Estimation of the local Hurst function of multifractional Brownian motion

A second difference increment ratio estimator

Simon Edvinsson
Abstract

In this thesis, a specific type of stochastic processes displaying time-dependent regularity is studied. Specifically, multifractional Brownian motion processes are examined. Due to their properties, these processes have gained interest in various fields of research. An important aspect when modeling using such processes are accurate estimates of the time-varying pointwise regularity. This thesis proposes a moving window ratio estimator using the distributional properties of the second difference increments of a discretized multifractional Brownian motion. The estimator captures the behaviour of the regularity on average. In an attempt to increase the accuracy of single trajectory pointwise estimates, a smoothing approach using nonlinear regression is employed. The proposed estimator is compared to an estimator based on the Increment Ratio Statistic.

Sammanfattning

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1 Introduction

Multifractional Brownian motion (mBm) processes have become an area of interest in various fields of research, such as finance, internet traffic, image processing, and terrain modeling (see, Bianchi (2005); Ayache et al. (2000); Echelard et al. (2010)). The mBm is a special case of a process with variable smoothness as independently introduced by Peltier and Véhel (1995) and Benassi et al. (1997). Described by Mandelbrot (1983), fractal geometry can be used to describe the irregularities observed in nature. This notion motivates the application of stochastic fractal processes to model various phenomena as observed in the real world. This thesis proposes a pointwise Second Difference Ratio (SDR) estimator for the estimation of the time-varying irregularity function of mBm processes.

1.1 Background

A classic example of a stochastic fractal process is the fractional Brownian motion (fBm) first studied by Kolmogorov (1940) and later defined by Mandelbrot and Van Ness (1968). The fBm is characterized by the Hurst exponent, which describes the regularity of the process. This property makes the fBm suitable to model various processes in a parsimonious way, in the sense that their properties can be adjusted by varying a single parameter. In image processing, fractal processes have been used to generate images, classify textures, and calculate the length of coastlines (see, e.g., Fournier et al. (1982); Peleg et al. (1984); Mandelbrot (1975)). Furthermore, the property of long term dependence makes the fBm a more realistic model for modeling financial assets compared to classical models (see Mandelbrot and Van Ness (1968)).

However, a limitation of the fBm is that the regularity of the process remains constant at all time points. Generalizing the fBm to incorporate time-varying pointwise regularity enables a more natural modeling environment. Multifractional Brownian motion (mBm) is a generalization of the fBm, introducing a time-varying Hurst function enabling the fractional process to dynamically change its pointwise regularity. The mBm process was independently introduced by Peltier and Véhel (1995) and Ayache and Vehel (2000) and has since then attracted increased interest.

The research regarding mBm processes is an area that has gained traction as an increasing number of fields of research are considering the possibility of time-varying regularity in processes. The mBm process, in the same sense as the fBm, provides a parsimonious modeling environment using a time-varying Hurst function, which enables a wide range of possible applications which makes it a interesting area of research.

1.2 Problem definition

The starting point to mBm modeling is obtaining accurate estimates of the process time-varying regularity. There are several proposed estimators (see, e.g., Benassi et al. (1998); Istas and Lang (1997)). A pointwise moving window approximation method using separate estimation of a scaling constant has been proposed by Bianchi et al. (2013). Another example of these estimators is the Increment Ratio Statistic (IRS) as developed by Surgailis et al. (2008) and applied to multifractional Brownian motion processes by Bardet and Surgailis (2013). The IRS estimator will be used for comparison purposes in this thesis.

In this thesis the distributional properties of the second difference increments of mBm processes and the moving window approach of Bianchi et al. (2013) are combined to derive the SDR estimator, which belongs to the class of ratio estimators defined by Benassi et al. (1998). The reason for basing the estimator on the second difference increments is that it will be possible to estimate the Hurst function on its whole range. This would not be possible if the estimator was based on first difference increments (see Bardet and Surgailis (2011)). The idea of deriving the SDR estimator was originally proposed by professor Yuliya Mishura.

When evaluating the SDR estimator, it can be concluded that it manages on average to estimate the Hurst function. The computed error measures for the SDR estimator and the IRS estimator are comparable. However, the single trajectory estimates are volatile, which motivates a smoothing approach in an attempt to increase the accuracy of the pointwise estimates. The application of nonlinear regression as a smoothing technique manages to reduce the error of the pointwise estimates when the Hurst function is specified as a power function.
This thesis is organized as follows. Firstly in Section 2, fractional Brownian motion and multifractional Brownian motion processes are defined. Secondly, local properties of a increment mBm process and distributional properties of the increments are presented. Lastly, the section is concluded with the definition of the SDR estimator as well as the introduction of the IRS estimator. In Section 3, the main results are presented. Firstly, the results for estimation of the Hurst exponent are evaluated. Secondly, the estimators are evaluated for various Hurst functions. Lastly, the section is concluded with the results from the smoothing of the pointwise estimates using nonlinear regression. The thesis is concluded with a discussion of the results and future research. The appendix concludes with a complete collection of figures and tables that are not displayed in the main body of the thesis.

## 2 Fractional Brownian processes and local pointwise regularity

In this section, the underlying theory needed for deriving the estimator is presented. With the properties of fractional Brownian motion (fBm) as an outset, a generalization to multifractional Brownian motion (mBm) with a time-varying Hurst function is made. In the following section, the distributional properties of the second difference increments of a discretized mBm process are shown. Finally, the SDR estimator is defined and presented as well as an estimator based on the Increment Ratio Statistic.

### 2.1 Stochastic processes

With the definition of stochastic processes as an outset, some important properties are defined that will be of use in the proceeding sections of this thesis.

**Definition 1.** A *stochastic process* \( X = \{X(t, \omega), t \in T, \omega \in \Omega\} \), is a set of random variables on a common probability space \((\Omega, \mathcal{F}, P)\) indexed by a parameter \( t \in T \subset \mathbb{R} \).

Clearly, if a fixed \( t \) is considered \( \{X(t, \cdot), t \in T\} \) will be a collection of random variables. Correspondingly, if a fixed \( \omega \) is considered \( \{X(\cdot, \omega), \omega \in \Omega\} \) is a realization of the stochastic process. For notational convenience we will use \( \{X(t), t \in T\} = \{X(t, \omega), t \in T, \omega \in \Omega\} \) when referring to a stochastic process.

**Definition 2.** We say that a stochastic process \( \{X(t), t \in T\} \) is a *Gaussian process* if for every \( n \in \mathbb{N}^+ \) and every finite subset \( \{t_1, \ldots, t_n\} \) of \( T \), the random vector \((X(t_1), \ldots, X(t_n))\) is multivariate normally distributed.

A property of Gaussian processes is that they are uniquely defined by their mean and covariance functions.

**Definition 3.** We say that a stochastic process \( \{X(t), t \in T\} \) is *selfsimilar* with index \( H > 0 \) if for any \( \alpha > 0 \), \( \{X(\alpha t)\} \overset{d}{=} \{\alpha^H X(t)\} \).

It is of importance to note that the Hurst exponent, which quantifies the pointwise regularity of the process, is denoted by \( H \).

### 2.2 Fractional Brownian motion

In this section, the fractional Brownian motion (fBm) is defined following the definition given in [Mishura (2008)](#).

