that they overpass their initial positions, and another electric field will be produced in the opposite direction. This process produce a collective oscillations around the massive ions.

The frequency of these oscillations is known as the plasma frequency and is defined as [9]

$$\omega_p^2 = \frac{n_0 e^2}{m_e \varepsilon_0}, \tag{2.17}$$

where m_e is the mass of the electron.

2.2 Kinetic Theory

We are interested in studing plasma phenomena. This means that we need to find the position of each particle and how they move as the time evolves. One way to do it is to write down the equations of motion for each particle and solve them. The problem with this basic idea is that we are dealing with a huge amount of particles inside the plasma, moreover these particles interact with each other. This generates so many coupled equations that it is impossible to solve them.

In order to be able to have a theoretical description in plasma physics, there are two approaches with different approximations to simplify the problem.

- **Fluid Theory**. From a macroscopic point of view the plasma interacting with external fields can be seen as a fluid containing electrical charges. The properties of such a fluid are described by combining the hydrodynamic equations for the fluid and Maxwell's equations to describe the fields.
- **Kinetic Theory**. From a microscopic point of view the plasma consist of an assembly of particles interacting through known forces and it is possible to adopt a statistical approach. In the kinetic theory, it is necessary to know the distribution function for the particles. So now it is important to solve the proper kinetic equations to find the evolution of the distribution function in phase space.

In this work we will not cover the fluid theory in detail since our main interest is to analyze Landau damping and this is only possible in the kinetic approach. Based on a chapter 2 in [10] we will make a summary how the fluid equations can be derived from the kinetic theory.

2.2.1 Phase Space

As was mentioned before, to describe and have a clear picture of a classical system it is important to be able to identify the instantaneous position and the velocity of all the particles at a given time t. For example, if we want to describe a system formed by N particles it is necessary to specify a set of $3N$ positions coordinates $\{x_i\}|_{i=1}^{i=3N}$ and a set of $3N$ velocity coordinates $\{v_i\}|_{i=1}^{i=3N}$. Geometrically we can define a six dimensional space defined by the positions and the velocities of each particle, this space is called the "phase space". The instantaneous state of a system is represented by a point in this space. Furthermore, a trajectory in the phase space give us the evolution of the system.

In a plasma, as in any system built by many particles, at any given time t each particle will have a specific position and velocity. In the phase space this will be depicted as a bunch of representative points, one point for each particle. As time pass these points will move along their respective trajectories. Therefore, it is possible to characterize the instantaneous configuration by specifying the density of particles at each point in the phase space. The function that allow us to describe the density is known as the distribution function $f(\mathbf{r}, \mathbf{v}, t)$.

Due to the fact that particle positions and velocities are dependent of time, the distribution function also will vary with time. The evolution of the distribution function gives a more detailed description than the fluid theory, but a description less detailed than the one following the trajectories of each individual particle.

2.2.2 The Boltzmann and Vlasov Equation

In the last section we have mentioned that we can describe the evolution of a system of N particles if we know the distribution function $f(\mathbf{r}, \mathbf{v}, t)$. One of the main problems of kinetic theory consist to find the right function $f(\mathbf{r}, \mathbf{v}, t)$ for a given system.

To find the correct distribution function, we have to solve the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c ; \tag{2.18}$$

where $\mathbf{F}$ is the force applied to the system, and the right side is the rate of change of the function f due to collisions.

We have here introduced the notation for the gradient operator in the velocity space as

$$\frac{\partial}{\partial \mathbf{v}} = \hat{\mathbf{x}} \frac{\partial}{\partial v_x} + \hat{\mathbf{y}} \frac{\partial}{\partial v_y} + \hat{\mathbf{z}} \frac{\partial}{\partial v_z}. \tag{2.19}$$

In the case where we neglect the internal collisions of the particles, we set the right hand side of (2.18) to zero. This will give us the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{2.20}$$

This equation can be written as

$$\frac{Df}{Dt} = 0, \tag{2.21}$$

where the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}}, \tag{2.22}$$

represents the total derivative with respect to time in the phase space.

The fact that we can re-write the Boltzmann equation (2.18) in the form of equation (2.21) in abscence of collisions, means the particles follow the contours of constant f as they move around the phase space.

The Boltzmann equation give us the right distribution function for a system of many particles, when the right force is applied to the system. In a plasma the dominant force is electromagnetic, that is why we can use the the Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{2.23}$$

in (2.20), which can be written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{2.24}$$

This equation is one of the most important equations in plasma physics.

It is important to point out that part of the electric and magnetic fields in the Lorentz force can be due to externally applied fields and the other part is due to internally generated fields. In order to have a set of closed equations, we must be able to compute the electromagnetic response to the internal current and charge sources.

In this spirit the fields are calculated by the Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} = & \quad \tfrac{\rho}{\varepsilon_0}, \\ \nabla \cdot \mathbf{B} = & \quad 0, \\ \nabla \times \mathbf{E} = & \quad -\tfrac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} = & \quad \mu_0 \mathbf{j} + \tfrac{1}{c^2}\tfrac{\partial \mathbf{E}}{\partial t}, \end{aligned} \tag{2.25}$$

where we have introduced the charge and current density as sources. We can compute these densities from the appropriate integrals of the distribution function

$$\rho = q \int f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}, \tag{2.26}$$

and

$$\mathbf{j} = q \int f(\mathbf{r}, \mathbf{v}, t) \mathbf{v} d\mathbf{v}. \tag{2.27}$$

2.2.3 Fluid Equations

As was mentioned at the begining of this section we will derive some of the fluid equations from the Boltzmann equation, to show that starting from the microscopic regime of a plasma we are able to reach the fluid regime. From now on we are going to define the charge as $q = -e$ in our equations.

To find the fluid equations we have to calculate the statistical moments over the velocity space. The lowest moment can be calculated integrating (2.24) over the velocity, i.e.

$$\int \frac{\partial f}{\partial t} d\mathbf{v} + \int \mathbf{v} \cdot \nabla f d\mathbf{v} - \int \frac{e}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = \int \left(\frac{\partial f}{\partial t}\right)_c d\mathbf{v}. \tag{2.28}$$

The first term can be re-written as

$$\int \frac{\partial f}{\partial t} d\mathbf{v} = \frac{\partial}{\partial t} \int f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} = \frac{\partial n(\mathbf{r}, t)}{\partial t}. \tag{2.29}$$

Due to the fact that the velocity $\mathbf{v}$ is not affected by the operator ∇, the second term gives

$$\int \mathbf{v} \cdot \nabla f d\mathbf{v} = \nabla \cdot \int \mathbf{v} f d\mathbf{v} = \nabla \cdot (n\bar{\mathbf{v}}) = \nabla \cdot (n\mathbf{u}). \tag{2.30}$$

Here we have defined the velocity $\mathbf{u}$, which is the average velocity and by definition is the velocity of the fluid.

Since collisions cannot change the total number of particles, the right side of (2.18) is

$$\int \left(\frac{\partial f}{\partial t} \right)_c d\mathbf{v} = 0. \tag{2.31}$$

We will now focus on the third term of (2.28). We separate the integral in the electric and magnetic part. In the electrical part we have

$$\int \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = \int \frac{\partial (f\mathbf{E})}{\partial \mathbf{v}} d\mathbf{v} = \int_S f\mathbf{E} \cdot d\mathbf{S}, \tag{2.32}$$

where we have used Gauss theorem in the velocity space. If we integrate $f\mathbf{E}$ on the surface at $\mathbf{v} = \infty$, the integrand vanishes if $f \to 0$ faster than v^{-2} as $v \to \infty$. This is a necessary condition for any distribution with finite energy. Thus

$$\int_S f\mathbf{E} \cdot d\mathbf{S} = 0. \tag{2.33}$$

The magnetic part can be written using vectorial properties as

$$\int (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = \int \frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{v} \times \mathbf{B})\, d\mathbf{v} - \int f \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B}). \tag{2.34}$$

The first integral in (2.34) has the same form as the electrical part, for that reason we conclude that this term vanishes for the same reason. The second integral also vanishes because $\frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B}) = 0$. Finally we get

$$\int (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = 0. \tag{2.35}$$

Replacing all the previous equations into (2.28) we get

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0. \tag{2.36}$$

This is the continuity equation.

We can calculate the first moment multiplying equation (2.24) by the momentum $m\mathbf{v}$ and integrating again in the velocity space. So our equation is

$$m \int \mathbf{v} \frac{\partial f}{\partial t} d\mathbf{v} + m \int \mathbf{v} (\mathbf{v} \cdot \nabla) f d\mathbf{v} - e \int \mathbf{v} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = m \int \mathbf{v} \left(\frac{\partial f}{\partial t} \right)_c d\mathbf{v}. \tag{2.37}$$

Since the calculation of the integrals are lenghty to show, we are going to skip the procedure and we will give only the final result. The reader is referred to [8] [10] for details. After all the calculation the final equation is

$$mn \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -en (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla \cdot \bar{\mathbf{P}} + \mathbf{R}, \tag{2.38}$$

where $\bar{\mathbf{P}}$ is the stress tensor and $\mathbf{R}$ is the momentum gained by collisions between the particles. This is the standard fluid equation. The last equation has to be written for each species part of the plasma and we will have the multi fluid model.

2.3 Waves in Plasmas

The study of wave propagation in plasma physics is an important part for the understanding of many plasma phenomena. We have mentioned that the natural motion of plasma often generates waves. Also, waves propagating into laboratory plasmas can be used to heat and probe them.

By convention we can write an oscillatory quantity as

$$g = g_0 \exp\left[i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right]; \tag{2.39}$$

the quantity g can be a vector or scalar, furthermore the real part, $\mathfrak{Re}(g)$ is the physical quantity. This exponential notation is very useful for analysis of linear systems where it is possible to apply Fourier analysis and the superposition principle is valid.

2.3.1 Phase Velocity

A point of constant phase in the wave moves with a distinctive velocity. This is, in one dimension

$$kx - \omega t = constant. \tag{2.40}$$

Solving the last equation for the position

$$x = \left(\frac{\omega}{k}\right) t + \frac{const.}{k}. \tag{2.41}$$

Now

$$v_\phi = \frac{dx}{dt} = \frac{\omega}{k} \quad \text{in 1-D,} \tag{2.42}$$

this is known as the phase velocity. This quantity can be greater than the light velocity since an infinite wave with constant amplitude does not carry information.

The generalization for a three dimensional case is

$$\mathbf{v}_\phi = \hat{\mathbf{k}}\frac{\omega}{|\mathbf{k}|} \quad \text{in 3-D.} \tag{2.43}$$

2.3.2 Group Velocity

The information usually is encoded on a carrier wave. This wave has to be modulated and the information travels with the group velocity. Thus the group velocity is defined as

$$v_g = \frac{d\omega}{dk} \quad \text{in 1-D;} \tag{2.44}$$

or

$$\mathbf{v}_g = \frac{d\omega}{d\mathbf{k}} \quad \text{in 3-D,} \tag{2.45}$$

where we have used the notation of the gradient in the k-space. This quantity does not exceed the velocity of light. Moreover, both velocities can be determined from its dispersion function $\omega(k)$.

2.3.3 Linearization

All the equations we have mentioned so far that describe the movement of the particles, are non linear differential equations. This kind of equations in very few cases can be solved explicitly. But it is possible to make some approximation to first order using the procedure of linearization.

In this method, we assume that the amplitude of oscillation is small and we can neglect terms containing higher orders in the Taylor expansion around some equilibrium value. In the case we are interested, the perturbed variables are the wave related quantities. Thus for example the density in one dimension can be written as

$$n(x,t) = n_0 + n_1(x,t)\exp\left[i\left(kx - \omega t\right)\right], \tag{2.46}$$

where n_0 is the equilibrium part and n_1 indicates a perturbation. The unperturbed quantity express the state of the plasma without any oscillations. In the next part we will use the linearization procedure to present some of the different waves that are present in a plasma and we will examine the dispersion functions that characterize these waves.

2.3.4 Linear Plasma Waves

At the begining we said that a small perturbation in the positions of the electrons generate oscillations in the plasma with a frequency ω_p. There are other effects that can cause the plasma to oscillate.

Electron Plasma Waves

The thermal motion of the electrons also contributes what is happening in the oscillating region. We assume that there is no magnetic field, and we neglect changes due collisions. With these assumptions in mind our equation (2.38) can be written for the electrons as

$$mn_e\left[\frac{\partial \mathbf{v_e}}{\partial t} + \left(\mathbf{v_e}\cdot\nabla\right)\mathbf{v_e}\right] = -en_e\mathbf{E} - \nabla\cdot\bar{\mathbf{P}}. \tag{2.47}$$

To solve this equation we need to find or known the stress tensor. In this case we will consider an isotropic Maxwellian distribution function then we can write $\nabla\cdot\bar{\mathbf{P}} = \nabla p$, where p is the scalar pressure. We recall the pressure can be obtained from the thermodynamic equation of state relating p and the density n. This is written

$$p = Cn^{\gamma} \quad \text{where} \quad \gamma = C_p/C_v, \tag{2.48}$$

where C_p and C_v are the heat capacity at constant pressure and volume respectively.

