Tychonoff’s theorem and its equivalence with the axiom of choice

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Abstract
In this essay we give an elementary introduction to topology so that we can prove Tychonoff’s theorem, and also its equivalence with the axiom of choice.

Sammanfattning
Denna uppsats tillhandahåller en grundläggande introduktion till topologi för att sedan bevisa Tychonoff’s theorem, samt dess ekvivalens med urvalsaxiomet.
## Contents

List of Figures

1. Introduction 1

2. Basic topology 3

3. Product spaces 9

4. Compact spaces 13

5. Tychonoff’s theorem and its equivalence with the axiom of choice 17

6. Acknowledgements 21

References 23
List of Figures

1.1 Flowchart, dashed lines indicate equivalence. 2
2.1 Four different topologies for the set $X = \{a, b, c\}$. 4
2.2 Mind map of bases and covers in a topological space. 7
3.1 The product space of two arbitrary topological spaces $X_1$ and $X_2$. 10
3.2 The product space of a class of sets $\{X_i\}$, where each $X_i$ is equal to the closed interval $[0, 1]$. 11
5.1 An example which illustrates the axiom of choice. From the non-empty class $\{S_i\}$ we can create the set $\{x_i\}$. 18
5.2 An illustration of some of the sets, classes and $n$-tuples used in the current proof. In this example, the $n$-tuple $a$ has $n = 2$. The dashed shapes indicates arrays. 20
1. Introduction

At a first glance the simple statement that the product of any non-empty class of compact spaces is compact is considered by many the most important result in topology. This was first proved in 1930 for the case of intervals by the Russian mathematician Andrey Nikolayevich Tikhonov ([7]), and in 1935 he stated that the original proof was still valid in the general case ([8]). Tikhonov’s surname is also transliterated as "Tychonoff", supposedly since he originally published in the German language. The first transcribed and detailed proof for the general case of Tychonoff’s theorem seems to be by Eduard Čech ([1]).

Our main goal of this essay is to give an elementary topological background in order to give a detailed proof of our star:

**Theorem 5.1 (Tychonoff’s Theorem).** The product of any non-empty class of compact spaces is compact in the product topology.

The converse of Tychonoff’s theorem is also true, as we shall see in Proposition 5.2. The impacts of Tychonoff’s theorem are several, e.g. the Heine-Borel theorem, the Banach-Alaoglu theorem, and the Arzelà-Ascoli theorem. The proofs of these theorems can easily be found in elementary text books (see e.g. [2, 6]). In this essay we shall instead look at the so called Kakutani conjecture that was first solved by Kelley in 1950 ([3]). This conjecture states a quite remarkable connection between Tychonoff’s theorem and the axiom of choice. Recall that by assuming that the axiom of choice is true, in a logical sense, the Banach-Tarski paradox is true. The Banach-Tarski paradox states that given a solid sphere in dimension three there exists a decomposition of the sphere into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original sphere ([9]). This is obviously strongly counterintuitive, but the reader has to remember that the decomposition of the mentioned sphere is into unmeasurable elements. Therefore, if one wants to dismiss the axiom based purely on intuition, one has to provide an intuitive argument which does not rely on the concept of volume for disbelieving the previously mentioned decomposition, and it is not clear that such an argument exists ([5]). We end this essay by proving the following theorem, and leave the philosophical debate concerning if we should assume the axiom of choice or not to the reader.

**Theorem 5.4:** Tychonoff’s theorem is true if, and only if, the axiom of choice is true.

The overview of the essay is as follows. In section 2, we state and prove some basic topological facts. In section 3, we focus on product spaces. In section 4, the attention is on compact spaces, and in section 5 we prove the above theorems. For a detailed flow-chart of the proof of Tychonoff’s theorem see Figure 3.1.

This essay, up to Theorem 5.4, is based on [6] with some inspiration from [4]. The last part is based on [3, 5].
Figure 1.1. Flowchart, dashed lines indicate equivalence.
2. Basic topology

When working with spaces such as $\mathbb{R}^n$ and $\mathbb{C}^n$, we are used to having a defined concept of distance between any two elements, also known as a metric. Spaces like this are called metric spaces, but they are only special cases of topological spaces. Thus, when we transition into topology we have to abandon the rather intuitive concept of distances.

In this section we will provide an introduction to topology. We include the basic definitions needed to construct a topological space, and the theorems that follow. We then define different versions of bases, and lastly the concept of covers. We begin however, by defining what we mean by a topology.