**Definition 4.** The two-sided, normalized *fractional Brownian motion* with Hurst exponent \( H \in (0, 1) \) is a Gaussian process \( B_{H,K} = \{B_{H,K}(t), t \in \mathbb{R}\} \), on the complete probability space \((\Omega, \mathcal{F}, P)\), having the properties

(i) \( B_{H,K}(0) = 0 \), \( \text{a.s.} \),

(ii) \( \mathbb{E}B_{H,K}(t) = 0 \), \( t \in \mathbb{R} \),

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The integral representation of fBm can be written as

\[ B_{H,t}(t) := KV_{Ht}^{1/2} \int_{\mathbb{R}} f_t(s) dB_{1/2}(s), \]

where \( f_t(s) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \left(|t - s|^{H-1/2} 1_{(-\infty,t]}(s) - |s|^{H-1/2} 1_{(-\infty,0]}(s)\right), \)

and \( 1_{(\cdot)} \) denotes the indicator function. \( V_{Ht} = \Gamma(2H + 1) \sin(\pi H) \) is a normalizing factor, and \( K > 0 \) is a scaling constant (see [Peltier and Véhel (1995)]. Choosing the normalizing factor to be as above gives

\[ \mathbb{E}\left[(B_{H,K}(t) - B_{H,K}(s))B_{H,K}(u) - B_{H,K}(v)\right] = \frac{1}{2} K^2(|s - u|^{2H} + |t - v|^{2H} - |t - u|^{2H} - |s - v|^{2H}). \]

The regularity of the fBm can be varied by specifying values of the Hurst exponent \( H \). For example, taking \( H \) equal to 1/2 gives the process commonly referred to as **standard Brownian motion** or **Wiener process**, \( B_{1/2,K} \). This process is the only case where the fBm has stationary independent increments. For \( H \in (0,1/2) \), the increments are negatively correlated, whereas for \( H \in (1/2,1) \) the increments are positively correlated. For positively correlated increments, the process \( B_{H,K} \) is said to have the property of long term dependence (see [Mishura (2008)]).

In Figure 4, three sample fBm trajectories for \( H \in \{0.2,0.5,0.8\} \) are plotted illustrating the regularity of processes with various Hurst exponents. It is evident that a higher value for \( H \) produces a more regular trajectory.

### 2.3 Multifractional Brownian motion

The multifractional Brownian motion (mBm) as defined by [Levy-Vehel (1995)] is a process with time-varying pointwise regularity. The mBm is a generalization of the fBm obtained by letting the pointwise regularity parameter \( H \) vary over time.

**Definition 5.** We say that a function \( f : X \to Y \) is **Hölder continuous** of exponent \( \beta > 0 \), if for each \( x, y \in X \) such that \( |x - y| < 1 \), we have \( |f(x) - f(y)| \leq C|x - y|^\beta \) for some positive constant \( C \).

**Definition 6.** Let the Hurst function \( H_t \in (0,1) \) for \( t \in \mathbb{R} \) be Hölder continuous with some \( \beta > 0 \). The two-sided, normalized **multifractional Brownian motion** (mBm) is a Gaussian process \( B_{H_t} = \{B_{H_t}(t), t \in \mathbb{R}\} \) on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), having the properties

(i) \( B_{H_0,K}(0) = 0 \), a.s.,

(ii) \( \mathbb{E}B_{H_0,K}(t) = 0 \), \( t \in \mathbb{R} \),

(iii) \( \mathbb{E}B_{H_0,K}(t)B_{H_0}(s) = D(H_t, H_s)K^2 \left(|t|^{H_t+H_s} + |s|^{H_t+H_s} - |t-s|^{H_t+H_s}\right), \)

for \( t, s \in \mathbb{R} \), where

\[ D(H_t, H_s) = \frac{\sqrt{\Gamma(2H_t + 1)\Gamma(2H_s + 1) \sin(\pi H_t) \sin(\pi H_s)}}{2\Gamma(H_t + H_s + 1) \sin(\pi(H_t + H_s)/2)}. \]

The integral representation of mBm can be written as follows,

\[ B_{H_t,K}(t) := KV_{H_t}^{1/2} \int_{\mathbb{R}} f_t(s) dB_{1/2}(s), \]

\[ f_t(s) = \frac{1}{\Gamma\left(H_t + \frac{1}{2}\right)} \left(|t - s|^{H_t-1/2} 1_{(-\infty,t]}(s) - |s|^{H_t-1/2} 1_{(-\infty,0]}(s)\right). \]
Figure 1: Sample trajectories of fBm processes with $H \in \{0.2, 0.5, 0.8\}$ on $t \in (0, 1)$. In the upper panel, the fBm sample trajectory for $H = 0.2$ is shown. In the middle panel, a sample trajectory for $H = 0.5$, and in the lower panel, a sample trajectory for $H = 0.8$ are shown.

Figure 2: In the upper panel, the Hurst function $H_t = 0.35 + 0.4t^2$ on the interval $t \in (0, 1)$ is displayed. In the lower panel, a corresponding mBm realization is presented illustrating the time-varying regularity.
Here, $\mathbb{1}_\{{x}\}$ denotes the indicator function, $V_{H_t} = \Gamma(2H_t + 1) \sin(\pi H_t)$ is a normalizing function, and the $K > 0$ is a scaling constant (see Bianchi et al. (2013)). Note that other integral representations of the mBm exists (see, e.g., Coeurjoly (2005); Mandelbrot and Van Ness (1968)).

In Figure 2 an example of a time-varying Hurst function and a corresponding mBm realization are shown. It is evident that an increasing value of $H_t$ corresponds to an increasingly regular mBm trajectory. The effect of a varied Hurst exponent, as observed in Figure 1, is integrated into one process.

### 2.4 MBm process regularity estimation

This section begins with the explanation of important assumptions regarding the local properties of a discretized mBm process. Furthermore, distributional properties of the second difference increments of a mBm process are presented. Using these results the SDR estimator is defined. For comparison, a ratio estimator based on the Increment Ratio Statistic as developed by Surgailis et al. (2008) is presented. The section is concluded with a short explanation of nonlinear regression.

#### 2.4.1 Approximate local behaviour

Denote a increment mBm process as $Y(t, au) = B_{H_t+u} - B_{H_t}$ for some scaling factor $a$, then

\[ a^{-H_t} Y(t, au) \mathop{\rightarrow}^{d} B_{H_t}(u) \quad \text{as} \quad a \rightarrow 0^+, \]

where $u \in \mathbb{R}$. The increments of the mBm process asymptotically converges in distribution to those of an fBm when the scaling factor $a \rightarrow 0^+$ (see Benassi et al. (1998)). The variance of a infinitesimal increment of a mBm process thus becomes $\mathbb{E} \left( B_{H_t}(u) \right)^2 = K^2 u^{2H_t}$ considering the scaling constant $K$. This means that in the neighborhood of $t$, an increment of a mBm process assumes to behave locally as a fBm with Hurst exponent $H_t$ (see Bianchi and Pianese (2014)). The variance of the increments of the mBm is then

\[ \text{var}(B_{H_t}(t) - B_{H_t}(s)) = \mathbb{E}[(B_{H_t}(t) - B_{H_t}(s))^2] \]

\[ = \mathbb{E}[B_{H_t}^2(t) + B_{H_t}^2(s) - 2B_{H_t}(t)B_{H_t}(s)] \]

\[ = K^2 t^{2H_t} + K^2 s^{2H_t} - 2 \mathbb{E}[B_{H_t}(t)B_{H_t}(s)] \]

\[ = K^2 [t^{2H_t} + s^{2H_t} - 2D(H_t, H_s)(t^{H_t + H_s} + s^{H_t + H_s} - |t - s|^{H_t + H_s})]. \]