With this relation the term ∇p is given by

$$\frac{\nabla p}{p} = \gamma\frac{\nabla n}{n}. \tag{2.49}$$

The factor γ has the value $\gamma = \left(2+N\right)/N$, where N is the number of degrees of freedom. Then for an ideal gas with $p = k_BTn$ we get

$$\nabla p = \gamma p\frac{\nabla n}{n} = \gamma k_BT\nabla n. \tag{2.50}$$

Replacing this equation for the electrons in our equation of motion, we have

$$mn_e\left[\frac{\partial \mathbf{v}_e}{\partial t} + \left(\mathbf{v_e}\cdot\nabla\right)\mathbf{v}_e\right] = -en_e\mathbf{E} - \gamma k_BT\nabla n_e. \tag{2.51}$$

To solve this equation we use perturbation theory so we propose a solutions for the velocity, the density and the electric field of the form (2.46), this is

$$\begin{aligned} n_e = &\quad n_{e0} + n_1, \\ \mathbf{v}_e = &\quad \mathbf{v_{e0}} + \mathbf{v_{e1}}, \\ \mathbf{E} = &\quad \mathbf{E}_0 + \mathbf{E}_1. \end{aligned} \tag{2.52}$$

Also we assume a uniform neutral plasma at rest before the perturbation, so we set the variables

$$\nabla n_{e0} = \mathbf{v_{e0}} = \mathbf{E}_0 = 0, \tag{2.53}$$

and

$$\frac{\partial n_{e0}}{\partial t} = \frac{\partial \mathbf{v}_0}{\partial t} = \frac{\partial \mathbf{E}_0}{\partial t} = 0. \tag{2.54}$$

Taking only the linear terms, our equation (2.51) is

$$mn_e \frac{\partial \mathbf{v_{e1}}}{\partial t} = -en_e\mathbf{E}_1 - \gamma k_B T \nabla n_1. \tag{2.55}$$

Now we propose an ansatz of the form

$$\begin{aligned} n_1 = &\quad n_1 \exp\left[i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right], \\ \mathbf{v_{e1}} = &\quad \mathbf{v_{e1}} \exp\left[i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right], \\ \mathbf{E}_1 = &\quad \mathbf{E}_1 \exp\left[i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right]. \end{aligned} \tag{2.56}$$

Thus our equation (2.55) can be written as

$$mn_{e0}\left(-i\omega\right)\mathbf{v}_1 = -en_{e0}\mathbf{E}_1 - \gamma k_B T\left(i\mathbf{k}n_1\right). \tag{2.57}$$

The continuity equation (2.36) similarly linearized and simplified is written

$$-i\omega n_1 + n_{e0} i\mathbf{k}\cdot\mathbf{v}_1 = 0. \tag{2.58}$$

In Poisson's equation, we note that $n_{i0} = n_{e0}$ in equilibrium and since the ions are fixed $n_{i1} = 0$, so we have

$$\nabla\cdot\mathbf{E}_1 = -\frac{en_1}{\varepsilon_0}. \tag{2.59}$$

Replacing the ∇ operator by $i\mathbf{k}$ in the last equation due to our oscillating ansatz we have

$$i\mathbf{k}\cdot\mathbf{E}_1 = -\frac{en_1}{\varepsilon_0}. \tag{2.60}$$

Restricting the movement of the electrons to $\mathbf{k} \parallel \mathbf{E_1}$ and then solving the system of equations formed by eqs. (2.57), (2.58) and (2.60), we find the dispersion function

$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2. \tag{2.61}$$

The group velocity in this case is

$$v_g = \frac{d\omega}{dk} = \frac{3}{2}\frac{v_{th}^2}{v_\phi}. \tag{2.62}$$

Ion Waves

Because ions are massive they do not move as easily electrons do, but even so they can produce low-frequency oscillations. We start by writing the equation of motion (2.51) for ions

$$Mn_i\left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla)\,\mathbf{v}_i\right] = en_i\mathbf{E} - \gamma_i k_B T_i \nabla n_i. \tag{2.63}$$

Now we replace $\mathbf{E} = -\nabla\phi$, linearizing, introducing the oscillatory ansatz, and assuming motion in one dimension. We then get

$$-\,i\omega M n_0 v_{i0} = -en_0 ik\phi_1 - \gamma_i k_B T_i ikn_1. \tag{2.64}$$

Now the distribution for the electrons is given by (2.13). If we linearize it we find that the perturbation has the form

$$n_1 = n_0\frac{e\phi_1}{k_B T_e}. \tag{2.65}$$

To close the equations we use the continuity equation (2.58) for $\mathbf{k} \parallel \mathbf{v}$

$$i\omega n_1 = n_0 ikv_{i0}. \tag{2.66}$$

Solving (2.64), (2.65) and (2.66) together, the dispersion function is

$$\frac{\omega}{k} = \left(\frac{k_B T_e}{M} + \frac{\gamma_i k_B T_i}{M}\right)^{1/2}. \tag{2.67}$$

The equation for the sound velocity in the air is

$$c_s = \sqrt{\frac{\gamma k_B T}{M}}. \tag{2.68}$$

By analogy we define

$$v_s = \left(\frac{k_B T_e}{M} + \frac{\gamma_i k_B T_i}{M}\right)^{1/2}, \tag{2.69}$$

as the sound velocity in the plasma. The electrons have the time to equalize the temperature everywhere since they move faster than these waves, therefore the electrons reach an isothermal distribution easily and hence $\gamma_e \approx 1$. By contrast ions are adiabatic and thus $\gamma_i = 3$.

Electromagnetic waves with $\mathbf{B}_0 = 0$

To consider electromagnetic waves in vacuum the relevant Maxwell's equations are

$$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t}, \tag{2.70}$$

$$\nabla \times \mathbf{B}_1 = \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t}. \tag{2.71}$$

The corresponding dispersion relation is

$$\omega^2 = k^2 c^2, \tag{2.72}$$

and $\omega/k = c$ is the phase velocity of light waves. In a plasma with $\mathbf{B}_0 = 0$ we introduce the current density into (2.71) due to the motion of the charged particles

$$\nabla \times \mathbf{B}_1 = \mu_0 \mathbf{j}_1 + \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t}. \tag{2.73}$$

Now we take the time derivative of this equation getting

$$\nabla \times \frac{\partial \mathbf{B}_1}{\partial t} = \mu_0 \frac{\partial \mathbf{j}_1}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_1}{\partial t^2}. \tag{2.74}$$

Replacing (2.70) into (2.74), using the identity $\nabla \times (\nabla \times \mathbf{E}_1) = \nabla (\nabla \cdot \mathbf{E}_1) - \nabla^2 \mathbf{E}_1$ and introducing our plane wave ansatz we have

$$-\mathbf{k}(\mathbf{k}\cdot\mathbf{E}_1) + k^2\mathbf{E}_1 = i\mu_0\omega\mathbf{j}_1 + \frac{\omega^2}{c^2}\mathbf{E}_1. \tag{2.75}$$

Now we decide whether we have a transverse or longitudinal electric field perturbation in reference to the direction of the vector $\mathbf{k}$. In this case we will consider transverse waves which means $\mathbf{k}\cdot\mathbf{E}_1 = 0$ and hence (2.75) becomes

$$\left(\omega^2 - k^2c^2\right)\mathbf{E}_1 = -\frac{i\omega\mathbf{j}_1}{\varepsilon_0}. \tag{2.76}$$

For high frequencies waves the ions can be assumed they are fix. Thus the current is due to the movement of the electrons

$$\mathbf{j}_1 = -en_0\mathbf{v}_{\mathbf{e}1}. \tag{2.77}$$

From the linearized electron equation of motion (2.57) for $k_BT_e = 0$ we get

$$\mathbf{v}_1 = \frac{e}{im\omega}\mathbf{E}_1. \tag{2.78}$$

Replacing these equations into (2.75), the result is

$$\omega^2 = \omega_p^2 + k^2c^2. \tag{2.79}$$

This is the dispersion relation for electromagnetic waves traveling in a plasma with no external magnetic field.

There are more waves due the appearence of an external magnetic field $\mathbf{B}_0$. Furthermore, we can consider ion oscillations in the presence of a magnetic field. But we will not cover all these waves here. For people interested in plasma waves in more general situations I strongly recommend [8] which has a demonstration of the dispersion function, or [11] that does not calculate the dispersion functions but has a very good discussion about them.

Chapter 3

Linear Landau Damping in Unmagnetized Plasmas

We have mentioned that our equations are non-linear differential equations and in order to have a first approximation valid for small amplitudes we can linearize them. In the end of Chapter 2 we used this method in the fluid equations to find the dispersion relations of three wave modes. Now it is time to linearize the kinetic equation. This procedure will lead us to find a type of damping of the waves which is not found using the fluid equations.

3.1 Linearized Vlasov Equation

First we consider the Vlasov equation (2.24) without the magnetic term. This is:

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f - \frac{e}{m}\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{3.1}$$

We propose a solution of the form

$$\begin{aligned} f(\mathbf{r}, \mathbf{v}, t) = &\quad f_0(\mathbf{v}, t) + f_1(\mathbf{r}, \mathbf{v}, t), \\ \mathbf{E}(\mathbf{r}, t) = &\quad \mathbf{E}_0 + \mathbf{E}_1(\mathbf{r}, t). \end{aligned} \tag{3.2}$$

In this case $\mathbf{v}$ is an independent variable, so it is not neccessary to linearize this variable. Furthermore, we can notice that the undisturbed distribution function does not depend on the position so the gradient of this quantity will disappear and also we will assume that $\mathbf{E}_0 = 0$. With this in mind our equation will be

$$\frac{\partial f_1}{\partial t} + (\mathbf{v} \cdot \nabla) f_1 - \frac{e}{m}\mathbf{E_1} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0. \tag{3.3}$$

This equation is the linearized Vlasov equation.

3.2 Solution of the Linear Vlasov Equation

We now make a plane wave ansatz $f_1 \to f_1(\mathbf{v}) \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t])$, and $\mathbf{E_1} \to \mathbf{E_1} \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t])$ which means that we can replace $\nabla \to i\mathbf{k}$ and $\partial/\partial t \to -i\omega$. Hence the equation (3.3) is written as

$$-i\omega f_1 + i(\mathbf{k} \cdot \mathbf{v}) f_1 - \frac{e}{m}\mathbf{E_1} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0. \tag{3.4}$$

Solving for the perturbed distribution function we obtain

$$f_1 = \frac{ie}{m}\mathbf{E_1} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v})}. \tag{3.5}$$

Now from the linearized Poisson's equation we have

$$\nabla \cdot \mathbf{E_1} = -\frac{e}{\varepsilon_0} n_1. \tag{3.6}$$

Using our oscillatory solution we get

$$i\mathbf{k} \cdot \mathbf{E_1} = -\frac{e}{\varepsilon_0} n_1. \tag{3.7}$$

We recall that the particle number density is the integral over the velocity space

$$n_1 = \int f_1 d\mathbf{v}, \tag{3.8}$$

combining (3.8), (3.7) and (3.5) we get

$$\mathbf{k} \cdot \mathbf{E_1} = -\frac{e^2}{\varepsilon_0 m} \int \mathbf{E_1} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v})} d\mathbf{v}. \tag{3.9}$$

Without loosing generality we are going to suppose that the plane waves propagates in the x-direction. Thus we get

$$k_x E_{1x} = -\frac{e^2}{\varepsilon_0 m} \int E_{1x} \frac{\partial f_0'}{\partial v_x} \frac{1}{(\omega - k_x v_x)} d\mathbf{v}. \tag{3.10}$$

Since the electric field does not depend on the velocity it is possible to simplify (3.10) getting

$$1 = -\frac{e^2}{\varepsilon_0 m k_x} \int \frac{\partial f_0'}{\partial v_x} \frac{1}{(\omega - k_x v_x)} d\mathbf{v}. \tag{3.11}$$

The unperturbed distribution function can be written as $f_0' = n_0 \hat{f}_0$, where $\hat{f}_0$ is a normalized distribution function, then

$$1 = -\frac{\omega_p^2}{k_x} \int \frac{\partial \hat{f}_0}{\partial v_x} \frac{1}{(\omega - k_x v_x)} dv_x; \tag{3.12}$$

here we have used the definition of the plasma frequency (2.17), and we have integrated over the variables v_y and v_z since the distribution is assumed to be separable, i.e $f_0(v_x, v_y, v_z) = f_0'(v_x)G(v_y, v_z)$. The integral in the last equation is not straightforward to calculate because the denominator vanishes when $\omega = k_x v_x$.

The first person to treat this problem properly was Landau in 1946. He said that the linearized Vlasov equation should be treated as an initial condition problem instead of a normal mode problem as we have been doing so far.

Before we consider the solution suggested by Landau it is important that we mention some features of the Laplace transform.

3.3 The Laplace Transform

The Laplace transform of the function $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt = F(s). \tag{3.13}$$

If s is complex, then the real part of this variable has to be positive, $\mathfrak{Re}(s) > 0$, so the integral converges. However, it is possible that $f(t)$ contains exponential growing factors of the form

$e^{\alpha t}$. For very long times, $t \to \infty$, this growing factor will dominate other factors of $f(t)$ making the integral (3.13) divergent. To avoid this problem it is important to fullfil the condition $\mathfrak{Re}(s) > \alpha$, so our factor e^{-st} always will dominate the growing factor and make our integral finite.

