The Riesz Representation Theorem For Positive Linear Functionals

Jimmy Norqvist

VT 2016
Examensarbete, 15hp
Kandidatexamen i matematik, 180hp
Institutionen för matematik och matematisk statistik

Abstract

This essay serves as an elementary introduction to measure theory and topology, which is then used to prove the famed Riesz representation theorem.

Sammanfattning

Denna uppsats består av en elementär introduktion till måtteori och topologi, som sedan används för att bevisa den berömda Riesz representationssats.

Contents

1. Introduction

In 1909 the Hungarian mathematician Frigyes Riesz [7] proved a theorem stating that all continuous linear functionals can be represented by Riemann-Stieltjes integrals on the unit interval. Using Riesz original notation it looked like this:

$$A[f(x)] = \int_0^1 f(x)d\alpha(x),$$

where α is a function of bounded variation on the unit interval.

This has become known as the Riesz representation theorem. Since Riesz's originial proof, mathematicians have been able to extend this theorem to concern more general spaces, and thus they have created a family of theorems which all go by the name Riesz representation theorem. In 1938 the Russian mathematician Andrey Markov extended Riesz's result to some non-compact spaces [5], and three years after that, in 1941, the Japanese-American mathematician Shizuo Kakutani proved a theorem regarding compact Hausdorff spaces [4]. The main purpose of this essay is to give a complete proof of the following theorem, which is known as the Riesz-Markov-Kakutani representation theorem.

Theorem 4.1. *Let X be a locally compact Hausdorff space, and let I be a positive linear functional on the space of continuous functions with compact support. Then there is a unique regular Borel measure μ on X such that*

$$I(f) = \int f d\mu$$

holds for each function f in the space of continuous functions with compact support on X.

This particular version of the theorem can be found in [2], which is the book that inspired this essay. J.D. Gray begins his essay [3] about the history of the theorem with the remark:

> "Only rarely does the mathematical community pay a theorem the accolade of transforming it into a tautology. The Riesz representation theorem has received this accolade."

This suggests that the mentioned theorem has had a big impact on the world of mathematics. An obvious reason for this is that the theorem allows one to change the point of view of a problem, from linear functionals to integrals and measures, and vice versa. Another contributing factor is that there are many ways to alter the theorem; like which type of functionals, measures, spaces, etc. that is being assumed. Because of this, the Riesz representation theorem turns up in many areas of mathematics. Some examples are category theory, linear algebra, real analysis, complex analysis and probability theory. An exampel from probability theory is that Anthony Narkawicz of NASA Langley Research Center published a complete proof of the theorem in [6], as part of a project about "formalizing probabilistic models of aircraft behaviour in the airspace".

This essay is meant to be a self contained proof of Theorem 4.1 from a perspective of topology and measure theory, i.e. the reader is not assumed to possess prior knowledge of either. Some familiarity with analysis is however needed, but only a minor part of what is covered by any introductory course. Forgetful readers; fear not! There will be reminders of the most important results, and anything that may have been omitted can be easily looked up elsewhere (see e.g. [1], [8]).

The Riesz representation theorem (later referred to by RRT) belongs to functional analysis, which is a branch of mathematics that partly is about vector spaces endowed with a topology. In this essay we deal with the vector spaces consisting of continuous functions with compact support defined on a locally compact Hausdorff space. The only thing however, that we need from functional analysis (which is neither covered by measure theory nor topology) is the definition of a linear functional. Because of this, we have dedicated one chapter each for measure theory and topology, but none for functional analysis. These two chapters includes all the anticipatory theory that we need to prove RRT, but they also serve as elementary introductions to these fields.

Chapter 2 has a clear goal in Theorem 2.12; which says that measures can be constructed by a certain restriction on outer measures. This is the key idea in the proof of RRT, and it is clear by comparing the definitions of measures and outer measures why this is the case; outer measure are simply easier to construct from scratch than measures.

Since the proof is about constructing a measure; the purpose of Chapter 3 (which is about topology) does not have the same clarity as Chapter 2. There are however, three theorems from this chapter that will be used directly in the proof of RRT. Of these three, Theorem 3.20 might be the most influential one, since it is not only used to prove the uniqueness of the measure, but it also inspires the initial construction of the function used to create the coveted measure. The other two main theorems of this chapter are Theorem 3.17 and Theorem 3.19, both of which are used to show that the constructed measure indeed is a regular Borel measure.

2. Measure theory

Like the name advocates, measures are supposed to quantify the size of things. Our intuition of this is very good if we are in $\mathbb{R}^1$, $\mathbb{R}^2$ or $\mathbb{R}^3$; but for measuring objects in a general set, our intuition does not take us very far. Therefore some abstractions needs to be made that agrees with our intuition of the concept. We will go through these abstractions, which are basically that we assign values to subsets of the space we are measuring in. This chapter consist mostly of definitions, but it is concluded with one of the most important theorems of this essay, which says that measures can be created via outer measures, and this is essential in the constructive proof we present of RRT in Chapter 4.

Definition 2.1. Let X be an arbitrary set. A collection $\mathscr{A}$ of subsets of X is called a *σ-algebra* if

(a) $X \in \mathscr{A}$,

(b) if $A \in \mathscr{A}$, then the complement of A, denoted by A^c is also in $\mathscr{A}$,

(c) if $A_1, A_2, ...$ are members of $\mathscr{A}$, then $\cup A_n$ is also a member of $\mathscr{A}$.

The properties stated in (b) and (c) are called *closed under complementation* and *closed under countable unions* respectively. The following two requirements can be excluded since they follow from the definition above, but they can be nice to keep in mind as alternative definitions for intuitive reasons.