Assuming that $|H_t - H_s| = 0$ as $|t - s| \rightarrow 0$, we have that $D(H_t, H_s) \approx \frac{1}{2}$ for $t \approx s$, which gives the variance

\[ \text{var}(B_{H_t}(t) - B_{H_t}(s)) \approx K^2 |t - s|^{2H_t}. \]

Note that for notational convenience, $B_{H_t}(t) = B_{H_t}(t)$. The variance of the second difference increments of a mBm process is then

\[ \mathbb{E}[(B_{H_t}(u) - B_{H_t}(t)) - (B_{H_t}(t) - B_{H_t}(s))]^2 \]

\[ = \mathbb{E}[(B_{H_t}(u) - B_{H_t}(t))^2] + \mathbb{E}[(B_{H_t}(t) - B_{H_t}(s))^2] + 2 \mathbb{E}[(B_{H_t}(u) - B_{H_t}(t))(B_{H_t}(t) - B_{H_t}(s))]. \]

Since the increment variance can be calculated using (3), what remains is to calculate the covariance term.

\[ 2 \mathbb{E}[(B_{H_t}(u) - B_{H_t}(t))(B_{H_t}(t) - B_{H_t}(s))] \]

\[ = 2K^2 \left[ D(H_u, H_t) \left( |u|^{H_t + H_s} + |t|^{H_t + H_s} - |u - t|^{H_t + H_s} \right) \right. \]

\[ - D(H_u, H_s) \left( |u|^{H_t + H_s} + |s|^{H_t + H_s} - |u - s|^{H_t + H_s} \right) \]

\[ + D(H_t, H_s) \left( |s|^{H_t + H_s} + |t|^{H_t + H_s} - |t - s|^{H_t + H_s} \right) - |u|^{2H_t} \right]. \]
Utilizing the assumption that $H_u \approx H_t \approx H_s$, all the $D(\cdot, \cdot)$ terms are equal to $1/2$. Then it is possible to calculate the covariance term of the second difference increments as

$$
2\mathbb{E}[(B_{H_u}(u) - B_{H_t}(t))(B_{H_t}(t) - B_{H_s}(s))]
$$

$$
= 2K^2\left[\frac{1}{2}(|u|^{2H_u} + |t|^{2H_u} - |u - t|^{2H_u}) - \frac{1}{2}(|u|^{2H_u} + |s|^{2H_u} - |u - s|^{2H_u}) + \frac{1}{2}(|u|^{2H_u} + |s|^{2H_u} - |t - s|^{2H_u}) - |t|^{2H_u}\right] = K^2\left[|u - s|^{2H_u} - |u - t|^{2H_u} - |t - s|^{2H_u}\right].
$$

We can then continue to calculate the variance of the second difference increment

$$
\mathbb{E}[(B_{H_u}(u) - B_{H_t}(t)) - (B_{H_t}(t) - B_{H_s}(s))]^2 \approx K^2|u - t|^{2H_u} + K^2|t - s|^{2H_u} + K^2\left[|u - s|^{2H_u} - |u - t|^{2H_u} - |t - s|^{2H_u}\right].
$$

Considering a discretized mBm process sampled at $t_i = i(n - 1)^{-1}$ for $i = 0, \ldots, n - 1$, where $n \in \mathbb{N}^+$ we have

$$
\mathbb{E}[(B_{H_{t_{j+2}}}(t_{j+2}) - B_{H_{t_{j+1}}}(t_{j+1})) - (B_{H_{t_{j+1}}}(t_{j+1}) - B_{H_{t_j}}(t_j))]^2 
\approx K^2\left(\frac{1}{n - 1}\right)^{2H_{t_i}} + K^2\left(\frac{1}{n - 1}\right)^{2H_{t_i}} + K^2\left(\frac{2}{n - 1}\right)^{2H_{t_i}} - 2K^2\left(\frac{1}{n - 1}\right)^{2H_{t_i}} = K^2\left(\frac{2}{n - 1}\right)^{2H_{t_i}}
$$

for $j = i - \delta, \ldots, i - 2$ and $i = \delta + 2, \ldots, n - 2$, where $\delta$ is the size of a window. Note that $\delta < n$. Now, assuming a moving window consisting of $\delta < n$ observations of a discretized mBm process, and considering the variance of the second difference increments as in (5), the following distributional property is obtained,

$$
B_{H_{t_{j+2}}}(t_{j+2}) - 2B_{H_{t_{j+1}}}(t_{j+1}) + B_{H_{t_j}}(t_j) \overset{d}{\sim} \mathcal{N}\left(0, K^2\left(\frac{2}{n - 1}\right)^{2H_{t_i}}\right)
$$

for $j = i - \delta, \ldots, i - 2$ and $i = \delta + 2, \ldots, n - 2$. This distributional result is the basis of the SDR estimator as it enables the estimation of the Hurst function using a moving window approach. The moving window is positioned to the left of a time point $t_i$ and contains $\{t_j\}_{j=i-\delta}^{i-2}$ time points. It can be noted that the variance of the second difference increments is dependent on the number of observations $n$ and the scaling constant $K$.

In Figure 3 the second difference increments are displayed for a window of size $\delta$. Within the window, two increments are shown. Finally, the second difference is calculated as $\Delta_2 B_H = \Delta_1 B_H = B_{H_{t_{j+2}}}(t_{j+2}) - B_{H_{t_{j+1}}}(t_{j+1})) - (B_{H_{t_{j+1}}}(t_{j+1}) - B_{H_{t_j}}(t_j))$. Note that within the illustrated window all second difference increments have the distributional property as according to equation (6).
2.4.2 Derivation of the Second Difference Ratio estimator

In order to arrive at the SDR estimator, we start from the formula providing the \( k \)th absolute moment of \( Y \sim N(0, \sigma^2) \),

\[
E[|Y|^k] = \frac{2^{k/2} \Gamma((k + 1)/2)}{\Gamma(1/2)} \sigma^k, \quad k \in \mathbb{N}. \tag{7}
\]

Define the following average of the second difference increments in the \( \delta \)-area of a discretized mBm process,

\[
S^k_{\delta,n}(i) := \frac{1}{\delta} \sum_{j=i-\delta}^{i-2} |B_{H_{t_{j+2}}} - 2B_{H_{t_{j+1}}} + B_{H_{t_j}}|^k
\tag{8}
\]

for \( i = \delta + 1, \ldots, n \). By equation (7) and distributional property (5), we get

\[
E[S^k_{\delta,n}(i)] = \frac{1}{\delta} \sum_{j=i-\delta}^{i-2} |B_{H_{t_{j+2}}} - 2B_{H_{t_{j+1}}} + B_{H_{t_j}}|^k \approx \frac{2^{k/2} \Gamma((k + 1)/2)}{\Gamma(1/2)} K^k \left( \frac{2}{n - 1} \right)^{kh_{t_i}}.
\]

The ratio

\[
\frac{S^k_{\delta,n}(i)}{E[S^k_{\delta,n}(i)]} \approx \frac{\sqrt{\pi} S^k_{\delta,n}(i)}{2^{k/2} \Gamma((k + 1)/2) K^k (2/(n - 1))^{kh_{t_i}}} \tag{9}
\]

tends to 1 in probability as \( \delta \to \infty \). It can be noted that since \( \delta \) is the window size, it is assumed that \( \delta < n \).