The inverse transform integral or Bromwhich integral is defined as

$$f(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} F(s) e^{st} ds, \tag{3.14}$$

in this case we have taken the general case in which s is a complex number. This notations specify that the real part of s remains constant at a value β which is greater than α, while the imaginary part goes from -infinity to infinity. For a clear and complete demonstration of this see Ref.[12].

Before we return to our problem we have to recall that the Laplace transform of a derivative is given by

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\mathcal{L}\left\{f(t)\right\} - f(t=0). \tag{3.15}$$

Since the last definition contains information about the initial value, this method is better suited for initial value problems. With this information we can return to our main problem and have a better understanding of the solution proposed by Landau.

3.4 The Landau Solution

We start from the linearized Vlasov equation

$$\frac{\partial f_1}{\partial t} + (\mathbf{v} \cdot \nabla) f_1 - \frac{e}{m} \mathbf{E_1} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0. \tag{3.16}$$

Now we assume a spatial dependence $\propto e^{i\mathbf{k}\cdot\mathbf{r}}$ of the perturbed quantities, thus we have

$$\frac{\partial f_1}{\partial t} + i\mathbf{k} \cdot \mathbf{v} f_1 - \frac{e}{m} \mathbf{E_1} \cdot \frac{\partial f_1}{\partial \mathbf{v}} = 0. \tag{3.17}$$

Applying the Laplace transform in time to the last equation, ordering and solving for the transform of the perturbed distribution, we get

$$\mathcal{L}\left\{f_1\right\} = \frac{1}{s + i\mathbf{k} \cdot \mathbf{v}} \left[f_1(\mathbf{v}, t=0) + \frac{e}{m} \mathcal{L}\left\{\mathbf{E_1}\right\} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \right]. \tag{3.18}$$

We can mention that in the last equation we have explicitly included the initial condition $f_1(\mathbf{v}, t = 0)$ in contrast to equation (3.5). We have to mention that the variable s it is related to the frequency as $s = -i\omega$.

Now it is possible to take the Laplace transform in time of the equations (3.7) and (3.8), since the spatial dependence is oscillatory. Doing this and combining both equations we have

$$i\mathbf{k} \cdot \mathcal{L}\left\{\mathbf{E_1}\right\} = -\frac{e}{\varepsilon_0} \int \mathcal{L}\left\{f_1\right\} d\mathbf{v}. \tag{3.19}$$

Replacing (3.18) into (3.19) we obtain

$$i\mathbf{k} \cdot \mathcal{L}\left\{\mathbf{E_1}\right\} = -\frac{e}{\varepsilon_0} \int \frac{1}{s + i\mathbf{k} \cdot \mathbf{v}} \left[f_1(\mathbf{v}, t=0) + \frac{e}{m} \mathcal{L}\left\{\mathbf{E_1}\right\} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \right] d\mathbf{v}. \tag{3.20}$$

Now we suppose that $\mathbf{k} = k\hat{\mathbf{x}}$ and $\mathbf{E_1} = E_{1x}\hat{\mathbf{x}}$, also the integrals over the velocities v_y and v_z are easy to compute due that our distributions can be factorized for these variables. So the last equation can be written as

$$ik\mathcal{L}\left\{E_{1x}\right\} = -\frac{e}{\varepsilon_0}\int \frac{1}{s+ikv_x}\left[f_1(v_x, t=0) + \frac{e}{m}\mathcal{L}\left\{E_{1x}\right\}\frac{\partial f_0}{\partial v_x}\right] dv_x. \tag{3.21}$$

Re-ordering we have

$$\mathcal{L}\left\{E_{1x}\right\}\left(1 - i\frac{e^2}{\varepsilon_0 mk}\int \frac{\partial f_0/\partial v_x}{s+ikv_x}dv_x\right) = \frac{ie}{\varepsilon_0 k}\int \frac{f_1(v_x, t=0)}{s+ikv_x}dv_x. \tag{3.22}$$

Finally we can write it as

$$\mathcal{L}\left\{E_{1x}\right\} = \frac{N(s,k)}{D(s,k)}, \tag{3.23}$$

where

$$N(s,k) = \frac{ie}{\varepsilon_0 k}\int \frac{f_1(v_x, t=0)}{s+ikv_x}dv_x, \tag{3.24}$$

and

$$D(s,k) = 1 - i\frac{\omega_p^2}{k}\int \frac{\partial \hat{f}_0/\partial v_x}{s+ikv_x}dv_x. \tag{3.25}$$

Here again we have introduced a normalized distribution function. Replacing (3.23) into the x-direction of the equation (3.18) we have:

$$\mathcal{L}\left\{f_1\right\} = \frac{1}{s+ikv_x}\left[f_1(v_x, t=0) + \frac{e}{m}\frac{N(s,k)}{D(s,k)}\frac{\partial f_0}{\partial v_x}\right] \tag{3.26}$$

Since we are interested to solve the problems for f_1 and E_{1x} we have to take the inverse transform (3.14) of the equations (3.23) and (3.26). However the integrals of the form

$$\int_{\beta-i\infty}^{\beta+i\infty} \frac{N(s,k)}{D(s,k)}e^{st}ds, \tag{3.27}$$

are difficult to evaluate exactly. But it is possible to have an approximate dispersion relation derived from this integral in the long-time asymptotic limit.

If $D(s,k)$ is equal to zero for some values of s, these values are called the poles of E_{1x} and the maximum contribution to the integral will come from the poles. For that reason we will study the integral (3.27).

According to the definition of the Bromwhich integral the path of integration has to pass to the right of the poles of the function. In this case the poles are given by the values of s where $D(s,k) = 0$, since we proposed s is complex, we can depict the path of integration in Figure 3.1.

We can relate the variable s with the frequency by the relation $s = -i\omega$. This implies that ω also can be imaginary, which states $\mathfrak{Re}(s) = \mathfrak{Re}(-i\omega) = \mathfrak{Im}(\omega)$. We can conclude from this result that the complex plane of ω is just a rotation of 90 degrees relative to the plane of s as it can be seen in Figure 3.2, then $\mathfrak{Im}(\omega) = \beta$. Thus the poles of the function are situated below the path of integration.

The behavior of the plasma is determined by the poles of $\mathcal{L}\left\{E_{1x}\right\}$, since these values will determine the long term behavior of the electric field. If the poles are assumed to be simple poles

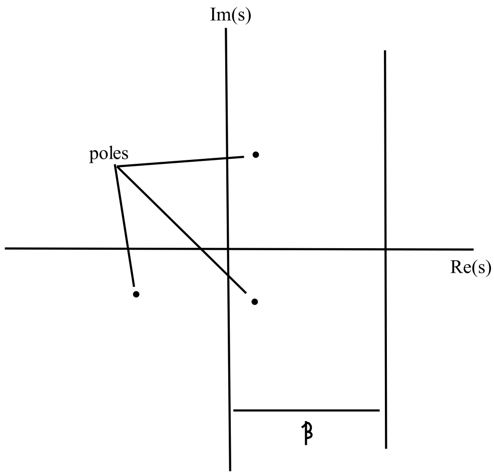


Figure 3.1: A picture about the poles of the function in relation with the path of integration of the inverse Laplace transform in the complex plane of s

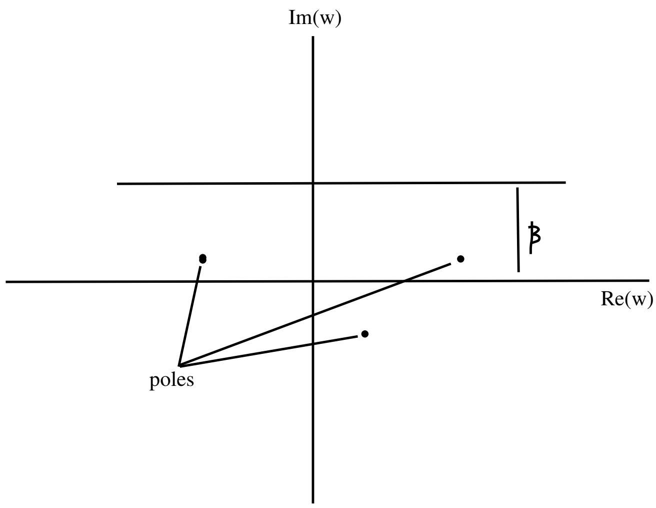


Figure 3.2: A picture about the poles of the function in relation with the path of integration of the inverse Laplace transform in the complex plane of ω

at ω_n we can evaluate the integral

$$\begin{aligned} E_{1x} &= \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{N(s,k)}{D(s,k)}e^{st}ds, \\ &= \sum_n \lim_{s\to -i\omega_n}\frac{(s+i\omega_n)}{D(-i\omega,k_x)}\frac{ie}{\varepsilon_0 k_x}\int\frac{f_1(v_x,t=0)}{s+ik_xv_x}e^{-i\omega_n t}dv_x. \end{aligned} \tag{3.28}$$

In order to write the expression in (3.28) as we have done, we need to know the values ω_n. These values are the roots of the dispersion function $D(-i\omega,k)$. Thus the equation to solve is

$$D(-i\omega,k) = 1 - i\frac{\omega_p^2}{k}\int\frac{\partial\hat{f}_0/\partial v_x}{-i\omega+ikv_x}dv_x = 0; \tag{3.29}$$

that is

$$1 = -\frac{\omega_p^2}{k_x}\int\frac{\partial\hat{f}_0/\partial v_x}{\omega-k_xv_x}dv_x. \tag{3.30}$$

which is the same equation as in (3.12). Now we have to integrate over the velocity, in this case we have to integrate along the real axis of the velocity. The pole in the complex velocity space according to (3.30) is given by $v_x = \omega/k$. This means that we can set k as real and ω as complex, with this idea in mind the imaginary part of $\omega_i = \gamma$ so the in the velocity space we have $v_x = i\gamma/k$. From Figure (3.2) we notice the poles are passing over the path of integration. In our case we have a single pole passing by γ/k in the velocity space, so the contour prescribed by Landau is a straight line along the real axis of the velocity and a small semicircle below the pole. This path comes from demanding a convergence in the Laplace transform.

Going around the pole we will have $2\pi i$ times half the residue in this point, hence our last equation can be written as

$$1 = \frac{\omega_p^2}{k^2}\left[P\int \frac{\partial \hat{f}_0/\partial v_x}{v_x - (\omega/k)} dv_x + i\pi \left.\frac{\partial \hat{f}_0}{\partial v_x}\right|_{v=\omega/k}\right], \tag{3.31}$$

where P referes to the Cauchy principal value. To calculate this integral, we integrate over all the space except for a region where the pole is present. If the phase velocity is large enough the contribution due to the neglected part is small. Thus we can integrate the first term by parts, getting

$$P\int \frac{\partial \hat{f}_0/\partial v_x}{v_x - (\omega/k)} dv_x = \left.\frac{\hat{f}_0}{v_x - (\omega/k)}\right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \hat{f}_0 \frac{1}{(v_x - (\omega/k))^2} dv_x. \tag{3.32}$$

The first term vanishes since the denominator goes to zero when $v_x \to \pm\infty$. The second term is the average of $(v_x - (\omega/k))^{-2}$ over the distribution function. Therefore equation (3.31) can be written as

$$1 = \frac{\omega_p^2}{k_x^2}\left[\left\langle (v_x - (\omega/k))^{-2}\right\rangle + i\pi \left.\frac{\partial \hat{f}_0}{\partial v_x}\right|_{v=\omega/k}\right]. \tag{3.33}$$

In order to calculate the integral analytically we have mentioned that the phase velocity ω/k should be large enough. In that case we are able to expand the first term

$$\begin{aligned}\left\langle (v_x - (\omega/k))^{-2}\right\rangle &= \frac{k_x^2}{\omega^2}\left\langle \left(1 - \frac{kv_x}{\omega}\right)^{-2}\right\rangle \\ &= \frac{k_x^2}{\omega^2}\left\langle \left(1 + 2\frac{kv_x}{\omega} + 3\left(\frac{kv_x}{\omega}\right)^2 + \dots\right)\right\rangle.\end{aligned} \tag{3.34}$$

Taking the average the odd terms will disappear and finally we will get

$$\left\langle (v_x - (\omega/k))^{-2}\right\rangle \approx \frac{k_x^2}{\omega^2}\left(1 + 3\frac{k_x^2\langle v_x^2\rangle}{\omega^2}\right). \tag{3.35}$$

Now let to be $\hat{f}_0$ a Maxwellian distribution, we can relate the average kinetic energy by equation (2.6)

$$\frac{1}{2}m\langle v_x^2\rangle = \frac{1}{2}k_B T_e. \tag{3.36}$$

Combining (3.35) and (3.36) and replacing into (3.33) we obtain

$$1 = \frac{\omega_p^2}{k_x^2}\left[\frac{k^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{k_B T_e}{m}\right) + i\pi \left.\frac{\partial \hat{f}_0}{\partial v_x}\right|_{v=\omega/k}\right]. \tag{3.37}$$

Re-ordering the terms of (3.37) the equation becomes

$$1 = \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{k_B T_e}{m}\right) + i\pi\frac{\omega_p^2}{k^2}\left.\frac{\partial \hat{f}_0}{\partial v_x}\right|_{v=\omega/k}. \tag{3.38}$$

Then

$$\omega^2\left\{1 - i\pi\frac{\omega_p^2}{k^2}\left.\frac{\partial \hat{f}_0}{\partial v_x}\right|_{v=\omega/k}\right\} = \omega_p^2 + 3k^2\frac{\omega_p^2}{\omega^2}\frac{k_B T_e}{m}. \tag{3.39}$$