**Definition 2.1.** Let $X$ be a non-empty set. A class $T$ of subsets of $X$ is called a topology on $X$ if it satisfies the following conditions:

(i) The union of every class of sets in $T$ is a set in $T$.
(ii) The intersection of every finite class of sets in $T$ is a set in $T$.
(iii) $X$ and $\emptyset$ are in $T$.

This means that a topology on $X$ is closed under the formation of arbitrary unions and finite intersections. We continue by defining the connection between topology and spaces.

**Definition 2.2.** A topological space consists of two objects: a non-empty set $X$ and a topology $T$ on $X$. The sets in $T$ are called the open sets of the topological space $(X, T)$.

It is customary to refer to the topological space $(X, T)$ as just $X$. Thus, we see that in order to define a topological space one must first specify a non-empty set, which subsets are to be considered the open sets, and then verify that the resulting class of subsets satisfies condition (i) and (ii) in Definition 2.1. We illustrate this with some examples.

**Example 2.3.** Let $X$ be any non-empty set, and let the topology be the class of all subsets of $X$. This is called the discrete topology on $X$, and a topological space whose topology is the discrete topology is called a discrete space.

**Example 2.4.** Let $X$ be any non-empty set, and let the topology be the class which consists only of the empty set $\emptyset$ and the full space $X$, i.e. the class $\{\emptyset, X\}$.

**Example 2.5.** Let $X$ be a set consisting of the elements $\{a, b, c\}$. There are many different topologies for $X$. We illustrate some of them in Figure 2.1. In the upper left corner, the topology consists only of $X$ and $\emptyset$. The topology in the upper right corner consists of $X$, $\emptyset$, $\{a\}$ and $\{a, b\}$. In the lower left corner the topology consists of $X$, $\emptyset$, $\{b\}$, $\{a, b\}$ and $\{b, c\}$. The last figure in the lower right corner has topology which consists of every subset of $X$. 

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3
Definition 2.6. Let $X$ and $Y$ be two topological spaces, and $f$ a mapping of $X$ into $Y$. The mapping $f$ is called **continuous** if $f^{-1}(G)$ is open in $X$ whenever $G$ is open in $Y$.

Our definition of a topological space allows us to define another important property, namely that of relative topology.

Definition 2.7. Let $X$ be a topological space and let $Y$ be a non-empty subset of $X$. The **relative topology** on $Y$ is the class of all intersections with $Y$ of open sets in $X$, and when $Y$ is equipped with its relative topology it is called a **subspace** of $X$.

In Definition 2.2 we defined what an open set is, but we know from the theory of metric spaces that it is useful to have some notion of both open and closed sets at our disposal, which leads us to our next definition.

Definition 2.8. A **closed set** in a topological space is a set whose complement is open. The complement of a set $X$ is denoted as $X'$.

It would actually have been possible for us to start our definition of topological spaces with the closed set as our basic undefined concept, and still end up with the same theory. This is however an approach which we do not explain further since it is of no use to us, for more information about it see e.g. [6].

With our definition of topology and closed sets as it is, we can state the following theorem.

**Theorem 2.9.** Let $X$ be a topological space. Then the following holds:

(i) Any intersection of closed sets in $X$ is closed.

(ii) Any finite union of closed sets in $X$ is closed.

**Proof.** Part (i): Let $\{A_i\}$ be a class of closed sets in $X$, where $i \in I$ and $I$ is an index set. From Definition 2.8 we see that it is equivalent to show that $(\bigcup_{i \in I} A_i)'$ is
open in order to prove (i). We have that
\[
\left( \bigcap_{i \in I} A_i \right)' = \bigcup_{i \in I} A_i'
\] (2.1)
and by assumption we have that \(A'\) is open. It then follows from the definition of topology that \(\bigcup_{i \in I} A'_i\) is open and thus \(\bigcap_{i \in I} A_i\) is closed. This proves the first part of the theorem.

Part (ii): This part can be shown in the same way, by using that
\[
\left( \bigcup_{i=1}^n A_i \right)' = \bigcap_{i=1}^n A'_i
\]
instead of equation (2.1).

We now move on to define different types of bases in our topological space. The concept of bases might at first sight seem superfluous, but it will eventually prove itself to be of great assistance as it allows us to solve some otherwise long and tedious proofs by simple consequences of the properties of these bases.

**Definition 2.10.** Let \(X\) be a topological space. An open base for \(X\) is a class of open sets with the property that every open set in \(X\) is a union of sets in this class. The sets in an open base is referred to as basic open sets.

**Definition 2.11.** Let \(X\) be a topological space. An open subbase is a class of open subsets of \(X\) whose finite intersections form an open base. This open base is called the open base generated by the open subbase, and the sets in an open subbase are called subbasic open sets.