This implies that if \( \delta \to \infty \), then \( n \to \infty \).
Remark 1. The convergence in probability of (9) holds for $H \in [0,1]$. See Bardet and Surgailis (2011).

Note that Remark 1 is a consequence of the SDR estimator being based on the second difference increments. It is possible to construct an estimator based on the first difference increments. However, it would be limited to estimating the Hurst function for $H_t \in [0,3/4]$, (see e.g., Bianchi (2005)). Specifying quantity (9) using $2n$ points gives

$$
\frac{S^k_{\delta,2n}(i)}{E[S^k_{\delta,2n}(i)]} \xrightarrow{P} 1 \quad \text{as} \quad \delta, n \to \infty.
$$

Using (9) and (10) it is possible to write

$$
\frac{S^k_{\delta,2n}(i)}{S^k_{\delta,n}(i)} \xrightarrow{P} \left( \frac{1}{2} \right)^{kH_t(i)} \quad \text{as} \quad \delta, n \to \infty,
$$

from which it follows that

$$
\frac{S^k_{\delta,2n}(i)}{S^k_{\delta,n}(i)} \xrightarrow{P} \left( \frac{1}{2} \right)^{kH_t(i)} \quad \text{as} \quad \delta, n \to \infty.
$$

Taking the logarithm of equation (11) we get

$$
\ln S^k_{\delta,2n}(i) - \ln S^k_{\delta,n}(i) \xrightarrow{P} -kH_t(i) \ln 2 \quad \text{as} \quad \delta, n \to \infty.
$$

Then equation (12) can be solved for $H_t(i)$, which gives the Second Difference Ratio estimator

$$
\hat{H}^k_{\delta,n}(t_i) = \frac{\ln S^k_{\delta,n}(i) - \ln S^k_{\delta,2n}(i)}{k \ln 2}, \quad i = \delta + 1, \ldots, n.
$$

The SDR estimator is a ratio estimator similar to the one defined in Benassi et al. (1998). Observe that the SDR estimator does not depend on the unknown scaling constant $K$.

2.4.3 Increment Ratio Statistic estimator

For comparison purposes the Increment Ratio Statistic (IRS) estimator as developed by Bardet and Surgailis (2013) is considered. In the following section, the IRS estimator is defined for regularity estimation of fBm and mBm processes. Considering a filter $a := (a_0, \ldots, a_q) \in \mathbb{R}^{q+1}$ such that for some $m \in \mathbb{N}$,

$$
\sum_{l=0}^{q} l^p a_l \quad \text{for} \quad p = 0, \ldots, (m-1),
$$

and

$$
\sum_{l=0}^{q} l^m a_l \neq 0,
$$

we denotes the class of filters by $\mathcal{A}(m,q)$. The corresponding generalized variations of $Z$ is defined by

$$
V^a_n Z(t) := \sum_{l=0}^{q} a_l Z(t + l/n).
$$

We now consider the filter $a = (1, -2, 1) \in \mathcal{A}(2,2)$ and a process $(Z(t))_{t \in (0,1)}$ from an observed sample $(Z(1/n), Z(2/n), \ldots, Z((n-1)/n))$. To be able to estimate the local Hurst function of $Z$ we define

$$
\xi(x,y) := \begin{cases} \frac{|x+y|}{|x|+|y|} & \text{if} \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 1 & \text{if} \quad (x,y) = (0,0). \end{cases}
$$
Using the function $\xi$, the second order Increment Ratio Statistic for measuring the roughness of a trajectory is defined as

$$IR_2(Z) := \frac{1}{n-2} \sum_{k=0}^{n-3} \xi \left( V_n^a Z \left( \frac{k}{n} \right), V_n^a Z \left( \frac{k+1}{n} \right) \right)$$

When considering $Z$ to be a stationary increment process of a fBm with local properties as described in equation (3) gives $IR_2(Z) \xrightarrow{P} \Lambda_2(H)$ as $n \to \infty$, where

$$\Lambda_2(H) = \frac{1}{\pi} \arccos (-\rho_2(H)) + \frac{1}{\pi} \sqrt{\frac{1 + \rho_2(H)}{1 - \rho_2(H)}} \log \left( \frac{2}{1 + \rho_2(H)} \right),$$

$$\rho_2(H) := \frac{-3^{2H} + 2^{2H+2} - 7}{8 - 2^{2H+1}}.$$

An estimator of $H$ is provided by $\Lambda^{-1}_2(\Lambda_2(H))$, where $\Lambda^{-1}_2(\cdot)$ is the inverse of $\Lambda_2(\cdot)$. In order to obtain a local estimator of the Hurst function, a bandwidth with parameter $\alpha \in (0, 1)$, on a neighborhood of $t \in \mathbb{R}$ is defined as

$$V_{n,\alpha}(t) := \{ k \in \{1, 2, \ldots, n - q - 1\}, \left| \frac{k}{n} - t \right| \leq n^{-\alpha} \}$$

where $\nu_{n,\alpha}(t) := \#V_{n,\alpha}(t)$.

The localized estimator of the Hurst function $\hat{H}_{IRS,\alpha,n}$ is then defined as

$$\hat{H}_{IRS,\alpha,n}(t) := \Lambda^{-1}_2 \left( \frac{1}{\nu_{n,\alpha}} \sum_{k \in V_{n,\alpha}(t)} \xi \left( V_n^a Z \left( \frac{k}{n} \right), V_n^a Z \left( \frac{k+1}{n} \right) \right) \right).$$

As stated above, the second difference increments, $a = (1, -2, 1)$, is used. For a thorough description, distributional properties and evaluation of the IRS estimator, see Bardet and Surgailis (2013).

### 2.4.4 Nonlinear regression

Fitting a smooth curve to the point-estimates is a way of approximating the true $H_t$ function of a mBm process. One way of achieving this is to use the common technique of nonlinear regression. If the true $H_t$ function is assumed to be a power function of the form $H_t = \phi + \psi t^\gamma$ nonlinear least squares can be applied to approximate the functional parameters, $\phi$, $\psi$, and $\gamma$.

Assume that a set of estimated points are denoted as $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$. One can then consider a function $y = f(x, \phi, \psi, \gamma)$ which depends on the variable $x$ and the parameters $\phi, \psi$, and $\gamma$. In order to acquire a smooth curve with the closest possible fit to the estimated points, the residuals $r_i = y_i - f(x_i, \phi, \psi, \gamma)$ needs to be minimized. In a least squares sense, this is achieved by calculating the optimal parameter values of $\phi, \psi$, and $\gamma$ to minimize the sum of squares

$$\min S(\phi, \psi, \gamma) = \sum_{i=1}^{m} r_i^2.$$

Since finding the optimal parameter values is a nonlinear problem, starting values are required. In this thesis, the fit() function in Matlab is used to calculate the optimal parameter values (see, Mathworks). This function heuristically chooses appropriate starting values based on the dataset and the specific functional form. More specifically a trust-region algorithm is chosen in order to estimate the parameter values. This is a type of iterative algorithm for solving nonlinear optimization problems. For a more detailed explanation of this kind of method, the reader is referred to Conn et al. (2000).
3 Main results

In this section, the performance of the SDR estimator, \( \hat{H}_{\delta,n} \), is evaluated using both fBm and mBm realizations of several Hurst functions, where \( H_t \in [0, 1] \). The SDR estimator is computed using \( k = 2 \) for all estimates. Therefore, for notational convenience, \( k \) is suppressed throughout the remaining part of this thesis. When considering the IRS estimator, the bandwidth parameter \( \alpha = 0.3 \) is used. In the final section where the Hurst function is specified as a power function, the estimates are smoothed using a nonlinear regression approach in an attempt to lower the estimation error.