Taking the square root of both sides and expanding to the first order the left side of (3.39) we obtain

$$\omega\left\{1 - i\frac{\pi}{2}\frac{\omega_p^2}{k}\left.\frac{\partial \hat{f}_0}{\partial v_x}\right|_{v=\omega/k}\right\} = \sqrt{\omega_p^2 + \frac{3}{2}k^2 v_{th}^2\frac{\omega_p^2}{\omega^2}}. \tag{3.40}$$

Finally solving for ω we get

$$\omega = \sqrt{\omega_p^2 + \frac{3}{2}k^2 v_{th}^2\frac{\omega_p^2}{\omega^2}} + i\frac{\pi}{2}\frac{\omega_p^2\omega}{k}\left.\frac{\partial \hat{f}_0}{\partial v_x}\right|_{v=\omega/k}. \tag{3.41}$$

If the thermal correction is small we may replace ω^2 by ω_p^2 in the second term in the right side of (3.39) and solving for ω we get

$$\omega = \sqrt{\omega_p^2 + \frac{3}{2}k_x^2 v_{th}^2} + i\frac{\pi}{2}\frac{\omega_p^3}{k^2}\left.\frac{\partial \hat{f}_0}{\partial v_x}\right|_{v=\omega/k}. \tag{3.42}$$

The imaginary part of the last equation depends on the rate of change of the undisturbed distribution function at the point ω/k. If the function $\hat{f}_0$ is a Maxwellian distribution and the phase velocity ω/k lies in the tail of the distribution we have a negative imaginary part of $\omega = \gamma$, which implies an exponential decay which can be understood as a damping on the plasma waves, this damped factor is calculated in [8] where the expression is given by

$$\frac{\gamma}{\omega_p} = -0.22\sqrt{\pi}\left(\frac{\omega_p}{k v_{th}}\right)^3 \exp\left(-\frac{1}{2k^2\lambda_D^2}\right). \tag{3.43}$$

This result is called Landau Damping, and we note that the damping is small for wavelengths greater than the Debye length λ_D, but it becomes important as the wavelengths approaches to λ_D.

3.5 Physical Picture of the Landau Damping

We have presented the Landau solution for the linearized Vlasov equation in the last section. We note that this solution comes from a completely mathematical approach with a clear lack of physical information about the process responsible for the damping of the wave. For that reason Dawson [6] analysed a simple model that gave us a better picture of the process involved in the damping efffect.

In this model we split the plasma into beams of velocity $\mathbf{u}_0 = u\hat{\mathbf{x}}$, and the system is driven by a sinusoidal forces from the wave field. Linearizing equation (2.38) around the beam velocity, i.e $\mathbf{v} = (u + v_1)\hat{\mathbf{x}}$ where $|v_1| \ll |u|$ we have

$$m\left(\frac{\partial v_1}{\partial t} + u\frac{\partial v_1}{\partial x}\right) = -eE_{1x}; \tag{3.44}$$

then

$$\mathbf{E}_1 = E_1 \sin\left(kx - \omega t\right) \hat{\mathbf{x}}. \tag{3.45}$$

Combining these equations we have

$$m\left(\frac{\partial v_1}{\partial t} + u\frac{\partial v_1}{\partial x}\right) = -eE_1 \sin\left(kx - \omega t\right). \tag{3.46}$$

A possible solution for this equation is

$$v_1 = -\frac{eE_1}{m}\frac{\cos\left(kx - \omega t\right)}{\omega - ku}. \tag{3.47}$$

We notice that this anzats have a problem, since it does not satisfy the initial condition $v_1 = 0$ at $t = 0$. In that case we can add a function of the form $-\cos(kx - kut)$. Then equation (3.47) becomes,

$$v_1 = -\frac{eE_1}{m}\frac{\left[\cos\left(kx - \omega t\right) - \cos(kx - kut)\right]}{\omega - ku}. \tag{3.48}$$

Next we apply the linearized continuity equation

$$\frac{\partial n_1}{\partial t} + u\frac{\partial n_1}{\partial x} = -n_u\frac{\partial v_1}{\partial x}, \tag{3.49}$$

where n_u is the density of particles with velocity u and n_1 is the perturbed particle density. To solve this equation it is possible to try a solution of the form

$$n_1 = \bar{n}_1 \left[\cos\left(kx - \omega t\right) - \cos\left(kx - kut\right)\right]. \tag{3.50}$$

Replacing this ansatz into (3.49) we have

$$\bar{n}_1 \sin\left(kx - \omega t\right) = -\frac{n_u ekE_1}{m}\frac{\sin\left(kx - \omega t\right) - \sin\left(kx - kut\right)}{\left(\omega - ku\right)^2}. \tag{3.51}$$

Since the term $\sin\left(kx - \omega t\right)$ in the left side and does not cancel with the terms of the right side for $t = 0$, it is necessary to add a term of the form $At\sin(kx - kut)$. The coefficient A must be proportional to $\left(\omega - ku\right)^{-1}$ to match with the factor of $\partial v_1/\partial x$. After some algebra we have

$$n_1 = -\frac{n_u ekE_1}{m}\frac{1}{\left(\omega - ku\right)^2}\left[\cos\left(kx - \omega t\right) - \cos(kx - kut) - \left(\omega - ku\right)t\sin\left(kx - kut\right)\right]. \tag{3.52}$$

Now that we have the terms for n_1 and v_1 we can calculate the work done by the wave on each beam. The force density in this case is given by

$$F_u = -eE_1 \sin\left(kx - \omega t\right)\left(n_u + n_1\right). \tag{3.53}$$

The change of the energy will be

$$\frac{dW}{dt} = F_u\left(u + v_1\right). \tag{3.54}$$

Combining equations (3.53) and (3.54) we have

$$\frac{dW}{dt} = -eE_1 \sin\left(kx - \omega t\right)\left(n_u u + n_u v_1 + n_1 u\right), \tag{3.55}$$

where we have neglected one term because it was a second order perturbation. Now we take the spatial average over a wavelength, since the first term is a constant this term will vanish in the average, so the contribution that remain is

$$\left\langle\frac{dW}{dt}\right\rangle = -eE_1 \left\langle\sin\left(kx - \omega t\right)\left(n_u v_1 + n_1 u\right)\right\rangle. \tag{3.56}$$

We have to recall the following properties

$$\langle \sin(kx-\omega t)\cos(kx-kut)\rangle = -\frac{1}{2}\sin(\omega t-kut) \tag{3.57}$$

and

$$\langle \sin(kx-\omega t)\sin(kx-kut)\rangle = \frac{1}{2}\cos(\omega t-kut). \tag{3.58}$$

Using these properties to calculate the averages of (3.56) and after some algebra we finally get

$$\left\langle \frac{dW}{dt} \right\rangle_u = n_u \frac{e^2 E_1^2}{2m}\left[\frac{\sin(\omega t-kut)}{\omega-ku} + ku\frac{\sin(\omega t-kut)-(\omega t-kut)\,t\cos(\omega t-kut)}{(\omega-ku)^2}\right]. \tag{3.59}$$

We are interested to find the total work done over the particles. In that case we have to sum over all the beams, which gives

$$\left\langle \frac{dW}{dt} \right\rangle_{tot} = \sum_u \left\langle \frac{dW}{dt} \right\rangle_u = \int \frac{f_0(u)}{n_u}\left\langle \frac{dW}{dt} \right\rangle_u du. \tag{3.60}$$

Introducing our normalized distribution function $\hat{f}_0$ to represent the continuous limit of infinately many beams we get

$$\left\langle \frac{dW}{dt} \right\rangle_{tot} = n_0 \int \frac{\hat{f}_0(u)}{n_u}\left\langle \frac{dW}{dt} \right\rangle_u du. \tag{3.61}$$

Replacing (3.59) into (3.61) and performing the corresponding algebra[1]

$$\left\langle \frac{dW}{dt} \right\rangle_{tot} = \frac{1}{2}\varepsilon_0 E_1^2 \omega_p^2 \int_{-\infty}^{\infty} du \hat{f}_0 \frac{d}{du}\left\{u\frac{\sin(\omega t-kut)}{\omega-ku}\right\}. \tag{3.62}$$

Because there has to be conservation of the energy, the last relation has to be equal to the rate of loss of energy density of the wave. The wave energy density is given by the sum of the energy density of the electrostatic field and the kinetic energy density of oscillation of the particles.

The kinetic energy density W_k of the beam will be

$$W_k = \frac{1}{2}m(n_u+n_1)(u+v_1)^2. \tag{3.63}$$

Expanding the last equation we have

$$\Delta W_k = \frac{1}{2}m\left(n_u v_1^2 + 2un_1v_1 + n_1u^2 + 2n_u u v_1 + n_1 v_1^2\right), \tag{3.64}$$

where we have defined

$$\Delta W_k = W_k - \frac{1}{2}mn_u u^2. \tag{3.65}$$

Now we take the average over a wavelenght. In this case the last three terms of (3.64) vanish because they have odd powers of the oscillating quantities. Then the equation remaining is

$$\langle \Delta W_k \rangle = \frac{1}{2}m\left\langle\left(n_u v_1^2 + 2un_1v_1\right)\right\rangle. \tag{3.66}$$

Since the contributions of the intial conditions does not contribute much to the total energy of the wave, the velocity modulation caused by the wave as the beam of electrons pass is

$$v_1 = -\frac{eE_1}{m}\frac{\cos(kx-\omega t)}{\omega-ku}. \tag{3.67}$$

[1]To the interested reader Ref. [8] has a detailed procedure.

We note that:

$$\left\langle \cos^2(kx-\omega t)\right\rangle = \left\langle \sin^2(kx-\omega t)\right\rangle = \frac{1}{2}. \tag{3.68}$$

Using the expression for the velocity and the properties mentioned it is possible to find that

$$n_u \left\langle v_1^2\right\rangle = \frac{1}{2} n_u \frac{e^2 E_1^2}{m^2 (\omega - ku)^2}. \tag{3.69}$$

In the same way it is possible to neglect the contribution of the initial conditions of (3.52), such that the density is given by

$$n_1 = -\frac{n_u e k E_1}{m} \frac{\cos(kx-\omega t)}{(\omega - ku)^2}. \tag{3.70}$$

Then we have

$$2u \left\langle n_1 v_1\right\rangle = n_u \frac{e^2 E_1^2 ku}{m^2 (\omega - ku)^3}. \tag{3.71}$$

Thus

$$\left\langle \Delta W_k\right\rangle_u = \frac{n_u}{4} \frac{e^2 E_1^2}{m(\omega - ku)^2}\left[1 + \frac{2ku}{\omega - ku}\right]. \tag{3.72}$$

Now the total kinetic energy is given by

$$\left\langle \Delta W_k\right\rangle_{tot} = \sum_u \left\langle \Delta W_k\right\rangle_u = \int \frac{f_0}{n_u}\left\langle \Delta W_k\right\rangle_u du. \tag{3.73}$$

Combining (3.72) and (3.73) we have

$$\left\langle \Delta W_k\right\rangle_{tot} = \frac{1}{4}\frac{e^2 E_1^2}{m}\int_{-\infty}^{\infty} \frac{f_0}{(\omega - ku)^2}\left[1 + \frac{2ku}{\omega - ku}\right] du. \tag{3.74}$$

In the limit $\omega/k \gg v_{th}$ the second term inside the brackets can be neglected, then

$$\left\langle \Delta W_k\right\rangle_{tot} = \frac{1}{4}\frac{e^2 E_1^2}{m}\int_{-\infty}^{\infty} \frac{f_0}{(\omega - ku)^2} du. \tag{3.75}$$

Now we use the Poisson's equation

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\varepsilon_0}\sum_u n_1, \tag{3.76}$$

replacing (3.45) and (3.70) in the last equation we get

$$1 = \frac{e^2}{\varepsilon_0 m}\sum_u \frac{n_u}{(\omega - ku)^2} = \frac{e^2}{\varepsilon_0 m}\int_{-\infty}^{\infty} \frac{f_0}{(\omega - ku)^2} du, \tag{3.77}$$

this is the dispersion relation. Combining (3.77) with (3.75) the final result for the kinetic energy of the beam is

$$\left\langle \Delta W_k\right\rangle_{tot} = \frac{1}{4}\varepsilon_0 E_1^2. \tag{3.78}$$

The energy density due to the electrostatic field is given by

$$\left\langle W_E\right\rangle = \frac{\varepsilon_0}{2}\left\langle \mathbf{E}_1^2\right\rangle = \frac{\varepsilon_0 E_1^2}{4} = \left\langle \Delta W_k\right\rangle_{tot}. \tag{3.79}$$

The total energy density of the wave is

$$W_w = \left\langle \Delta W_k\right\rangle_{tot} + \left\langle W_E\right\rangle = \frac{\varepsilon_0 E_1^2}{2}. \tag{3.80}$$

Due to energy conservation the rate of change of this quantity has to be set as

$$\frac{dW_w}{dt} = -\left\langle \frac{dW}{dt} \right\rangle_{tot}. \tag{3.81}$$

Thus,

$$\frac{dW_w}{dt} = -W_w \omega_p^2 \int_{-\infty}^{\infty} du \hat{f}_0 \frac{d}{du} \left\{ u \frac{\sin(\omega t - kut)}{\omega - ku} \right\}. \tag{3.82}$$

Integrating by parts and noting that $\hat{f}_0$ vanish in the limit $\pm\infty$, we have

$$\frac{dW_w}{dt} = W_w \omega_p^2 \int_{-\infty}^{\infty} u \frac{\partial \hat{f}_0}{\partial u} \left[\frac{\sin(\omega - ku)\, t}{\omega - ku} \right] du. \tag{3.83}$$

Now we recall the relation

$$\delta\left(u - \frac{\omega}{k}\right) = \frac{k}{\pi} \lim_{t \to \infty} \left[\frac{\sin(\omega - ku)t}{\omega - ku} \right]. \tag{3.84}$$

Hence,

$$\frac{dW_w}{dt} = W_w \pi \omega \frac{\omega_p^2}{k^2} \frac{\partial \hat{f}_0}{\partial u} \bigg|_{u=\omega/k}. \tag{3.85}$$

It is possible to say that $E_1 \propto e^{\gamma t}$, then we know that W_w is proportional to E_1^2, so

$$\frac{dW_w}{dt} = 2\gamma W_w. \tag{3.86}$$

Finally (3.85) is

$$\gamma / \omega = \frac{\pi}{2} \frac{\omega_p^2}{k^2} \frac{\partial \hat{f}_0}{\partial u} \bigg|_{u=\omega/k}. \tag{3.87}$$

In agreement with the imaginary part of equation (3.42) for $\omega = \omega_p$.