The two previous definitions leads us to our next theorem, which allows us to generate a topology from an arbitrary class of subsets of a non-empty set.

**Theorem 2.12.** Let \(X\) be a non-empty set, and let \(S\) be an arbitrary class of subsets of \(X\). Then \(S\) can serve as an open subbase for a topology on \(X\), in the sense that the class of all unions of finite intersections of sets in \(S\) is a topology.

**Proof.** Part I: Assume that \(S\) is empty. Here we have to remember that intersection of an arbitrary class of sets \(\{S_i\}\) in \(S\) is defined as
\[
\cap_i S_i = \{x : x \in S_i \text{ for every } i \in I\}.
\]
Then the class of all finite intersections of sets in \(S\) is \(\{X\}\). This follows from the definition, since we require of every element that it belongs to to each set in \(S\). Since there are no sets present, every element in \(X\) satisfies this requirement. We also know that the class of all finite unions is \(\{\emptyset, X\}\), by which we can conclude that the class of all finite unions and intersections of \(S\) is a topology.

Part II: Assume now that \(S\) is non-empty, then we define \(B\) to be the class of all finite intersections of sets in \(S\), and \(T\) to be the class of all finite unions of sets in \(S\). We now want to prove that \(T\) is a topology by showing that if
\[
\{G_1, G_2, ..., G_n\}
\]
is a non-empty finite class of sets in $T$, then

$$G = \bigcap_{i=1}^{n} G_i$$

is also in $T$. We already know that $\emptyset$ and $X$ is in $T$, and they are closed under the formation of arbitrary unions. Since the empty set is in $T$ we can assume that $G$ is non-empty. We let $x$ be a point in $G$, and it follows that $x$ is also contained in each $G_i$. From our definition of $T$ we have that for each $i$ there is a set $B_i$ in $B$ such that

$$x \in B_i \subseteq G_i.$$  

Each $B_i$ is a finite intersection of sets in $S$. This means that we can repeat this process for each $i$, and the intersection of all sets in $S$ which arises this way is a set in $B$. This set contains $x$ and is contained in $G$. This means that $G$ is a union of sets in $B$ and thus itself a set in $T$. □

We move on to define closed bases and subbases.

**Definition 2.13.** Let $X$ be a topological space. A class of closed subsets of $X$ is called a closed base if the class of all complements of its sets is an open base. In a similar manner, a class of closed subsets of $X$ is called a closed subbase if the class of all complements is an open subbase.

Lastly, we introduce the notation of open covers and subcovers.

**Definition 2.14.** Let $X$ be a topological space. A class $\{S_i\}$ of open subsets of $X$ is said to be an open cover of $X$ if

$$X = \bigcup_{i} S_i.$$ 

A subclass of an open cover which itself is an open cover is said to be a subcover.

**Definition 2.15.** Let $X$ be a topological space. An open cover of $X$ whose sets are all in some given open base is called a basic open cover, and if they all lie in some given open subbase it is called a subbasic open cover.

In the coming sections we shall frequently use the properties of bases and covers in our theorems and proofs. Since there are quite a lot of definitions for bases and covers, all relating to each other in different ways, this can easily become confusing. We therefore provide Figure 2.2, which is a mind map of all these definitions that can be useful to look back at from time to time when reading the forthcoming sections.
Topological space

A topological space $X$ with the topology $T = \{T_i\}$. Each set $T_i$ is open and $T_i'$ is closed. All elements $x \in X$.

Open cover
A class $A = \{A_i\}$ where each $A_i$ is open and $X = \cup_i A_i$. Each $x$ belong to at least one $A_i$.

Subcover
A class $B = \{B_i\}$ where each $B_i$ is open and in an open cover $A$. $B$ is also an open cover.

Basic open cover
An open cover $A$ where each $A_i$ is in an open base; $A_i \subseteq C$.

Subbasic open cover
An open cover $A$ where each $A_i$ is in an open subbase; $A_i \subseteq D$.

Closed base generated by the closed subbase
A close based $E$ where each $E_i$ is a finite unions of $F_i$'s; $E_i = F_1 \cap F_2 \cap ... \cap F_n$.

Open base
A class $C = \{C_i\}$ where each $C_i$ is open. All open sets in $X$ are unions of $C_i$.

Open subbase
A class $D = \{D_i\}$ where each $D_i$ is open, $D_i \subseteq X$, and whose finite intersections form an open base $C$.

Closed base
A class $E = \{E_i\}$ where each $E_i$ is closed and $\{E_i'\}$ is an open base; $\{E_i'\} = C$.