Throughout this section, the discretized realizations of both fBm and mBm trajectories are generated using the FracLab toolbox as developed by Jacques-Lévy Véhel, available at [http://fraclab.saclay.inria.fr/](http://fraclab.saclay.inria.fr/). The circulant matrix algorithm is used when generating mBm realizations (see Chan and Wood (1998)).

3.1 Fractional Brownian motion evaluation

It is reasonable to begin by considering fBm processes as they have a constant Hurst exponent \( H \). Examples of fBm realizations with various \( H \) are shown in Figure 1. Sets of 500 independent sequences of discretized fBms of length \( n = 4000 \) on the unit interval are generated for three cases where \( H \in [0, 1] \). The following three cases are considered

(C1) \( H = 0.35 \),

(C2) \( H = 0.75 \),

(C3) \( H = 0.95 \).

A pilot study was conducted where the SDR estimator was applied to fBm trajectories for various values of \( \delta \). An observed effect of increasing the window size \( \delta \) was lower variance of the pointwise estimates. Based on these preliminary results, a sufficiently large window size of \( \delta = 150 \) was chosen.

In Figure 4, it can be observed that the IRS estimator is underestimating the Hurst exponent in the case of \( H = 0.95 \). The mean pointwise estimates of the SDR estimator have a higher variance than the pointwise estimates of the IRS estimator. The probability normalized histograms of the mean pointwise estimates are shown in the lower panels of Figure 4. In the lower panels, the true value of \( H \) (black) and the mean value of the mean pointwise estimates (green) are shown. Comparing the two histograms, the bias of the IRS estimator becomes visible while the SDR estimator captures the true value of \( H \) for the case C3. However, for lower values of \( H \) the IRS estimator is less biased as can be observed in corresponding figures, see appendix.

In Table 1 the average value of the mean pointwise estimates of \( H \) and corresponding sample standard deviation for all three cases (C1, C2, and C3) are presented. It can be noted that the sample standard deviation of the mean pointwise estimates of the IRS estimator is lower than the corresponding mean pointwise estimates of the SDR estimator for all considered cases. Note that in case C3, the mean pointwise estimates of the IRS estimator is not within two standard deviations of the true value.

Table 1: Sample standard deviation and mean values of the mean pointwise estimates for \( H \in \{0.35, 0.75, 0.95\} \) using the estimators \( \hat{H}_{\delta,n} \) and \( \hat{H}_{IRS,\alpha,n} \), where \( \delta = 150 \) and \( \alpha = 0.3 \).

<table>
<thead>
<tr>
<th>( H )</th>
<th>( \hat{H}_{\delta,n} )</th>
<th>( \hat{\sigma} )</th>
<th>( \hat{H}_{IRS,\alpha,n} )</th>
<th>( \hat{\sigma} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.35</td>
<td>0.3455</td>
<td>0.0049</td>
<td>0.3474</td>
<td>0.0035</td>
</tr>
<tr>
<td>0.75</td>
<td>0.7452</td>
<td>0.0045</td>
<td>0.7487</td>
<td>0.0023</td>
</tr>
<tr>
<td>0.95</td>
<td>0.9460</td>
<td>0.0042</td>
<td>0.9408</td>
<td>0.0015</td>
</tr>
</tbody>
</table>
Figure 4: In the upper panel, the mean pointwise estimates of a fBm with Hurst exponent $H = 0.95$ using the SDR estimator with $\delta = 150$ (blue) and the IRS estimator with $\alpha = 0.3$ (red) are shown. In the lower panels, probability normalized histograms are shown for the mean pointwise estimates of the SDR estimator (left) and the IRS estimator (right). The estimates are based on 500 independent realizations.

3.2 Multifractional Brownian motion evaluation

The main focus of this thesis is studying stochastic processes that are assumed to follow a multifractional Brownian motion, which implies a time-varying Hurst function $H_t$. To evaluate the performance of the SDR estimator, three cases of mBm processes will be considered with the following underlying $H_t$ functions

(C4) $H_t = 0.35 + 0.4t, \quad t \in [0, 1],$
(C5) $H_t = 0.35 + 0.4t^2, \quad t \in [0, 1],$
(C6) $H_t = 0.1 + 0.8(1 - t)\sin^2(10t), \quad t \in [0, 1].$

For each $H_t$ function (C4, C5, C6), 100 mBm sequences of length $n \in \{2000, 4000, 6000\}$ on the unit interval are generated. The SDR estimator is compared to the IRS estimator using Monte Carlo simulation. For evaluation purposes, the mean integrated squared error (MISE) will be considered and is defined as follows

$$\text{MISE} := \int_0^1 \mathbb{E}(\hat{H}_t - H_t)^2 dt,$$

where $\hat{MISE}$ is the corresponding numerical approximation. The trapezoidal rule is used to approximate the integral over the average square errors of the individually estimated trajectories. The maximum mean squared error $\text{mMSE}$ is also considered

$$\text{mMSE} := \max_{t \in [0, 1]} \mathbb{E}(\hat{H}_t - H_t)^2,$$
where $H_t$ is the true Hurst function and $\hat{H}_t$ is its pointwise estimate. The numerical approximation of this error measure is calculated as the maximum value of the averaged pointwise square errors.

In the left panel of Figure 5, the mean pointwise estimates for case C6 for 100 trajectories of the SDR estimator is shown using window size $\delta = 150$ and $n = 6000$. It can be observed that the mean pointwise estimates are slightly shifted to the right. In the right panel, a single trajectory of the pointwise estimates for $\hat{H}_{\delta,n}$ using $\delta = 150$ and $n = 6000$ is shown. The variance of the pointwise estimates is large, creating a problem for empirical application where only one realization of a process is available.

In the left panel of Figure 6, the mean pointwise estimates of the IRS estimator on the same mBm trajectories as in Figure 5 are shown. In the right panel, a single trajectory of the pointwise estimates using the IRS estimator on a single mBm trajectory are shown. It can be observed in the right panel of Figure 6 that the IRS estimator is not allowed to assume negative values. Comparing Figure 5 and Figure 6 the SDR estimator has a higher variance when considering single trajectory estimation.