After all the mathematical machinery display so far in this picture, we are able to notice that we have particles traveling at velocity u and a wave with a phase velocity ω/k. The rate of growth of the wave depends on the slope of the distribution function $\hat{f}_0$. If the slope is positive the amplitude of the wave will grow exponentially, but on the other hand if the slope is negative the wave will be damped which is the same conclusion obtained by Landau. If we look into the region where the particles are close to the wave phase velocity these particle will enter in resonance.

Inside this resonant region there are some particles traveling faster than the wave and others traveling slower, see Fig.3.3. The particles traveling faster than the wave will on average transfer energy to the wave making it grow. On the other, particles traveling with a velocities slower than the phase velocity of the wave will on average gain energy from the wave, making the wave to lose energy and decrease the amplitude.

In the case where the distribution of velocities is Maxwellian there are more particles traveling slower than the wave. Consequently, as it was mentioned before, the wave will suffer a damping.

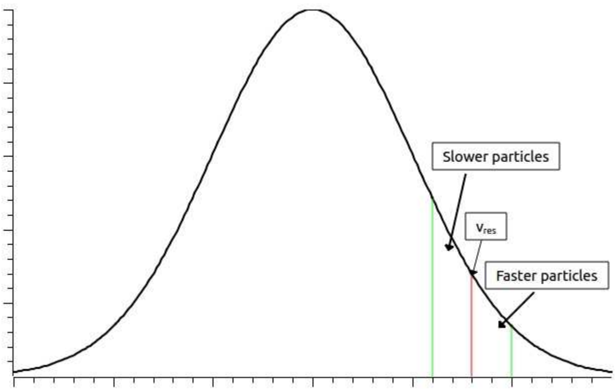


Figure 3.3: The resonant region compared to the the total distribution function.

3.6 A New Method for the Landau Damping

In 1995 it was presented by G. Brodin [7] a new way to handle the linear Landau problem. This new approach was intended to make it more easy to understand and solve this problem. He started proposing an anzats of the form

$$\mathbf{E} = E_1(t)e^{i(kx-\omega t)}\hat{\mathbf{x}} + c.c, \tag{3.88}$$

and

$$f_1 = \tilde{f}_1(t, \mathbf{v})e^{i(kx-\omega t)} + c.c, \tag{3.89}$$

where c.c stands for the complex conjugated. In this solution it is assumed that the time-dependent quantities are slow which means $|\partial E_1/\partial t| \ll |\omega E_1|$ and $\left|\partial \tilde{f}_1/\partial t\right| \ll |\omega \tilde{f}_1|$.

Introducing these expressions into the linearized Vlasov equation (3.3) we have

$$\frac{\partial \tilde{f}_1}{\partial t} - i(\omega - kv_x)\tilde{f}_1 - \frac{e}{m}E_1\frac{\partial f_0}{\partial v_x} = 0. \tag{3.90}$$

From the Poisson's equations we deduce

$$ikE_1 = -\frac{e}{m}\int \tilde{f}_1 d\mathbf{v}. \tag{3.91}$$

In most cases the first term of (3.90) is small compared to the second one, but close to the phase velocity where we have the resonant region the first term become more important. For that reason it is possible to think that $\tilde{f}_1$ has to be divided into two different expressions to be applied in different regions of the velocity space. Then

$$ikE_1 = -\frac{e}{m}\int_{nr} \tilde{f}_1 d\mathbf{v} - \frac{e}{m}\int_r \tilde{f}_1 d\mathbf{v}, \tag{3.92}$$

where one integral is over the non-resonant part (nr) and the second one is over the resonant region (r). Combining (3.92) and (3.90) we obtain

$$ikE_1 = -\frac{ie^2E_1}{\varepsilon_0 m}\int_{nr} \frac{\partial f_0}{\partial v_x}\frac{1}{(\omega - kv_x)}d\mathbf{v} + \frac{ie}{\varepsilon_0}\int_{nr} \frac{1}{(\omega - kv_x)}\frac{\partial \tilde{f}_1}{\partial t}d\mathbf{v} - \frac{e}{\varepsilon_0}\int_r \tilde{f}_1 d\mathbf{v}. \tag{3.93}$$

Re-ordering

$$\left(1 + \frac{e^2}{\varepsilon_0 km}\int_{nr} \frac{\partial f_0}{\partial v_x}\frac{1}{(\omega - kv_x)}d\mathbf{v}\right) ikE_1 = \frac{ie}{\varepsilon_0}\int_{nr} \frac{1}{(\omega - kv_x)}\frac{\partial \tilde{f}_1}{\partial t}d\mathbf{v} - \frac{e}{\varepsilon_0}\int_r \tilde{f}_1 d\mathbf{v}. \tag{3.94}$$

Since the left side of the equation has the form of eq.(3.30) it is possible to select values of ω and k to fulfill the dispersion function, but with the resonant region excluded. What remains is

$$\int_{nr} \frac{1}{(\omega - kv_x)} \frac{\partial \tilde{f}_1}{\partial t} d\mathbf{v} = -i \int_r \tilde{f}_1 d\mathbf{v}. \tag{3.95}$$

Inside the non-resonant region we know that $\left|\partial \tilde{f}_1/\partial t\right| \ll |\omega \tilde{f}_1|$ so using this fact in (3.90) we have

$$\tilde{f}_1 = \frac{ie}{m} E_1 \frac{\partial f_0}{\partial v_x} \frac{1}{(\omega - kv_x)}; \tag{3.96}$$

taking the time derivative we have

$$\frac{\partial \tilde{f}_1}{\partial t} = \frac{ie}{m} \frac{\partial E_1}{\partial t} \frac{\partial f_0}{\partial v_x} \frac{1}{(\omega - kv_x)}. \tag{3.97}$$

Substituting this into (3.95) we obtain

$$\frac{\partial E_1}{\partial t} = Q \int_r \tilde{f}_1 d\mathbf{v}, \tag{3.98}$$

where

$$Q = \left[-\frac{e}{m} \int_{nr} \frac{\partial f_0}{\partial v_x} \frac{dv_x}{(\omega - kv_x)^2} \right]^{-1}. \tag{3.99}$$

Thus we have a system of coupled equations

$$\frac{\partial \tilde{f}_1}{\partial t} - i\delta\omega(v_x) \tilde{f}_1 = H(v_x) E_1, \tag{3.100}$$

and

$$\frac{\partial E_1}{\partial t} = Q \int_r \tilde{f}_1 d\mathbf{v}; \tag{3.101}$$

where we have defined

$$H(v_x) = \frac{e}{m} \frac{\partial f_0}{\partial v_x} \quad \text{and} \quad \delta\omega(v_x) = (\omega - kv_x). \tag{3.102}$$

To have a well defined system of equations is important to define the resonant region. The upper limit is found by expanding the Maxwellian distribution

$$\begin{aligned} f_0(v_x + \delta v_x) &= A \exp[-\frac{(v_x + \delta v_x)^2}{v_{th}^2}] \\ &= A \exp\left[-\frac{v_x^2}{v_{th}^2}\right] \exp\left[-\frac{2 v_x \delta v_x}{v_{th}^2}\right]; \end{aligned} \tag{3.103}$$

since δv_x is small the second exponential can be Taylor expanded getting

$$f_0(v_x + \delta v_x) = A \exp\left[-\frac{v_x^2}{v_{th}^2}\right] \left\{1 - 2\frac{v_x \delta v_x}{v_{th}^2}\right\}. \tag{3.104}$$

The perturbation has to be small, thus $v_x \delta v_x \ll v_{th}^2$ in the resonant region gives $\delta v_x \ll k v_{th}^2/\omega$.

To find the lower limit we have to use the fact that $\partial \tilde{f}_1/\partial t$ is negligible compared to $(\omega - kv_x)\tilde{f}_1$ in the non-resonant region. Thus in the non-resonant region we can write that

$$(\omega - kv_x)\, \tilde{f}_1 \gg \frac{\partial \tilde{f}_1}{\partial t}. \tag{3.105}$$

Now to find the lower limit we can replace $v_x = v_x - \delta v_x$ in the last equation, thus we get

$$(\omega - kv_x)\,\tilde{f}_1 + k\delta v_x \tilde{f}_1 \gg \frac{\partial \tilde{f}_1}{\partial t}. \tag{3.106}$$

On the other hand, in the vecinity of the resonant region it is assumed that $\partial \tilde{f}_1/\partial t$ is of the order $\gamma \tilde{f}_1$. Thus in the resonant region the first term of 3.107 will be exactly zero then

$$k\delta v_x \tilde{f}_1 \gg \gamma \tilde{f}_1. \tag{3.107}$$

Solving for δv_x we define the complete resonant region as

$$\frac{\gamma}{k} \ll \delta v_x \ll \frac{kv_{th}^2}{\omega}. \tag{3.108}$$

In the majority of the cases these system of coupled equations are not possible to solved analitically so they have to be solved numerically. But Brodin showed that is possible to have an analitic solution if $E_1 = \tilde{E}\exp[\gamma t]$ where $\tilde{E}$ is a constant. The remarkable result of this is that the γ damping factor is given by[2]

$$\gamma = -\frac{\pi Q H(\omega/k)}{k}. \tag{3.109}$$

And this result agrees with the expresion (3.87) when Q is evaluated to the lowest order in an kv_{th}/ω_p expansion.

3.7 Conclusions

In this chapter we have discussed three different ways to approach to the linear Landau damping, the first two methods are widely covered in many introductory books about plasma physics. There are many details to take into considerations in both approaches that make them tricky to undestand at the beginning and specially for people who is starting to study this subject.

This new approach is in some sense more easy to follow, since we are separating two regions for the distribution function. The most important feature is that when the system of coupled equations is derived it is possible to see that this problem is also selfconsistent, since we need the distributions to calculate the fields and viceversa. Furthermore, the fact that the system has to be solved numerically lead us to incorporate duly initial conditions. Finally, as it is mentioned in the original paper additional effects can be incorporated easily.

[2]See [7] for a detailed explanation

Chapter 4

Cyclotron Damping in Magnetized Plasmas

The last chapter was devoted to the problem of Landau damping in plasmas. This study was performed not considering the presence of any magnetic field. In many cases studied both space and in laboratory plasmas a magnetic field is present. In this chapter we are going to present a brief study of the propagation of waves in a magnetized plasmas, with the main goal to use the approach of section 3.6. We are going to find that the problem of wave-particle interaction is still present when the magnetic field is introduced, and is now refered as cyclotron damping.

4.1 Linearized Equation and Dispersion Function

The total magnetic field will be given by

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r}, t), \tag{4.1}$$

where $\mathbf{B}_1$ is our perturbation due to the wave.

The linearized Vlasov equation introducing the magnetic effects has the form

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \nabla f_1 - \frac{e}{m} (\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_1}{\partial \mathbf{v}} = \frac{e}{m} [\mathbf{E}_1 + (\mathbf{v} \times \mathbf{B}_1)] \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \tag{4.2}$$

As we have assumed in the last chapter the perturbed quantities have harmonic dependence

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{E}_1 \exp\left[i\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right)\right], \\ \mathbf{B}_1 &= \mathbf{B}_1 \exp\left[i\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right)\right], \\ f_1 &= f_1(\mathbf{v}) \exp\left[i\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right)\right]. \end{aligned} \tag{4.3}$$

Thus equation (4.2) becomes

$$- i\omega f_1 + i(\mathbf{k} \cdot \mathbf{v}) f_1 - \frac{eB_0}{m} (\mathbf{v} \times \hat{\mathbf{z}}) \cdot \frac{\partial f_1}{\partial \mathbf{v}} = \frac{e}{m} [\mathbf{E}_1 + (\mathbf{v} \times \mathbf{B}_1)] \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \tag{4.4}$$

where we have defined the z-direction for the constant external magnetic field, and the oscillatory factors cancel out naturally.