Closed subbase
A class $F = \{F_i\}$ where each $F_i$ is closed and $\{F_i'\}$ is an open subbase; $\{F_i'\} = D$.

Figure 2.2. Mind map of bases and covers in a topological space.
3. Product spaces

When welding together the sets of a given class we call the resulting set their product, and in this section we define the main concepts which follows from this process. The reader will find that this section does not rely as heavily on definitions as the previous one, and does not produce a single theorem. Instead it serves to aid the reader in creating a visual image of the product of sets or spaces, in order to ease the understanding of the following sections. We begin by defining the product.

**Definition 3.1.** Let \{X_i\} be a class of \(n\) non-empty sets. Their product

\[ X = X_1 \times X_2 \times \ldots \times X_n \]

is defined to be the set of all ordered pairs in the array \((x_1, x_2, \ldots, x_n)\), where each \(x_i \in X_i\) for each \(i = 1, 2, \ldots, n\).

**Definition 3.2.** Let \(\{X_i\}\) be a class of \(n\) non-empty sets, \(X\) the product of the spaces in the form of \(X_1 \times X_2 \times \ldots \times X_n\), and \(x\) a point in the product in the form of

\[ x = (x_1, x_2, \ldots, x_n). \]

The mapping \(p_i\) of the product onto its \(i\)th coordinate set \(X_i\) is defined to be

\[ p_i(x) = x_i \]

and is called the projection onto the \(i\)th coordinate.

We now have a definition for the products of finite classes of sets. However, this will not be general enough for our forthcoming theorems which requires us to work with arbitrary classes of sets. To achieve a more general definition we note that the array \((x_1, x_2, \ldots, x_n)\) basically is a function, which we call \(x\). This function has the index set \(I\) as its domain, and the restriction that its value \(x(i) = x_i\) is an element of the set \(X_i\) for each \(i\) in \(I\). With this in mind we can state the following definition.

**Definition 3.3.** Let \(\{X_i\}\) be a non-empty class of non-empty sets, where each element \(i\) belongs to an index set \(I\). The products of the sets \(X_i\) is written as

\[ X = P_i X_i = \prod_i X_i, \]

and is defined to be the set of all functions \(x\) defined on \(I\) such that \(x(i)\) is an element of the set \(X_i\), for each \(i\).

Out of convenience we will use the subscript notation \(x_i\) instead of the function notation \(x(i)\) used in Definition 3.3.

Since we know that all topological spaces consist of non-empty sets we can use this theory for the product of topological spaces. The resulting product will be a non-empty set which allows us to define a new base for a topology as follows.

**Definition 3.4.** Let \(\{X_i\}\) be any non-empty class of topological spaces and consider the product

\[ X = P_i X_i. \]
The *product topology* on $X$ is defined as the topology generated by the class $S$ of all inverse images of open sets in the $X_i$’s, meaning the class $S$ of all open subsets of $X$ in the form

$$S = p_i^{-1}(G_i)$$

where $i$ is any index and $G_i$ is any open subset of $X_i$.

If $X$ is a topological space equipped with the product topology, we see from Definition 3.4 that the projections $p_i$ are continuous.

In order to visualize the concept introduced in Definition 3.4 we refer the reader to Figure 3.1, in which we have illustrated how one can imagine the open sets of $S$ in the case of

$$X = X_1 \times X_2$$

where $X_1$ and $X_2$ are arbitrary topological spaces, and $G_1$ and $G_2$ are two open sets in respective space.

![Diagram of product space](image)

**Figure 3.1.** The product space of two arbitrary topological spaces $X_1$ and $X_2$.

We see from Definition 3.4 and Figure 3.1 that $S$ can also be described as the class of all products of the form

$$S = P_i G_i$$

where $G_i$ is an open subset of $X_i$ which equals $X_i$ for all $i$’s but one.
**Definition 3.5.** The class $S$ is called the *defining open subbase* for the product topology, and the class of all complements of sets in $S$ is called the *defining closed subbase*.

The defining closed subbase can also be described as a class of all products of the form $P_i F_i$ where $F_i$ is a closed subset of $X_i$ which equals $X_i$ for all $i$’s but one. The class of all finite intersections of $S$ also generates an open base, which is called the *defining open base*.