Table 2: Computation of $\sqrt{\text{MISE}}$ and $\sqrt{\text{mMSE}}$ for $\hat{H}_{\delta,n}$ at $\delta \in \{50, 100, 150, 200, 250\}$ and $n \in \{2000, 4000, 6000\}$. The results are based on 100 independent realizations.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\hat{H}_{\delta,n}$</th>
<th>$\hat{H}_{\delta,n}$</th>
<th>$\hat{H}_{\delta,n}$</th>
<th>$\hat{H}_{\delta,n}$</th>
<th>$\hat{H}_{\delta,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 0.3$</td>
<td>$\delta = 50$</td>
<td>$\delta = 100$</td>
<td>$\delta = 150$</td>
<td>$\delta = 200$</td>
</tr>
<tr>
<td>2000</td>
<td>$\sqrt{\text{MISE}}$</td>
<td>0.198903</td>
<td>0.290085</td>
<td>0.243246</td>
<td>0.250047</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.441855</td>
<td>0.440837</td>
<td>0.493360</td>
<td>0.603704</td>
</tr>
<tr>
<td>4000</td>
<td>$\sqrt{\text{MISE}}$</td>
<td>0.177784</td>
<td>0.281986</td>
<td>0.209868</td>
<td>0.190578</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.422814</td>
<td>0.352665</td>
<td>0.305047</td>
<td>0.340481</td>
</tr>
<tr>
<td>6000</td>
<td>$\sqrt{\text{MISE}}$</td>
<td>0.164262</td>
<td>0.280148</td>
<td>0.201223</td>
<td>0.172951</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.404926</td>
<td>0.365219</td>
<td>0.260164</td>
<td>0.259685</td>
</tr>
</tbody>
</table>

In table 2 the $H_t$ function from case C6 is considered. Empirical MISE and empirical mMSE for the estimators $\hat{H}_{\delta,n}$ at $\delta \in \{50, 100, 150, 200, 250\}$ and $\hat{H}_{\delta,n}$ are shown for $n \in \{2000, 4000, 6000\}$. When considering this function the SDR estimator outperforms the IRS estimator for $n = 6000$ and $\delta \geq 200$, producing both lower empirical MISE and lower empirical mMSE. It can be noted, that the window size $\delta$ which produces the lowest observed value of MISE and mMSE increases as the number of observations $n$ increases. Figures and Tables for cases C4 and C5 can be found in the appendix.

The high volatility of the pointwise estimates from a single realization estimated by the SDR estimator poses a problem for empirical application. This motivates the application of a smoothing technique in order to reduce variance of the pointwise estimates and produce more accurate results.
Figure 5: In the left panel, mean pointwise estimates using $\hat{H}_{\delta,n}$ for 100 trajectories using the function $H_t = 0.1 + 0.8(1 - t)\sin(10t)^2$ are shown (blue) together with the true $H_t$-function (black). In the right panel, pointwise estimates using $\hat{H}_{\delta,n}$ for a single trajectory (blue) of the same process are shown together with the true $H_t$-function (black).

Figure 6: In the left panel, mean pointwise estimates using $\hat{H}_{IRS,\alpha,n}$ for 100 trajectories using the function $H_t = 0.1 + 0.8(1 - t)\sin^2(10t)$ are shown (red) together with the true $H_t$-function (black). In the right panel, point estimates for a single trajectory (red) of the same process are shown together with the true $H_t$-function (black).
3.3 Smoothing using nonlinear regression

In an attempt to smooth the pointwise estimates of the SDR estimator, a nonlinear regression approach is evaluated. First, assume that the Hurst function $H_t$ has the following functional form,

$$H_t = \phi + \psi t^\gamma, \quad t \in [0, 1].$$

Fixing $\phi = 0.35$, $\psi = 0.4$ and $\gamma = \{0.5, 1, 2, 3\}$, 100 sequences of length $n = 6000$ on the unit interval are generated. Nonlinear regression is applied to each individually estimated trajectory in an attempt to estimate the parameters $\phi, \psi,$ and $\gamma$. In Figure 7 histograms of the parameter estimates for $\gamma = 2$ are displayed. The estimated parameter means are displayed in the figure by the solid red line, the true value is correspondingly displayed by the solid black line. It can be noted that some extreme values inflate the standard deviations of the estimates for all parameters. Using the Kolmogorov-Smirnov normality test, the null hypothesis at 95% significance level of the parameters estimates being asymptotically normally distributed was rejected for all $\gamma \in \{0.5, 1, 2, 3\}$. Histograms of the parameter estimates when $\gamma \neq 2$ can be found in the appendix.

Table 3: Standard deviation and expected values of the Hurst function $H_t$ parameter estimates with true values $\phi = 0.35$, $\psi = 0.4$ and $\gamma \in \{0.5, 1, 2, 3\}$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$(\hat{\phi}, \hat{\sigma}_\phi)$</th>
<th>$(\hat{\psi}, \hat{\sigma}_\psi)$</th>
<th>$(\hat{\gamma}, \hat{\sigma}_\gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>(0.401638, 0.862357)</td>
<td>(0.336996, 0.869900)</td>
<td>(0.442998, 0.491895)</td>
</tr>
<tr>
<td>1</td>
<td>(0.316403, 0.359270)</td>
<td>(0.418037, 0.362328)</td>
<td>(1.030407, 0.730812)</td>
</tr>
<tr>
<td>2</td>
<td>(0.327104, 0.051746)</td>
<td>(0.407523, 0.058150)</td>
<td>(1.951444, 0.715267)</td>
</tr>
<tr>
<td>3</td>
<td>(0.329802, 0.092810)</td>
<td>(0.380914, 0.129264)</td>
<td>(3.664023, 2.696220)</td>
</tr>
</tbody>
</table>

In Table 3 parameter estimates of $\phi, \psi,$ and $\gamma$ are presented together with their standard deviation. The estimates are calculated using 100 individual trajectories, $\delta = 150$ and, $n = 6000$. It is interesting to note that the standard deviation of the $\phi$ and $\psi$ estimates decrease as the value of $\gamma$ increases. However, the standard deviation of $\hat{\gamma}$ is larger for increased $\gamma$ values. The mean values of the parameter estimates are within one standard deviation of the true values.

In the left panel of Figure 8 the pointwise estimates of the SDR estimator for a single trajectory (blue) are displayed together with the true $H_t$ function (black). In the right panel, the corresponding estimates have been smoothed using nonlinear regression. It can be observed that this method manages to reduce the volatility of the single trajectory estimates. For examples of the remaining $\gamma$ values the reader is referred to the appendix.

To further evaluate the performance of the smoothing procedure using nonlinear regression (NLR) the empirical MISE and empirical mMSE of the fitted curves is compared to the SDR estimator and IRS estimator. Using NLR on each trajectory, individually smoothed curves are acquired from which the empirical MISE and the empirical mMSE are calculated. The results are shown for the quadratic case $\gamma = 2$ in Table 4 for the other three cases, see appendix. The empirical MISE is significantly lower compared to the original estimates of the SDR estimator.

It can be observed that the NLR technique manages to lower the empirical MISE and the empirical mMSE significantly. It can also be noted that the empirical MISE not necessarily decreases as the window size $\delta$ increases, as was the case for the SDR estimator. It can be seen in Table 4 that the smoothed results are comparable or better to those of the IRS estimator both considering empirical MISE and empirical mMSE for $\gamma = 2$. To view the empirical MISE and mMSE for cases $\gamma \in \{0.5, 1, 3\}$, see appendix.
Figure 7: Histograms of the parameter estimates. In the upper panel the approximate distribution of the $\phi$ parameter estimates for the true $\phi = 0.35$ is shown, in the middle panel the $\psi$ parameter estimates for $\psi = 0.4$, and in the lower panel the $\gamma$ parameter estimates for $\gamma = 2$.