Next we introduce cylindrical coordinates $(v_\perp, \phi_v, v_z)$ in velocity space. Using these coordinates we have

$$v_x = v_\perp \cos\phi_v, \quad v_y = v_\perp \sin\phi_v \quad \text{and} \quad v_z = v_z. \tag{4.5}$$

Using (4.5) we find that

$$(\mathbf{v}\times\hat{\mathbf{z}})\cdot\frac{\partial f_1}{\partial \mathbf{v}} = -\left\{v_\perp\hat{\phi}_\mathbf{v}\right\}\cdot\left\{\frac{\partial f_1}{\partial v_\perp}\hat{\rho}+\frac{1}{v_\perp}\frac{\partial f_1}{\partial \phi_v}\hat{\phi}_\mathbf{v}+\frac{\partial f_1}{\partial v_z}\hat{\mathbf{z}}\right\} = -\frac{\partial f_1}{\partial \phi_v}. \tag{4.6}$$

So equation (4.4) can now be written as

$$-i\omega f_1 + i(\mathbf{k}\cdot\mathbf{v})f_1 + \frac{eB_0}{m}\frac{df_1}{d\phi_v} = \frac{e}{m}\left[\mathbf{E}_1 + (\mathbf{v}\times\mathbf{B}_1)\right]\cdot\frac{\partial f_0}{\partial \mathbf{v}}. \tag{4.7}$$

Now we will define the direction of the wave vector parallel to the direction of the magnetic field $\mathbf{B}_0$, i.e $\mathbf{k} = k\hat{\mathbf{z}}$. Thus equation (4.7) can be written as

$$-i\omega f_1 + ikv_z f_1 + \omega_c\frac{df_1}{d\phi_v} = \frac{e}{m}\left[\mathbf{E}_1 + (\mathbf{v}\times\mathbf{B}_1)\right]\cdot\frac{\partial f_0}{\partial \mathbf{v}}, \tag{4.8}$$

where we have introduced the cyclotron frequency $\omega_c = eB_0/m$.

Now we notice that in the left hand side of (4.8) all the terms are porportional to f_1. Thus we look for solutions of the form

$$\frac{\partial f_1}{\partial \phi_v} \propto f_1. \tag{4.9}$$

Without loosing generality we propose $f_1 \propto \exp[\pm i\phi_v]$ this particular choice will be explained later. Thus equation (4.8) becomes

$$if_1(kv_z - \omega \pm \omega_c) = \frac{e}{m}\left[\mathbf{E}_1 + (\mathbf{v}\times\mathbf{B}_1)\right]\cdot\frac{\partial f_0}{\partial \mathbf{v}}. \tag{4.10}$$

It is time to use the Maxwell's equations to find a relation for the electromagnetic fields. From the Faraday's law we have

$$\nabla\times\mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t}. \tag{4.11}$$

Thus,

$$i\mathbf{k}\times\mathbf{E}_1 = i\omega\mathbf{B}_1, \tag{4.12}$$

finally the magnetic field have the form

$$\mathbf{B}_1 = \frac{k}{\omega}\left(\hat{\mathbf{z}}\times\mathbf{E}_1\right). \tag{4.13}$$

Now we are interested to expand the term

$$\begin{aligned} \mathbf{v}\times\mathbf{B}_1 &= \frac{k}{\omega}(\mathbf{v}\times(\hat{\mathbf{z}}\times\mathbf{E}_1)), \\ &= \frac{k}{\omega}\left[(\mathbf{v}\cdot\mathbf{E}_1)\hat{\mathbf{z}} - v_z\mathbf{E}_1)\right]. \end{aligned} \tag{4.14}$$

Replacing this equation into (4.10) we have

$$if_1(kv_z - \omega \pm \omega_c) = \frac{e}{m}\left[\mathbf{E}_1\left\{1 - \frac{kv_z}{\omega}\right\} + \frac{k}{\omega}\left(\mathbf{v}\cdot\mathbf{E}_1\right)\hat{\mathbf{z}}\right]\cdot\frac{\partial f_0}{\partial \mathbf{v}}. \tag{4.15}$$

At this point we look for solutions with the electric field vector in the perpendicular plane to $\mathbf{B}_0$, in particular circularly polarized fields

$$\mathbf{E}_1 = E_\perp(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}). \tag{4.16}$$

The positive sign denotes the right hand polarization, which represent the vector of electric field rotating clockwise, and the negative sign is for electric field vector rotating counter clockwise and is denoted the left hand polarization.

With this in mind,

$$\mathbf{v} \cdot \mathbf{E}_1 = E_\perp v_\perp \exp[\pm i\phi_v], \tag{4.17}$$

where we have used the identity $e^{\pm i\phi_v} = \cos\phi_v \pm \sin\phi_v$.

Then equation (4.15) is

$$if_1(kv_z - \omega \pm \omega_c) = \frac{e}{m}\left[\left\{1 - \frac{kv_z}{\omega}\right\}\mathbf{E}_1 \cdot \frac{\partial f_0}{\partial \mathbf{v}} + \frac{(kE_\perp v_\perp)}{\omega}\frac{\partial f_0}{\partial v_z}\exp[\pm i\phi_v]\right]. \tag{4.18}$$

Then due to our polarization of the electric field vector it is possible to find

$$\mathbf{E}_1 \cdot \frac{\partial f_0}{\partial \mathbf{v}} = E_\perp \frac{\partial f_0}{\partial v_\perp}\exp[\pm i\phi_v]. \tag{4.19}$$

Replacing this equation into (4.18) and solving for the distribution function we have

$$f_1(\mathbf{v}) = i\frac{e}{m}\left[\left\{1 - \frac{kv_z}{\omega}\right\}\frac{\partial f_0}{\partial v_\perp} + \frac{(kv_\perp)}{\omega}\frac{\partial f_0}{\partial v_z}\right]\frac{E_\perp \exp[\pm i\phi_v]}{(kv_z - \omega \pm \omega_c)}, \tag{4.20}$$

which agrees with the proposal in equation (4.9) and the sign $\pm$ comes from the two polarizations of the electric field.

Since we have the expression for the perturbed distribution function, now we have to find an equation for the electric field related to this distribution. In this case is more helpful to use the Ampere's law than the Poisson's equation as we did in the last chapter, since we have electromagnetic waves rather than electrostatic waves. The Ampere's law is

$$\nabla \times \mathbf{B}_1 = \mu_0 \mathbf{j_1} + \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t}; \tag{4.21}$$

where μ_0 is the magnetic permeability, $\mathbf{j_1}$ is the perturbed current density and c^2 is the velocity of light.

Replacing (4.13) in to the last equation with our oscillatory solutions, using properties of the vectorial differentiation and assuming transverse waves, $\mathbf{k} \cdot \mathbf{E}_1$, we get

$$-i\frac{k^2}{\omega}\mathbf{E}_1 = \mu_0 \mathbf{j_1} - \frac{i\omega}{c^2}\mathbf{E_1}. \tag{4.22}$$

The current density is given by the expression

$$\mathbf{j_1} = -e\int \mathbf{v} f_1(\mathbf{v})d\mathbf{v}. \tag{4.23}$$

Combining (4.22) and (4.23) we obtain

$$\mathbf{E}_1\left(\frac{k^2}{\omega} - \frac{\omega}{c^2}\right) = -ie\mu_0\int \mathbf{v} f_1(\mathbf{v})d\mathbf{v} \tag{4.24}$$

We focus on the directions perpendicular to $\hat{\mathbf{z}}$ in (4.24), and substitute (4.20) into it. The integration over the angular part ϕ_v of the velocity space is performed straighfowardly, and we obtain

$$E_\perp\left(\frac{k^2}{\omega} - \frac{\omega}{c^2}\right) = -ie\mu_0\pi\int v_\perp^2 f_1(v_\perp, v_z)dv_\perp dv_z, \tag{4.25}$$

where we have written $f_1(\mathbf{v}) = f_1(v_\perp, v_z)e^{\pm i\phi_v}$. It is possible to identify $f_1(v_\perp, v_z)$ from equation (4.20), now replacing this into (4.25) we have after some algebra

$$\omega^2 = c^2k^2 - \omega_p^2\pi \int v_\perp^2 \left[\{\omega - kv_z\} \frac{\partial \hat{f}_0}{\partial v_\perp} + \frac{kv_\perp}{\omega}\frac{\partial \hat{f}_0}{\partial v_z} \right] \frac{dv_\perp dv_z}{(kv_z - \omega \pm \omega_c)}. \tag{4.26}$$

Here we have introduced the re-normalized distribution function$\hat{f}_0 = n_0 f_0$. Assuming $\hat{f}_0$ is a Maxwellian distribution function, the integrals over $v_\perp$ are evaluated directly and we get after simplifications

$$\omega^2 = c^2k^2 + \frac{\omega_p^2\omega}{v_{th}\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp[-v_z^2/v_{th}^2]}{(kv_z - \omega \pm \omega_c)} dv_z. \tag{4.27}$$

We can rewrite this equation in the following form

$$\omega^2 = c^2k^2 + \omega_p^2 \left(\frac{\omega}{kv_{th}} \right) Z(\zeta), \tag{4.28}$$

where the function $Z(\zeta)$ is known as the plasma dispersion function, and is defined as

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{s - \zeta} ds. \tag{4.29}$$

In this case we have introduced the dimensionless variables $s = v_z/v_{th}$ and $\zeta = (\omega \mp \omega_c)/kv_{th}$.

There are many properties of this function that can be find in the literature for example [13], [14] and Fried and Conte have tabulated numerical values for this function, but in this case we will use the fact that

$$Z(\zeta) = i\sqrt{\pi}\exp(-\zeta^2) - 2\int_0^\zeta \exp(s^2 - \zeta^2) ds. \tag{4.30}$$

Replacing this equation into (4.28) we get

$$\omega^2 = c^2k^2 + i\omega_p^2 \left(\frac{\omega}{kv_{th}} \right) \sqrt{\pi}\exp(-\zeta^2) - 2\omega_p^2 \left(\frac{\omega}{kv_{th}} \right) \int_0^\zeta \exp(s^2 - \zeta^2) ds, \tag{4.31}$$

this equation is the dispersion relation. It may seem that (4.31) is the full solution, given that the properties of the plasma dispersion function can be found. However, that is not quite the case. The problem with capturing everything in a single dispersion relation is that a cyclotron damping - similar to the Landau damping - of the electric field does not correspond to a damping of the distribution function, i.e. the distribution function in the resonant region have a different behavior from the electric field. From energy conservation actually the perturbed distribution function grows when the field decays. A more complete picture accounting also for the evolution of the resonant particles will be given in the next section.

4.2 Coupled Equations

Now we are going to find the coupled differential equations using the new approach, (see discussion of section 3.6), to deal with the damping of the waves but now taking into consideration the magnetic fields. In accordance with the results in section 4.1 we propose a solution for the perturbed distribution function of the form

$$f_1 = g_1(t, v_\perp, v_z) \exp\left[i\left(kz - \omega t \pm \phi_v\right)\right]; \tag{4.32}$$

substituting this function into (4.2) the left hand side of the equation will be

$$LHS = \left[\frac{\partial g_1}{\partial t} + i\left(kv_z - \omega \pm \omega_c\right) g_1\right] \exp\left[i\left(kz - \omega t \pm \phi_v\right)\right], \tag{4.33}$$

since we have not changed the expressions of perturbed fields, the right hand side can be obtained from equation (4.20)

$$RHS = \frac{e}{m}\left[\left\{1 - \frac{kv_z}{\omega}\right\}\frac{\partial f_0}{\partial v_\perp} + \frac{(kv_\perp)}{\omega}\frac{\partial f_0}{\partial v_z}\right] E_\perp \exp\left[i\left(kz - \omega t \pm \phi_v\right)\right]. \tag{4.34}$$

The unperturbed distribution function is Maxwellian. Using this and making simplifications we get an equation of the form

$$\frac{\partial g_1}{\partial t} + i\left(kv_z - \omega \pm \omega_c\right) g_1 = \frac{e}{m} E_\perp \frac{\partial f_0}{\partial v_\perp}. \tag{4.35}$$

The evolution of the electric field is given by the equation (4.24)

$$\mathbf{E}_1\left(\frac{k^2}{\omega} - \frac{\omega}{c^2}\right) = -ie\mu_0 \int \mathbf{v} f_1 d\mathbf{v}. \tag{4.36}$$

Substituing (4.32) into (4.36) we get

$$\mathbf{E}_1\left(\frac{k^2}{\omega} - \frac{\omega}{c^2}\right) = -ie\mu_0 \int \mathbf{v} g_1 \exp\left(\pm i\phi_v\right) d\mathbf{v}. \tag{4.37}$$

Using (4.16) and carrying out the integrals over the angular variable we obtain

$$E_\perp\left(\frac{k^2}{\omega} - \frac{\omega}{c^2}\right) = -i\pi e\mu_0 \int v_\perp^2 g_1 dv_\perp dv_z. \tag{4.38}$$