Since our goal is to prove a theorem dealing with products of an arbitrary number dimensions, it is advantageous to be able to visualize an open set in a space consisting of more than two dimensions. We do this in the following way; let the set $I$ consist of all real numbers $i$ on the closed unit interval $[0, 1]$ and let each index $i$ correspond to a topological space $X_i$. We let each $X_i$ be a replica of the same closed unit interval, $[0, 1]$, with its usual topology. The resulting product space, $X = P_i X_i$ can be viewed in Figure 3.2. We have in this picture defined the base to be the set $I$ and each vertical cross section represent the coordinate space $X_i$ corresponding the $i$ value on the horizontal base.

![Figure 3.2](image-url)  

**Figure 3.2.** The product space of a class of sets $\{X_i\}$, where each $X_i$ is equal to the closed interval $[0, 1]$.

An element in $X$ is an array of points where every point is an element of its corresponding $X_i$. Such an element is essentially a function defined on the set $I$ in this case, if we identify each function with its graph. This can be seen for two arbitrary examples, $f$ and $g$ in Figure 3.2. We now choose an arbitrary finite set of indices, $\{i_1, i_2, i_3\}$ and for each index we define an open set $\{G_1, G_2, G_3\}$ where
each

\[ G_i \subseteq X_i. \]

These sets are shown as the thick black rectangles along the corresponding vertical cross sections in Figure 3.2. Our basic open set then consists of all functions in \( X \) whose graphs cross each of vertical segments within the given open set on that segment. As we can see in our figure, \( f \) is a part of the basic open set, but \( g \) is not. Thus one way to visualize the basic open set is to imagine all elements in \( X \) as fibers, bundled together by the rectangles which represents some open sets in each corresponding space.
4. Compact spaces

We now introduce the concept of compactness, which is the central concept in Tychonoff’s theorem. The theorems and definitions concerning compact spaces relies heavily on the properties of covers and bases, and the reader is encouraged to look back at Figure 2.2 from time to time in this section. We begin by defining what we mean by a compact space.

**Definition 4.1.** A compact space is a topological space $X$ such that every open cover of $X$ has a finite subcover. A subspace which is compact as a topological space in its own right is called a compact subspace.

We illustrate this concept with some examples.

**Example 4.2.** The real line $\mathbb{R}$ is not a compact space, since the open cover $A = \{(n, n + 2) : n \in \mathbb{Z}\}$ does not have a finite subcover. □

**Example 4.3.** We let $X$ be a subspace of the real line $\mathbb{R}$ defined as $X = \{0\} \cup \{1/n : n \in \mathbb{Z} \text{ and } n > 0\}$. Then we know that $X$ is compact. That can be shown in the following way: for any given cover $A$ of $X$ we know that there exists an element $B$ in $A$ which contains 0. The set $B$ will contain all except finitely many points of $X$. We choose for each point in $X$ that is not in $B$ an element in $A$ containing it. The resulting collections of elements of $A$ combined with $B$ is a finite subcover of $A$ that covers $X$. □

**Example 4.4.** Any space $X$ which contains a finite amount of points is compact, since any open cover of $X$ is finite. □

We now move on to state our first theorem concerning compact spaces.

**Theorem 4.5.** Any closed subspace of a compact space is compact.

*Proof.* Let $Y$ be a closed subspace of a compact space $X$, and let $\{G_i\}$ be an open cover of $Y$ and $\{H_i\}$ be a class of open subsets in $X$. Each $G_i$, which is open in the relative topology of $Y$, is the intersection with $Y$ of some $H_i$. Since $Y$ is closed we know that $Y'$ is open. This means that the class composed of $Y'$ and all $H_i$’s is an open cover of $X$. Also, since $X$ is compact this open cover has a finite subcover. If $Y'$ occurs in this subcover we discard it, and what remains is a finite class of $H_i$’s whose union form $X$. The corresponding $G_i$’s then form a finite subcover of the original open cover of $Y$, which concludes the proof. □

Another theorem that will be of importance for us, when we later on will show that the converse of Tychonoff’s theorem also is true, is the following.

**Theorem 4.6.** Any continuous image of a compact space is compact.

*Proof.* Let $f : X \rightarrow Y$ be a continuous mapping of a compact space $X$ into an arbitrary topological space $Y$, and $\{G_i\}$ an open cover of $f(X)$. Each $G_i$ is then a
intersection with \( f(X) \) of an open subset \( H_i \) of \( Y \). The class \( \{f^{-1}(H_i)\} \) is an open cover of \( X \) since \( f \) is continuous and each \( H_i \) is open.

Since \( X \) is compact this class has a finite subcover. The union of the finite class of \( H_i \)'s of which these are the inverse images must contain \( f(X) \), so the class of corresponding \( G_i \)'s is a finite subcover of the original open cover of \( f(X) \). This means that \( f(X) \) is compact and that concludes the proof. \( \square \)

It can be quite difficult to show that a topological space is compact just by Definition 4.1, and therefore the following theorem might come in handy.