Figure 8: In the left panel, the pointwise estimates of a single trajectory of a mBm with $H_t = 0.35 + 0.4t^2$ using $\hat{H}_{\delta,n}$ (blue) and the corresponding true $H_t$ function (black) is displayed. In the right panel, the smoothed pointwise estimates $\hat{H}_{NLR}$ (blue) and the corresponding true $H_t$ function (black) is shown. The estimation is performed with $\delta = 150$ and $n = 6000$. 
Table 4: Computation of $\sqrt{\text{MISE}}$ and $\sqrt{\text{mMSE}}$ for $\hat{H}_{\delta,n}$ and $\hat{H}_{NLR}$ for $\delta \in \{50, 100, 150, 200, 250\}$ and $n \in \{2000, 4000, 6000\}$ using the power function $H_t = 0.35 + 0.4t^\gamma$ for $\gamma = 2$. The results are based on 100 independent realizations.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Est.</th>
<th>$\delta = 50$</th>
<th>$\delta = 100$</th>
<th>$\delta = 150$</th>
<th>$\delta = 200$</th>
<th>$\delta = 250$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>$\hat{H}_{\delta,n}$ $\sqrt{\text{MISE}}$</td>
<td>0.261559</td>
<td>0.182505</td>
<td>0.149340</td>
<td>0.131516</td>
<td>0.121446</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.348133</td>
<td>0.225278</td>
<td>0.184680</td>
<td>0.165876</td>
<td>0.145531</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}_{NLR}$ $\sqrt{\text{MISE}}$</td>
<td>0.059097</td>
<td>0.056474</td>
<td>0.057771</td>
<td>0.061296</td>
<td>0.065178</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.095441</td>
<td>0.093264</td>
<td>0.089932</td>
<td>0.091995</td>
<td>0.096883</td>
</tr>
<tr>
<td>4000</td>
<td>$\hat{H}_{\delta,n}$ $\sqrt{\text{MISE}}$</td>
<td>0.258760</td>
<td>0.180217</td>
<td>0.146576</td>
<td>0.127326</td>
<td>0.114631</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.331273</td>
<td>0.225122</td>
<td>0.184769</td>
<td>0.158281</td>
<td>0.139677</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}_{NLR}$ $\sqrt{\text{MISE}}$</td>
<td>0.043840</td>
<td>0.039513</td>
<td>0.039643</td>
<td>0.040601</td>
<td>0.042030</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.067233</td>
<td>0.066810</td>
<td>0.068075</td>
<td>0.070610</td>
<td>0.073152</td>
</tr>
<tr>
<td>6000</td>
<td>$\hat{H}_{\delta,n}$ $\sqrt{\text{MISE}}$</td>
<td>0.259858</td>
<td>0.181026</td>
<td>0.146904</td>
<td>0.127198</td>
<td>0.114100</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.343125</td>
<td>0.237888</td>
<td>0.188723</td>
<td>0.161178</td>
<td>0.139410</td>
</tr>
<tr>
<td></td>
<td>$\hat{H}_{NLR}$ $\sqrt{\text{MISE}}$</td>
<td>0.035932</td>
<td>0.030774</td>
<td>0.030034</td>
<td>0.030093</td>
<td>0.030354</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\text{mMSE}}$</td>
<td>0.061831</td>
<td>0.054601</td>
<td>0.053744</td>
<td>0.053412</td>
<td>0.053494</td>
</tr>
</tbody>
</table>

Table 5: Computation of $\sqrt{\text{MISE}}$ and $\sqrt{\text{mMSE}}$ for $\hat{H}_{\delta,n}$, $\hat{H}_{NLR}$, and $\hat{H}_{IRS,\alpha,n}$ for $\delta = 150$, and $n \in \{2000, 4000, 6000\}$ using the $H_t$ power function with $\gamma = 2$. The IRS estimator is considered for $\alpha = 0.3$. The results are based on 100 independent realizations.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\hat{H}_{\delta,n}$ $\sqrt{\text{MISE}}$</th>
<th>$\hat{H}_{NLR}$ $\sqrt{\text{MISE}}$</th>
<th>$\hat{H}_{IRS,\alpha,n}$ $\sqrt{\text{MISE}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>0.1493</td>
<td>0.1043</td>
<td>0.1353</td>
</tr>
<tr>
<td></td>
<td>0.1847</td>
<td>0.1536</td>
<td>0.1353</td>
</tr>
<tr>
<td>4000</td>
<td>0.1466</td>
<td>0.1043</td>
<td>0.1353</td>
</tr>
<tr>
<td></td>
<td>0.1848</td>
<td>0.1241</td>
<td>0.1353</td>
</tr>
<tr>
<td>6000</td>
<td>0.1469</td>
<td>0.1043</td>
<td>0.1353</td>
</tr>
<tr>
<td></td>
<td>0.1887</td>
<td>0.1020</td>
<td>0.1353</td>
</tr>
</tbody>
</table>
4 Conclusion

It is important to remember that there are several variables affecting the estimation of the Hurst function. For example, both the sample size of the mBm process and the functional form of the Hurst function affect the results of the pointwise estimates. Additionally, in the case of the SDR estimator, choosing a optimal window size, $\delta$, complicates matters even further.

In this thesis, the performance of a ratio estimator of the Hurst function based on a moving window average technique defined by [Bianchi 2005] has been evaluated. The SDR estimator is first derived and its performance is later compared to that of the IRS estimator. It is important to remember that one of the basic assumptions when deriving the SDR estimator is that the Hurst function is constant within a window of size $\delta$. Considering a constant sample size, the realism of this assumption decreases as the size of the window increases.

A consequence of specifying a moving window of size $\delta$ to the left, is that the SDR estimator produces a slight shifting bias in its pointwise estimates. This could be corrected by using a centered window, which would be an interesting problem for future research. Due to the size of the SDR estimators moving window being specified as a constant, the comparison to the IRS estimator becomes quite inaccurate. This is because the IRS estimator uses a bandwidth, which size depends on the number of observations. Therefore to achieve a fair comparison, it would be of interest to either specify a bandwidth approach for the SDR estimator or match the window size $\delta$ to the bandwidth size of the IRS estimator.

It can be observed that the SDR estimator on average estimates the Hurst function relatively well, when compared to the IRS estimator. This is verified by comparing empirical MISE and empirical mMSE between the two estimators. It should be noted that to achieve comparable results to that of the IRS estimator, a suitable value of the window size $\delta$ is required. Note that the IRS estimators bandwidth parameter, $\alpha$, has been kept constant at 0.3 in this thesis. This motivates that the results should be verified in a more rigorous comparison.

The pointwise estimates of the SDR estimator are highly volatile when considering single trajectory estimation. If the SDR estimator is to be applied to empirical data where only one realization is available, it is crucial to try to decrease the variance of the pointwise estimates. In order to address this problem, nonlinear regression is considered. This smoothing technique is applied and tested using a naive approach, where the Hurst function is assumed to follow a specific functional form. The results show that nonlinear regression is an effective way of reducing the variance of the estimates in this specific setting.

Building on this preliminary results, it would be interesting to further develop the nonlinear regression approach to be applicable in a more general setting. The development of efficient smoothing techniques when the underlying Hurst function is not a priori known is of great interest concerning future empirical application of the estimator. One possible smoothing technique that could be considered is for example smoothing splines.

The SDR estimator, building on the distributional results of the second difference increments of the mBm process, shows adequate ability in capturing the Hurst function in its whole range, $H_t \in [0, 1]$. It is shown that it outperforms the IRS estimator when considering a fBm process with a Hurst exponent of $H = 0.95$. Since the accuracy of the estimates are dependent on $\delta$, the matter of determining the optimal window size $\delta$ when applying the SDR estimator remains a question for future research. The evaluation of the SDR estimator in this thesis is limited to the case when $k = 2$, which also motivates further research.
References


Mandelbrot, B.B., 1975. Stochastic models for the Earth’s relief, the shape and the fractal dimension of the coastlines, and the number-area rule for islands. Proceedings of the National Academy of Sciences 72, 3825–3828.