We assume now that the distribution is factorable so $g_1(t, v_\perp, v_z) = n_0 \tilde{g}(t, v_z) h(v_\perp)$. Replacing this into the last equation we get

$$E_\perp\left(\frac{k^2}{\omega} - \frac{\omega}{c^2}\right) = -i\pi e n_0 \mu_0 \alpha \int_{-\infty}^{\infty} \tilde{g}(t, v_z) dv_z, \tag{4.39}$$

where

$$\alpha = \int_0^\infty v_\perp^2 h(v_\perp) dv_\perp. \tag{4.40}$$

It is time to separate our integral in the non-resonant and resonant part, which will give us

$$E_\perp\left(\frac{k^2}{\omega} - \frac{\omega}{c^2}\right) = -i\pi e n_0 \mu_0 \alpha \int_{nr} \tilde{g}(t, v_z) dv_z - i\pi e n_0 \mu_0 \alpha \int_r \tilde{g}(t, v_z) dv_z. \tag{4.41}$$

Since the electric field has a slow time dependence, in the left hand side of (4.41) we can replace $\omega \to \omega + i\partial/\partial t$ which will give us

$$\left(\frac{k^2}{\omega} - \frac{\omega}{c^2} - i\left(\frac{k^2}{\omega^2} + \frac{1}{c^2}\right)\frac{\partial}{\partial t}\right) E_\perp = -i\pi e n_0 \mu_0 \alpha \int_{nr} \tilde{g}(t, v_z) dv_z - i\pi e n_0 \mu_0 \alpha \int_r \tilde{g}(t, v_z) dv_z. \tag{4.42}$$

From equation (4.35) we can find for the non resonant region

$$\tilde{g} = -\left\{\frac{\partial \tilde{g}}{\partial t} - \frac{e}{m}\frac{E_\perp}{h(v_\perp)}\frac{\partial f_0}{\partial v_\perp}\right\}\frac{1}{i\left(kv_z - \omega \mp \omega_c\right)}. \tag{4.43}$$

Combining both equations, we obtain

$$\begin{aligned} E_{\perp}\left(\frac{k^2}{\omega}-\frac{\omega}{c^2}\right) =&\pi e n_0\mu_0\alpha\int_{nr}\left\{\frac{\partial\tilde{g}}{\partial t}-\frac{e}{m}\frac{E_{\perp}}{h(v_{\perp})}\frac{\partial f_0}{\partial v_{\perp}}\right\}\frac{1}{(kv_z-\omega\pm\omega_c)}dv_z \\ &- i\pi e n_0\mu_0\alpha\int_r \tilde{g}(t,v_z)dv_z + i\left(\frac{k^2}{\omega^2}+\frac{1}{c^2}\right)\frac{\partial E_{\perp}}{\partial t}. \end{aligned} \tag{4.44}$$

Re-ordering we have

$$\begin{aligned} E_{\perp}\left(k^2c^2-\omega^2+\pi\omega_p^2\omega\alpha\int_{nr}\frac{1}{h(v_{\perp})}\frac{\partial f_0}{\partial v_{\perp}}\frac{dv_z}{(kv_z-\omega\mp\omega_c)}\right) =&\pi\frac{e n_0}{\varepsilon_0}\alpha\omega\int_{nr}\frac{\partial\tilde{g}}{\partial t}\frac{dv_z}{(kv_z-\omega\pm\omega_c)} \\ &- i\pi\frac{e n_0}{\varepsilon_0}\omega\alpha\int_r\tilde{g}(t,v_z)dv_z \\ &+ i\left(k^2c^2+\omega^2\right)\frac{1}{\omega}\frac{\partial E_{\perp}}{\partial t}. \end{aligned} \tag{4.45}$$

We can notice that all the expression inside the brackets is similar to equation (4.27) since f_0 is a Maxwellian distribution. To be precise we want that both expressions to be the same, the only way that this is possible is if we set

$$h(v_{\perp}) = \frac{\alpha n_0}{v_{th}^2}\left(-\frac{2v_{\perp}}{v_{th}^2}\right)\exp\left(-v_{\perp}^2/v_{th}\right). \tag{4.46}$$

Thus our equation can be written as follows

$$\begin{aligned} E_{\perp}\left(c^2k^2-\omega^2+\frac{\omega_p^2\omega}{v_{th}\sqrt{\pi}}\int_{nr}\frac{\exp[-v_z^2/v_{th}^2]}{(kv_z-\omega\pm\omega_c)}dv_z\right) =&\pi\frac{e n_0}{\varepsilon_0}\alpha\omega\int_{nr}\frac{\partial\tilde{g}}{\partial t}\frac{dv_z}{(kv_z-\omega\pm\omega_c)} \\ &- i\pi\frac{e n_0}{\varepsilon_0}\omega\alpha\int_r\tilde{g}(t,v_z)dv_z \\ &+ i\left(k^2c^2+\omega^2\right)\frac{1}{\omega}\frac{\partial E_{\perp}}{\partial t}. \end{aligned} \tag{4.47}$$

The left hand side is the dispersion relation excluding the resonant part, since the resonant region is small compared to the rest of the distribution, is possible to find values for ω and k that fullfil the relation so this term vanishes. Leaving the equation

$$\int_{nr}\frac{\partial\tilde{g}}{\partial t}\frac{dv_z}{(kv_z-\omega\pm\omega_c)} + i\frac{\varepsilon_0}{\pi e n_0\alpha}\left(1+\left(\frac{kc}{\omega}\right)^2\right)\frac{\partial E_{\perp}}{\partial t} = i\int_r\tilde{g}(t,v_z)dv_z. \tag{4.48}$$

In the non resonant region we consider that $\partial\tilde{g}/\partial t \ll (kv_z-\omega\pm\omega_c)\tilde{g}$, thus we can conclude from (4.35) that

$$\tilde{g} = -i\frac{e}{mh(v_{\perp})}\frac{\partial f_0}{\partial v_{\perp}}\frac{E_{\perp}}{(kv_z-\omega\pm\omega_c)}, \tag{4.49}$$

then

$$\frac{\partial\tilde{g}}{\partial t} = -i\frac{e}{mh(v_{\perp})}\frac{\partial f_0}{\partial v_{\perp}}\frac{\partial E_{\perp}}{\partial t}\frac{1}{(kv_z-\omega\mp\omega_c)}. \tag{4.50}$$

Replacing (4.50) into (4.48) we have

$$\frac{\partial E_{\perp}}{\partial t} = C(k,\omega)\int_r\tilde{g}(t,v_z)dv_z, \tag{4.51}$$

where

$$C(k,\omega) = \left[-\frac{e}{mh(v_{\perp})}\int_{nr}\frac{\partial f_0}{\partial v_{\perp}}\frac{dv_z}{(kv_z-\omega\pm\omega_c)^2} + \frac{e}{m}\frac{1}{\pi}\frac{1}{\alpha\omega_p^2}\left(1+\left(\frac{kc}{\omega}\right)^2\right)\right]^{-1}. \tag{4.52}$$

Finally the set of coupled equations that we have to solve are

$$\begin{aligned}\frac{\partial E_\perp}{\partial t} &= C(k,\omega)\int_r \tilde{g}(t,v_z)dv_z \\ \frac{\partial \tilde{g}}{\partial t} &= -i\left(kv_z - \omega \pm \omega_c\right)\tilde{g} + \frac{e}{m}\frac{E_\perp}{h(v_\perp)}\frac{\partial \hat{f}_0}{\partial v_\perp}\end{aligned} \tag{4.53}$$

4.3 Numerical Solutions

Since the system of equations presented before are self-consistent is not possible to solve them analytically, for such reason we are to implement a numerical scheme to solve them, but before starting it is important to simplify the equation introducing normalized variable to get rid of the dimensions and make it more easy to handle numerically.

The variables that we introduce are

$$\begin{aligned}t_n &= \gamma_L t, \\ E_n &= \frac{e}{m}\frac{\partial f_0}{\partial v_\perp}\frac{E_\perp}{h(v_\perp)\gamma_L}, \\ v_n &= \frac{k}{\gamma_L}\left(v_z - \left(\frac{\omega \mp \omega_c}{k}\right)\right).\end{aligned} \tag{4.54}$$

where γ_L is the damping factor.

With this change of variable we obtain the normalized equations

$$\begin{aligned}\frac{\partial E_n}{\partial t_n} &= H(v_n)\int_r \tilde{g}(t,v_n)dv_n, \\ \frac{\partial \tilde{g}}{\partial t_n} &= -iv_n\tilde{g} + E_n.\end{aligned} \tag{4.55}$$

Where

$$H(v_n) = \left.\frac{e}{m}\frac{\partial f_0}{\partial v_\perp}\frac{1}{h(v_\perp)}\frac{C(k,\omega)}{k\gamma_L}\right|_r; \tag{4.56}$$

this factor is a constant value when we evaluate this function exactly in the resonant region[1].

Now that we have deduced the normalized system of equations we are interested to study the evolution of the electric field and the distribution function inside the resonant region. For such reason we are going to use a discrete scheme to solve the differential equations. The scheme we will implement is the staggered leapfrog method [15], we picked this method because is second order accurate in time.

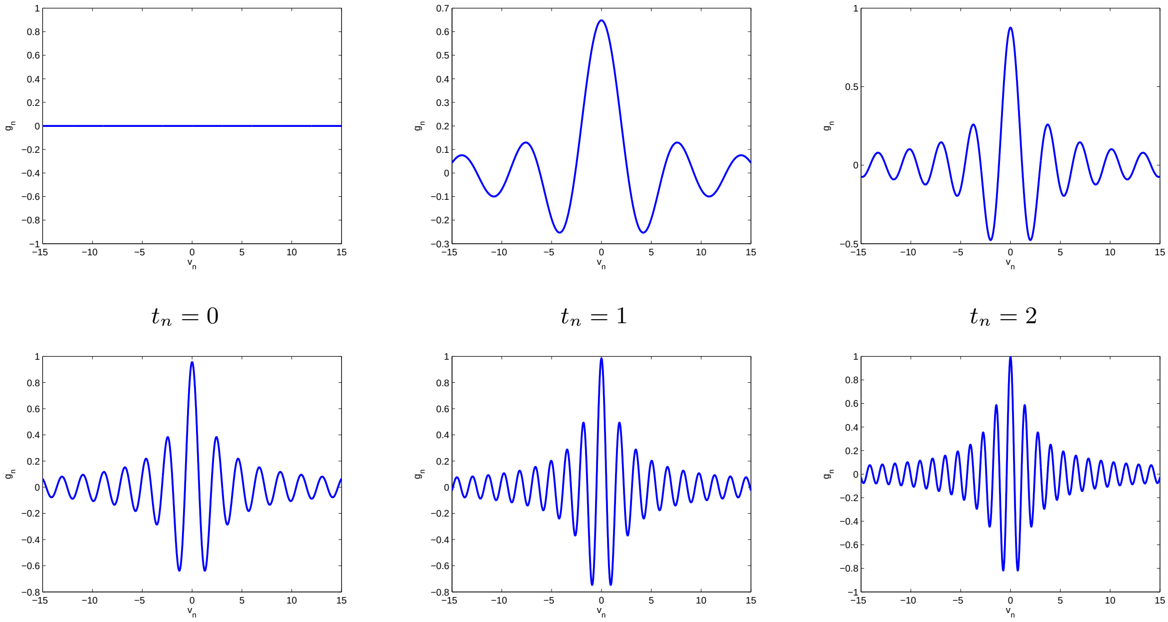
Our equations discretized are

$$\begin{aligned}\tilde{g}^{(j+1)} &= \tilde{g}^{(j)} - ih\tilde{g}^{(j+1/2)} + hE_n^{(j+1/2)}, \\ E_n^{(j+1)} &= E_n^{(j)} - \frac{h}{\pi}\int \tilde{g}^{(j+1/2)}dv_n, \\ \tilde{g}^{(j+1/2)} &= \tilde{g}^{(j-1/2)} - ih\tilde{g}^{(j)} + hE_n^{(j)}, \\ E_n^{(j+1/2)} &= E_n^{(j-1/2)} - \frac{h}{\pi}\int \tilde{g}^{(j)}dv_n.\end{aligned} \tag{4.57}$$

[1]For the numerical solutions we have found that the value of $H(v_n)$ in the resonant region is equal to $-1/\pi$.

The integration will be done using the Simpson's rule 1/3, The code for this scheme is presented in the appendix.

As first example we will consider an initial condition given by $\tilde{g}(t = 0, v_n) = 0$, the evolution of the distribution function for different times is given in Figure 4.1.



$t_n = 0$ $t_n = 1$ $t_n = 2$

$t_n = 3$ $t_n = 4$ $t_n = 5$

Figure 4.1: Evolution of the distribution function for different times.

We notice that the distribution function starts to oscillate according the time passes. This is due to the fact of "phase mixing", which is an indication that that we have different frequencies for different points in the velocity space. Since the evolution of the electric field according to equation (4.51) depends on the distribution function, this phase mixing will produce the reduction of magnitude of the electric field. The electric field is depicted in the next figure.