**Theorem 4.7.** A topological space \( X \) is compact if, and only if, every class of closed sets in \( X \) with empty intersections has a finite subclass with empty intersections.

**Proof.** Let \( \{G_i\} \) be an open cover of a compact space \( X \). We know that

\[
\bigcup_i G_i = X
\]

and therefore, we have that

\[
\left( \bigcup_i G_i \right)' = \bigcap_i G_i' = X' = \emptyset.
\]

Thus, for every class of closed sets with empty intersections we can find a corresponding class of open covers, and since \( \{G_i\} \) has a finite amount of subcovers we can find a corresponding finite amount of subclasses in the closed set. The same reasoning can be used to show the implication right to left. \( \square \)

**Definition 4.8.** Let \( A \) be a class of subsets of some topological space. We say that \( A \) has the **finite intersection property** if every finite subclass of \( A \) has non-empty intersection.

**Theorem 4.9.** A topological space \( X \) is compact if, and only if, every class of closed sets in \( X \) with the finite intersection property has non-empty intersection.

**Proof.** This theorem is a direct consequence of Theorem 4.7 and Definition 4.8. \( \square \)

**Theorem 4.10.** A topological space \( X \) is compact if every basic open cover of \( X \) has a finite subcover.

**Proof.** Let \( \{G_i\} \) be an open cover and \( \{B_j\} \) an open base. Each \( G_i \) is the union of certain \( B_j \)'s and the totality of those \( B_j \)'s is a basic open cover. By our hypothesis this class of \( B_j \)'s must have a finite subcover, and for each set in this finite subcover we can select a \( G_i \) which contains it. The class of \( G_i \)'s that arise this way is a finite subcover of the original open cover. \( \square \)

The next theorem we will show relating to compact spaces is of great importance to us, since it will simplify the proof of Tychonoff’s theorem greatly. The theorem provides a simple equivalence for compact topological spaces, but its proof is long and rather difficult. Before we can show it however, we must introduce the concept of partial order relations and Zorn’s lemma.
Definition 4.11. Let $P$ be a non-empty set. A partial order relation in $P$ is a relation which is symbolized by $\leq$ and assumed to have the following properties:

(i) $x \leq x$ for every $x$ (reflexivity);
(ii) $x \leq y$ and $y \leq x \implies x = y$ (antisymmetry);
(iii) $x \leq y$ and $y \leq z \implies x \leq z$ (transitivity).

A non-empty set $P$ in which there is defined a partial order relation is called a partially ordered set. Some partially ordered sets also possess a fourth property:

(iv) Any two elements are comparable.

If $P$ has this extra property then it is called a chain.

Lemma 4.12 (Zorn’s Lemma). If $P$ is a partially ordered set in which every chain has an upper bound, then $P$ possesses a maximal element.

It is not possible to prove Lemma 4.12 in the usual sense of the word. It is however possible to show that it is logically equivalent to the axiom of choice, which we mentioned in the introduction. We will look further at this in Section 5.

We are now in a position to show the last theorem of this section.

Theorem 4.13. A topological space $X$ is compact if every subbasic open cover of $X$ has a finite subcover, or equivalently, if every class of subbasic closed sets in $X$ with the finite intersection property has non-empty intersections.

Proof. The equivalence of the two statements follows directly from Theorem 4.7 and Theorem 4.9. We consider a closed subbase $\{S_i\}$ for our space, and let $\{B_i\}$ be its generated closed base. As in the statement we assume that every class of subbasic closed sets with the finite intersection property have a non-empty intersection, and by Theorem 4.10 it is then enough to prove that every class of $B_i$’s with the finite intersection property also have a non-empty intersection.

Let $B_j$ be a class of $B_i$’s with the finite intersection property. We must show that $\bigcap_j B_j$ is non-empty. We use Zorn’s lemma to show that $\{B_k\}$ is contained in some class $\{B_k\}$ of $B_i$’s which is maximal with respect to having the finite intersection property. This is in the sense that $\{B_k\}$ has this property and any class of $B_i$’s which properly contains $\{B_k\}$ fails to have this property.