5 Appendix

Table 6: Computation of $\sqrt{\hat{\text{MISE}}}$ and $\sqrt{\hat{\text{mMSE}}}$ for $\hat{H}_{IRS,\alpha}$ and $\hat{H}_{\delta,n}$ using the linear function $H_t = 0.35 + 0.4t$, for $\delta \in \{50, 100, 150, 200, 250\}$ and $n \in \{2000, 4000, 6000\}$. The results are based on 100 independent trajectories.

<table>
<thead>
<tr>
<th>$n$</th>
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<td>2000</td>
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Table 7: Computation of $\sqrt{\hat{\text{MISE}}}$ and $\sqrt{\hat{\text{mMSE}}}$ for $\hat{H}_{IRS,\alpha}$ and $\hat{H}_{\delta,n}$ using the power function $H_t = 0.35 + 0.4t^2$, for $\delta \in \{50, 100, 150, 200, 250\}$ and $n \in \{2000, 4000, 6000\}$. The results are based on 100 independent trajectories.

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Table 8: Computation of $\sqrt{\hat{\text{MISE}}}$ and $\sqrt{\hat{\text{mMSE}}}$ for $\hat{H}_{\delta,n}$ and $\hat{H}_{\text{NLR}}$ for $\delta \in \{50, 100, 150, 200, 250\}$ and $n \in \{2000, 4000, 6000\}$ using the linear function $H_t = 0.35 + 0.4t\gamma$ for $\gamma = 0.5$. The results are based on 100 independent realizations.

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Table 9: Computation of $\sqrt{\hat{\text{MISE}}}$ and $\sqrt{\hat{\text{mMSE}}}$ for $\hat{H}_{\delta,n}$ and $\hat{H}_{\text{NLR}}$ for $\delta \in \{50, 100, 150, 200, 250\}$ and $n \in \{2000, 4000, 6000\}$ using the linear function $H_t = 0.35 + 0.4t\gamma$ for $\gamma = 1$. The results are based on 100 independent realizations.

<table>
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<tr>
<th>$n$</th>
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Table 10: Computation of $\sqrt{\text{MISE}}$ and $\sqrt{\text{mMSE}}$ for $\hat{H}_{\delta,n}$ and $\hat{H}_{NLR}$ for $\delta \in \{50, 100, 150, 200, 250\}$ and $n \in \{2000, 4000, 6000\}$ using the linear function $H_t = 0.35 + 0.4t^\gamma$ for $\gamma = 3$. The results are based on 100 independent realizations.

<table>
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<th>$n$</th>
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Table 11: Computation of $\sqrt{\text{MISE}}$ and $\sqrt{\text{mMSE}}$ for $\hat{H}_{\delta,n}$, $\hat{H}_{NLR}$, and $\hat{H}_{IRS,\alpha,n}$ for $\delta = 150$, and $n \in \{2000, 4000, 6000\}$ using the $H_t$ power functions with $\gamma \in \{0.5, 1, 2\}$. The IRS estimator is considered with $\alpha = 0.3$. The results are based on 100 independent trajectories.

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Figure 9: In the upper panel, the mean pointwise estimates of a fBm with Hurst exponent $H = 0.35$ using the SDR estimator (blue) and the IRS estimator (red) are shown. In the lower panels, probability normalized histograms are shown for the mean pointwise estimates of the SDR estimator (left) and the IRS estimator (right). The estimates are based on 500 independent realizations.

Figure 10: In the upper panel, the mean pointwise estimates of a fBm with Hurst exponent $H = 0.75$ using the SDR estimator (blue) and the IRS estimator (red) are shown. In the lower panels, probability normalized histograms are shown for the mean pointwise estimates of the SDR estimator (left) and the IRS estimator (right). The estimates are based on 500 independent realizations.
Figure 11: In the left panel, mean pointwise estimates using $\hat{H}_{\delta,n}$ for 100 trajectories using the function $H_t = 0.35 + 0.4t$ are shown (blue) together with the true $H_t$-function (black). In the right panel, pointwise estimates using $\hat{H}_{\delta,n}$ for a single trajectory (blue) of the same process are shown together with the true $H_t$ function (black).

Figure 12: In the left panel, Monte Carlo estimates using $\hat{H}_{IRS,\alpha,n}$ for 100 trajectories using the function $H_t = 0.35 + 0.4t$ are shown (red) together with the true $H_t$-function (black). In the right panel, point estimates for a single trajectory (red) of the same process are shown together with the true $H_t$ function (black).
Figure 13: In the left panel, mean pointwise estimates using $\hat{H}_{\alpha,n}$ for 100 trajectories using the function $H_t = 0.35 + 0.4t^2$ are shown (blue) together with the true $H_t$-function (black). In the right panel, pointwise estimates using $\hat{H}_{\alpha,n}$ for a single trajectory (blue) of the same process are shown together with the true $H_t$-function (black).

Figure 14: In the left panel, Monte Carlo estimates using $\hat{H}_{IRS,\alpha,n}$ for 100 trajectories using the function $H_t = 0.35 + 0.4t^2$ are shown (red) together with the true $H_t$-function (black). In the right panel, point estimates for a single trajectory (red) of the same process are shown together with the true $H_t$-function (black).
Figure 15: Histograms of the parameter estimates for $\gamma = 0.5$. In the upper panel the approximate distribution of the $\phi$ parameter estimates for the true $\phi = 0.35$ is shown, in the middle panel the $\psi$ parameter estimates for $\psi = 0.4$, and in the lower panel the $\gamma$ parameter estimates are shown.

Figure 16: In the left panel, the pointwise estimates of a single trajectory of a mBm with $H_t = 0.35 + 0.4t^{0.5}$ using $\hat{H}_{\delta,n}$ (blue) and the corresponding true $H_t$ function (black) is displayed. In the right panel, the smoothed pointwise estimates $\hat{H}_{NLR}$ (blue) and the corresponding true $H_t$ function (black) is shown. The estimation is performed with $\delta = 150$ and $n = 6000$. 
Figure 17: Histograms of the parameter estimates for $\gamma = 1$. In the upper panel the approximate distribution of the $\phi$ parameter estimates for the true $\phi = 0.35$ is shown, in the middle panel the $\psi$ parameter estimates for $\psi = 0.4$, and in the lower panel the $\gamma$ parameter estimates are shown.

Figure 18: In the left panel, the pointwise estimates of a single trajectory of a mBm with $H_t = 0.35 + 0.4t$ using $\hat{H}_{\delta,n}$ (blue) and the corresponding true $H_t$ function (black) is displayed. In the right panel, the smoothed pointwise estimates $\hat{H}_{NLR}$ (blue) and the corresponding true $H_t$ function (black) is shown. The estimation is performed with $\delta = 150$ and $n = 6000$. 
Figure 19: Histograms of the parameter estimates for $\gamma = 3$. In the upper panel the approximate distribution of the $\phi$ parameter estimates for the true $\phi = 0.35$ is shown, in the middle panel the $\psi$ parameter estimates for $\psi = 0.4$, and in the lower panel the $\gamma$ parameter estimates are shown.

Figure 20: In the left panel, the pointwise estimates of a single trajectory of a mBm with $H_t = 0.35 + 0.4t^3$ using $\hat{H}_{\delta,n}$ (blue) and the corresponding true $H_t$ function (black) is displayed. In the right panel, the smoothed pointwise estimates $\hat{H}_{NLR}$ (blue) and the corresponding true $H_t$ function (black) is shown. The estimation is performed with $\delta = 150$ and $n = 6000$. 