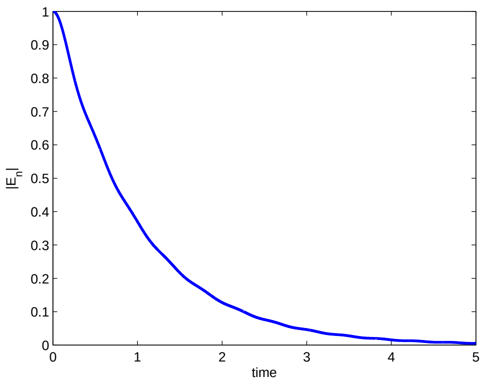


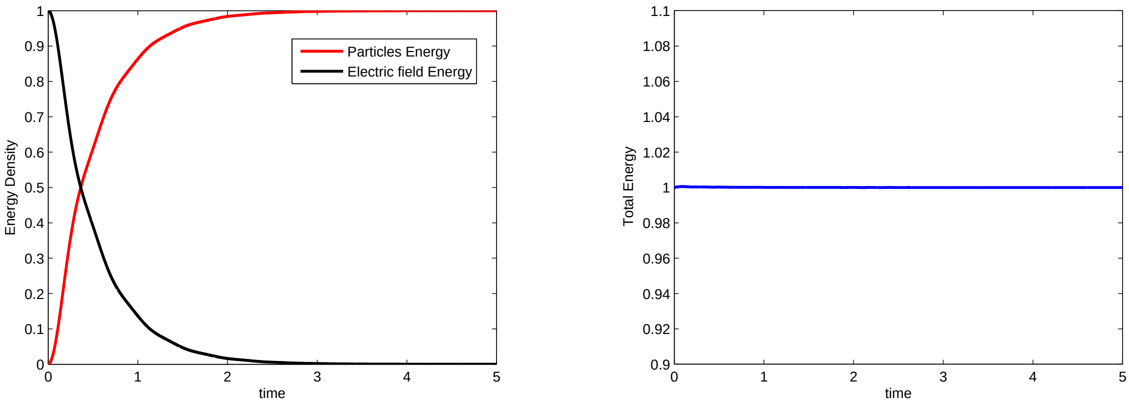
Figure 4.2: Evolution of the electric field.

As we expected the we have a damping of the electric field due to the phase mixing. Now it is important to consider the energy of the system, since in our case we are dealing with a collisionles plasma there has to be conservation of the total energy density. Figure 4.3 a) shows the relation between the particle density energy and the electric density energy. As the electric energy starts to decay the energy of the particles start to grow. This feature of our problem also gave us the opportunity to test the reliability of our code. In Figure 4.3 b) we can observe the total energy of the system. There is a small variation at the beginning in the Figure 4.3 b), this is due to the fact that we have to guess the first step for our code.

As a second example we use an intial distribution of the form:

$$\tilde{g}(t_n = 0) = \frac{E_n}{(iv_n - \gamma)} \tag{4.58}$$

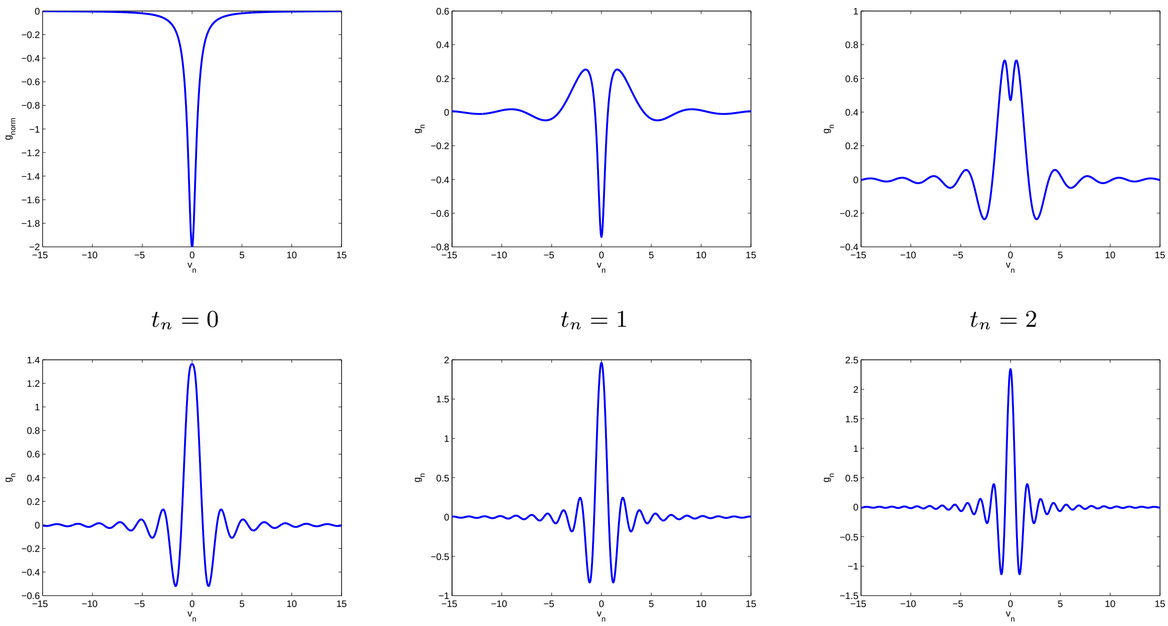
The evolution of the distribution function is depicted in Figure 4.4.



a *b*

Figure 4.3: Conservation energy of the system for an initial condition $\tilde{g}(t = 0, v_n) = 0$.

We can see in Figure 4.4 the oscillations are not so prominent in the first 2 seconds, this indicates the phase mixing is not dominating in the beginning, but after some time it starts to gain importance.



$t_n = 0$ $t_n = 1$ $t_n = 2$

$t_n = 3$ $t_n = 4$ $t_n = 5$

Figure 4.4: Evolution of the distribution function for different times.

The evolution of the electric field is shown in Figure 4.5. In this picture we can see how the electric field starts to grow at the beginning reaching its maximum, then as the time goes on the magnitude of the electric field start to decay. This is accordance what we have seen in the evolution of the distribution function, the phase mixing start to gain importance after the electric field reach the maximun.

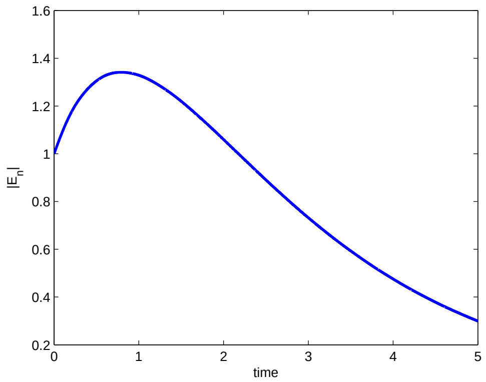
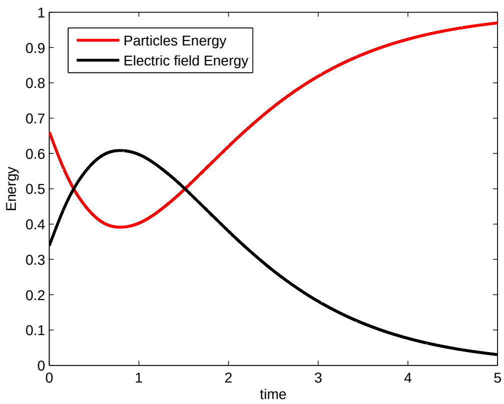
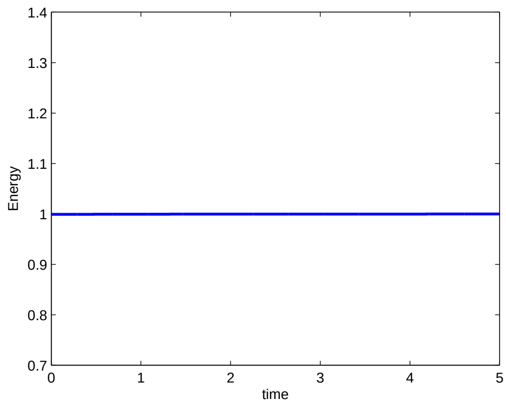


Figure 4.5: Evolution of the electric field.

This means that for energy conservation we have that the energy density of the particles decreases at the beginning and the electric energy density rises, until the electric field reaches the maximum. Then the particles start to gain energy and the electric field start to lose energy. This behavior is depicted in Figure 4.6



a)



b)

Figure 4.6: a) Conservation energy of the system for the second example. b) The numerical conservation of energy.

Chapter 5

Summary and Conclusions

In this thesis we have focused on the study of weak cyclotron damping. For such reason we have made a review of basic concepts of plasma physcis, we covered the standard theory of Landau damping in unmagnetized plasmas. Furthermore, we have compared these techniques with the one proposed by Brodin [7] to show the efficiency and reliability of the method.

The cyclotron damping is different from the Landau damping in many aspects, first for example the probabillity to have have trapped particles inside the potential is minimum. Second the presence of the magnetic field will produce a Doppler shifting to its cyclotron frequency. This resonance depend on the polarization of the electric field and how the particles rotate around the magnetic field. Finally since the electron are rotating around the external magnetic field in the opposite direction of the electric field, some of these particles will in some point feel the oscillations at their own cyclotron frequency and this will produce the damping of the plasma wave. This effect is present only when we have left polarization when $\omega \to \omega_c$.

Since the new approach described in [7] worked with the case of Landau damping, we have applied the same technique to the case where the plasma is embeded in a magnetic field. Using perturbation theory we have derived a couple set of linearized equations, wich describe the coupled evolution of the distribution function for resonant particles and the electric field amplitude of the electromagnetic waves.

We have performed the analysis of the evolution of the distribution function in the resonant region, where we notice the oscillations come in part from first term of the equation (4.53), wich will produce frequencies that will depend on the velocity. This is why we have phase mixing in the velocity space. The phase mixing induces the particles to take energy from the electromagnetic wave making to the wave decrease its amplitude.

In the literature the treatment of the cyclotron damping is related to the computation of the dispersion relation of the wave using the dielectric tensor, but the problem with this approach is that we have no a clear picture of the behavior of the distribution function. In this work we have shown the evolution of the perturbed distribution function and the magnitude of the electric field.

This method allow us to have an explicit dependence on the initial conditions. Computing the dispersion relation as in the standard theory we expect damping of the electric field and distribution function in all the cases, but we have noticed that this behavior is not always true. In both of our examples we have seen the formation of oscillation in the distribution functions due to the phase mixing. Moreover, in the second example we can notice that for certain intial conditions it is possible to have a growing electric field untill it reaches a maximum and then start to decay.

The code we have implemented with the staggered leapfrog method allows to check the energy conservation of the system. This method is well suited for this case because we use a staggered grid wich has shown stability when it is applied to problems involving electromagnetic fields. It is important to point out that the method need small steps sizes, specially because the distribution function develops fine structures in the velocity space. Thus, the numerical results will become inaccurate when the grids are not fine enough to resolve these fine velocity distribution structures.

Finally, in this project mainly we have worked with the linear problem the next work would be to include higher harmonics of the the distribution function in the treatment. Also, we have worked with a non degenerate plasma with a Maxwellian background distribution and could be interesting to modify the problem introducing the Fermi-Dirac distribution and study the case of degenerate plasma.

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[1] There is a misprint in formula (17) of this reference (p. 104): replace $e^{-(ka)^2/2}$ by $e^{-1/(2(ka)^2)}$.

Appendix

Here we found all the numerical codes in MATLAB used in this project. The main program is:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%  DIFFERENTIAL EQUATIONS %%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

close all
clear all
clc

x0=-15;
xf=15;
N=1000;
h=(xf-x0)/N;
vz=x0:h:xf;
gvzo=distribucion(vz);
[r,c]=size(gvzo);
dt=0.05;
to=0;
tf=2.5;
Nt=tf/dt;
E=zeros(Nt,1);
E_two=zeros(Nt,1);
%%%% Initial Conditions
E(1)=1;
g_one=gvzo;
%%%% Distribution square
gsqr=g_one.*conj(g_one);
%%%% Initial energy
We=0.5*E(1)*conj(E(1));
Wp=(0.5/pi)*simpsrule(h,gsqr);
Winit=We+Wp;
%%%% first step
E_two(1)=E(1)+(0.5*(dt/pi)*simpsrule(h,g_one));
for k=1:c
    g_two(k)=g_one(k)*(1+j*0.5*dt*vz(k))-dt*E(1);
end
Energy=zeros(Nt,1);
%%%% All the steps
for l=1:Nt
    for y=1:c
        g_one(y)=g_one(y)-j*dt*(vz(y)*g_two(y))+dt*E_two(l);
```

```
        gsqr(y)=g_one(y)*conj(g_one(y));
        g_two(y)=g_two(y)-j*dt*(vz(y)*g_one(y))+dt*E(l);
    end
    E(l+1)=E(l)-(dt/pi)*simpsrule(h,g_two);
    E_two(l+1)=E_two(l)-(dt/pi)*simpsrule(h,g_one);
    Wp(l)=(0.5/pi)*simpsrule(h,gsqr)/Winit;
    WE(l)=0.5*E(l)*conj(E(l))/Winit;
    Wtot(l)=Wp(l) + WE(l);
    plot(real(g_one))
    pause(0.1)
end
%%%% Las salidas son los vectores Wtot, Wp, WE
```

Inside the main program we have two functions for the integration using the simpson's rule and the initial distribution function

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%  SIMPSON'S RULE 1/3  %%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function res = simpsrule(h,y)
N = length(y);

acods = 0;
acevns = 0;

for i=2:2:N
    acevns = acevns + 4*y(i);
end

for i = 3:2:N-1
    acods = acods+2*y(i);
end

res = (h/3)*(y(1) + acevns + acods+y(N));
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Initial distribution functions %%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function rolo=distribucion(x)
[n,m]=size(x);
h=x(2)-x(1);
% Initial conditions
%rolo=(1./(j*x - 0.5));
rolo=((1/pi)./(x.*x + 0.5));
end
```