We do this by considering the family of all classes of $B_i$’s which contain $\{B_j\}$ and have the finite intersection property. This is a partially ordered set with respect to set inclusion. If we consider a chain in this partially ordered set, then the union of all classes in it is a class of $B_i$’s that contains every member of the chain and has the finite intersection property. This follows from the fact that every finite class of sets in this class of $B_i$’s is contained in some member of the chain, and that member has the finite intersection property. It follows that every chain in our partially ordered set has an upper bound. Therefore Zorn’s lemma guarantees that the partially ordered set has a maximal element. This shows the existence of a class $\{B_k\}$ with the properties stated above, and since

$$\bigcap_k B_k \subseteq \bigcap_j B_j$$
it is now enough to show that $\cap_k B_k$ is non-empty.

Each $B_k$ is a finite union of sets in our closed subbase, e.g.

$$B_1 = S_1 \cup S_2 \cup \ldots \cup S_n.$$  

It now suffices to show that at least one of the sets $S_1, S_2, \ldots, S_n$ belongs to the class $\{B_k\}$. If we obtain such a set for each $B_k$ then the resulting class of subbasic closed sets will have the finite intersection property since it is contained in $\{B_k\}$. Therefore, by our hypothesis relating to the subbasic closed sets, it will have non-empty intersection. Since this non-empty intersection will be a subset of $\cap_k B_k$ we shall know that $\cap_k B_k$ is non-empty.

We now show that at least one of the sets in $S_1, S_2, \ldots, S_n$ does in fact belong to the class $\{B_k\}$ by assuming that each of these sets is not in this class, and then deduce a contradiction from this assumption. Since $S_1$ is a subbasic closed set it is also a basic closed set. We also know that it is not in the class $\{B_k\}$ and therefore the class $\{B_k, S_1\}$ is a class of $B_i$’s which properly contains $\{B_k\}$. By the maximal property of $\{B_k\}$ the class $\{B_k, S_1\}$ lacks the finite intersection property, and thus $S_1$ is disjoint from the intersection of some finite class of $B_k$’s.

We repeat this argument for each of the sets $S_1, S_2, \ldots, S_n$. We then see that $B_1$, the union of these sets, is disjoint from the intersection of the total finite class of all the $B_k$’s which arise in this way. This contradicts the finite intersection property for the class $\{B_k\}$ and thus completes the proof. $\Box$

The proof of Theorem 4.13 is quite complex, but it is all worthwhile since this theorem will considerably simplify our proof of Tychonoff’s theorem.
5. Tychonoff’s theorem and its equivalence with the axiom of choice

We are now ready to state and prove Tychonoff’s theorem.

**Theorem 5.1 (Tychonoff’s theorem).** The product of any non-empty class of compact spaces is compact in the product topology.

**Proof.** Let \( \{X_i\} \) be a non-empty class of compact spaces, and form the product

\[
X = P_iX_i.
\]

Let \( \{F_j\} \) be a non-empty subclass of the defining closed subbase for the product topology on \( X \), which means that each \( F_j \) is a product of the form

\[
F_j = P_iF_{ij}
\]

where \( F_{ij} \) is a closed subset of \( X_i \) which equals \( X_i \) for all \( i \)'s but one. We assume that the class \( \{F_j\} \) has the finite intersection property, and therefore by Theorem 4.13 we only have to show that

\[
\bigcap_j F_j \neq \emptyset.
\]

For a given fixed \( i \) the class \( \{F_{ij}\} \) is a class of closed subsets of \( X_i \) with the finite intersection property, and by the assumed compactness of \( X_i \) and Theorem 4.9 there exists a point \( x_i \) in \( X_i \) which belongs to \( \cap_j F_{ij} \). If we repeat this argument for each \( i \), we obtain a point

\[
x = \{x_i\}
\]

in \( X \). As we can see this point also has the property that

\[
x \in \bigcap_j F_j,
\]

which concludes the proof. \( \square \)

The converse of Tychonoff’s theorem is also true, as we shall see below.

**Proposition 5.2.** Let \( X \) be a non-empty product of any non-empty class of spaces \( \{X_i\} \), where each \( i \) belongs to a index set \( I \). If \( X \) is compact in the product topology, then \( X_i \) is compact for every \( i \).

**Proof.** Let \( \alpha \in I \), and \( x_\alpha \in X_\alpha \). Then it follows by Definition 3.2 that

\[
p_\alpha(x) = x_\alpha
\]

and therefore that

\[
p_\alpha(P_iX_i) = p_\alpha(X) = X_\alpha.
\]

We know that the projections are continuous by the construction of the product topology. By Theorem 4.6 and the fact that \( X \) is compact, we see that \( X_\alpha \) is compact. \( \square \)

In our proof of Tychonoff’s theorem we used Theorem 4.13 which is derived using Zorn’s lemma. This lemma is in turn logically equivalent with the axiom of choice (see e.g. [5]), that states the following.
**Axiom 5.3 (Axiom of choice).** Given any non-empty class of non-empty sets, a set can be formed which contains precisely one element taken from each set in the given class.

The axiom of choice is usually considered the most important of the choice axioms. It has an important impact in many parts of pure mathematics, and is included as an assumption in the axiomatic set theory most commonly studied today. This theory is known as the as the ZFC (Zermelo–Fraenkel set theory with the axiom of choice). The concept of this axiom is illustrated in Figure 5.1.

![Figure 5.1](image)

**Figure 5.1.** An example which illustrates the axiom of choice. From the non-empty class \( \{S_i\} \) we can create the set \( \{x_i\} \).

We know that the axiom of choice implies Tychonoff's theorem, but as we shall see the reverse implication is also true. Hence,

**Theorem 5.4.** Tychonoff’s theorem is true if, and only if, the axiom of choice is true.

**Proof.** $\Rightarrow$: Let \( \{A_i\} \) be a class of non-empty sets where $i \in I$ and $I$ is an arbitrary index set. We want to show that

\[ P_i A_i \neq \emptyset. \]

since this means, by Definition 3.3, that there exists an element in each $A_i$. We define a new class \( \{X_i\} \) where for each $i$ we let

\[ X_i = A_i \cup \{i\}, \]

in other words every $X_i$ is the disjoint union of an $A_i$ and its corresponding index element $i$. We define the product of these sets to be

\[ X = P_i X_i \]

and the natural projection $p_i$ follows from this, which takes an element of $X$ and maps to its $i$th term.

We define the topology on each $X_i$ to be the topology $T_i$ which contains all subsets of $X_i$ whose complement in $X_i$ is a finite set (this is known as the cofinite...
topology), plus the empty set and the singleton \{i\}. Numerically we can write this as

\[ T_i = \{ \{ A \subseteq X : A \setminus X \text{ is finite} \}, \emptyset, \{i\} \}. \]

This means that the only closed sets in each \(X_i\) is finite and that each \(X_i\) is compact. By Tychonoff’s theorem we then know that their product \(X\) also is compact.

We know that every singleton \(\{i\}\) is a part of the topology on each \(X_i\) and thus an open set, which means that

\[ \{i\}' = A_i \]

is closed, for each \(\{i\}\) and \(A_i\) in \(X_i\). From this it follows that each \(p_i^{-1}(A_i)\) is a closed subset of \(X\) since it is the inverse projection of a closed set. We note that

\[ P_i A_i = \bigcap_{i \in I} p_i^{-1}(A_i) \] (5.1)

and so we only need to show that each

\[ p_i^{-1}(A_i) \neq \emptyset \]

and that the class \(\{p_i^{-1}(A_i)\}\) has the finite intersection property.

We let

\[ i_1, i_2, \ldots, i_n \]

be a finite collection of indices in \(I\). Then we know that the finite product

\[ A_{i_1} \times A_{i_2} \times \ldots \times A_{i_n} \]

is non-empty; it consists of \(n\)-tuples and we let

\[ a = (a_1, a_2, \ldots, a_n) \]

be such an \(n\)-tuple. We now extend \(a\) to cover all \(i\)'s, by defining the function

\[ f(j) = \begin{cases} a_k & \text{if } j = i_k, \\ j & \text{otherwise.} \end{cases} \]

Here we see the use of our definition of \(X_i\) with the extra point \(\{i\}\), since each

\[ j \neq i_k \]

is in \(X_j\) and thus \(f\) is defined for everything outside the \(n\)-tuple. This is illustrated in Figure 5.2.

We see that

\[ p_{i_k}(f) = a_k \in A_{i_k} \]

and it follows that the intersections of the inverse images corresponding to the \(n\)-tuple is non-empty, i.e.

\[ \bigcap_{k=1}^{n} p_{i_k}^{-1}(A_{i_k}) \neq \emptyset. \]

This means that the class \(\{p_i^{-1}(A_i)\}\) has the finite intersection property, and since it is a class of closed subset of the compact space \(X\) we know by Theorem 4.9 that

\[ \bigcap_{i \in I} p_i^{-1}(A_i) \neq \emptyset \]
and it follows from equation (5.1) that

$$P_i A_i \neq \emptyset$$

which concludes the proof of the equivalence from left to right.

$\Leftarrow$: In the proof of Tychonoff’s theorem we used Theorem 4.13, which is proved using Zorn’s lemma. This lemma is in turn logically equivalent with the axiom of choice, which concludes our proof of the equivalence from right to left. \(\square\)
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