(a′) $\emptyset \in \mathscr{A}$

(c′) If $A_1, A_2, ...$ are members of $\mathscr{A}$, then $\cap A_n$ is also a member of $\mathscr{A}$.

Specifically, we end up with equivalent definitions by changing (a) to (a′) or (c) to (c′). This follows from (b) together with the properties $X^c = \emptyset$ and $\cap A_i = (\cup A_i^c)^c$.

Remark. $\mathscr{A}$ is called an *algebra* if (c) is restricted to the formation of *finite* unions instead of *countable* unions.

Example 2.2. Let X be a set and let $\mathscr{A}$ be the collection of all subsets of X. Then $\mathscr{A}$ is a σ-algebra on X.

Example 2.3. Let X be a set and let $\mathscr{A} = \{X, \emptyset\}$. Then $\mathscr{A}$ is a σ-algebra on X.

Example 2.4. Let X be a set and let $\mathscr{A}$ be the collection of all subsets A of X such that either A or A^c is countable. Then $\mathscr{A}$ is a σ-algebra.

Definition 2.5. Let $\mathscr{A}$ be a σ-algebra, let $\mu : \mathscr{A} \to \mathbb{R}$ be a function and let I be an index set. Then we say that

(i) μ is *additive* if

$$\mu\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} \mu(A_i),$$

where $\{A_i\}$ is a collection of disjoint sets in $\mathscr{A}$, and that

(ii) μ is *subadditive* if

$$\mu\left(\bigcup_{i\in I} B_i\right) \leq \sum_{i\in I} \mu(B_i),$$

where $\{B_i\}$ is a collection of arbitrary sets in $\mathscr{A}$.

To these terms we can add the words *finitely*, *countably* or *uncountably* to specify the index set. In this essay we will almost exclusively deal with countably additive and countably subadditive functions.

Definition 2.6. Let X be a set and $\mathscr{A}$ be a σ-algebra on X. Then a *measure* on X is a function $\mu : \mathscr{A} \to [0, \infty]$ which is countably additive and satisfies $\mu(\emptyset) = 0$.

Note that this definition agrees with our intuition; the measure of the empty set should be zero, since it consists of nothing. And if we cut up an object into pieces, the measures of these pieces should add up to the measure the object had when still in one piece.

Definition 2.7. Let X be a set and let μ be a measure on X. If A is a subset of X such that $\mu(A)$ is defined, then A is said to be a *μ-measurable set*, or a *measurable set* with respect to the measure μ.

In Chapter 3 we will define what an *open set* is in a more general setting. For now the reader can think of X as a metric space whenever we deal with open sets (see e.g. [1] or [8] for a reminder of metric spaces).

Definition 2.8. Let X be a set, let $\mathscr{A}$ be a σ-algebra on X and let μ be a measure on $\mathscr{A}$. Then μ is said to be *regular* if

(a) each compact subset $K \subseteq X$ satisfies $\mu(K) < \infty$,

(b) each set $A \in \mathscr{A}$ satisfies

$$\mu(A) = \inf\{\mu(U) : A \subseteq U \text{ and } U \text{ is open}\},$$

(c) and each open set $U \subseteq X$ satisfies

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \text{ is compact}\}.$$

The property described by (b) is called *outer regularity*, and says that every set in $\mathscr{A}$ can be approximated from the outside by open measurable sets. The property described by (c) is called *inner regularity*, and says that every open subset of X can be approximated from within by compact measurable sets.

For the next definition; recall that the set consisting of all subsets of X is called the *power set* of X, and denote it by $\mathscr{P}(X)$.

Definition 2.9. Let X be a set. An *outer measure* on X is a function μ^*:$\mathscr{P}(X) \to [0, \infty]$ which satisfies

(a) $\mu^*(\emptyset) = 0$,

(b) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B \subseteq X$,

(c) $\mu^*(\cup A_i) \leq \sum \mu^*(A_i)$, where $\{A_i\}$ is an infinite sequence of subsets of X.

This means that outer measures are countably subadditive and *monotonically increasing* (property (b)). An observation we can make here is that a measure is also an outer measure only if $\mathscr{P}(X)$ is the σ-algebra that is its domain. A question that will occur later is whether or not a set is outer measurable. The definition for a set B to be *outer measurable*, or μ^**-measurable* given an outer measure μ^*; is for the equation

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) \tag{2.1}$$

to hold for all $A \in \mathscr{P}(X)$.

In this essay our primary concern regarding the μ^*-measurability of sets will be to confirm whether they are μ^*-measurable or not. Note that in equation (2.1) both $A \cap B$ and $A \cap B^c$ are disjoint sets whose union equals A. Since outer measures are subadditive by definition the inequality

$$\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

follows. Hence we only need to verify the reverse inequality, which leads to the following proposition.

Proposition 2.10. *Let X be a set and let μ^* be an outer measure on X. Then a subset B of X is μ^*-measurable if*

$$\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

holds for every $A \subseteq X$ such that $\mu^*(A) < \infty$.

The reason why we only need to check for sets with a finite measure is that the inequality is trivially true for sets of infinite measure. Next follows some important definitions we need in order to know what a regular Borel measures is, which of course is the type of measure considered in RRT.

A *Borel set* is a set that can be represented by any countable combination of unions, intersections and relative complements of open sets. An equivalent definition can be made by using closed sets instead of open sets, since the complement of an open set U relative to X is a closed set.

The σ-algebra generated by the collection of all the Borel subsets of X is called the *Borel σ-algebra* on X, and we denote it by $\mathscr{B}(X)$. If a measure has $\mathscr{B}(X)$ as its domain, it is called a *Borel measure*, and finally; a Borel measure that is regular is called a *regular Borel measure*.

That was all the definitions we need from measure theory. Next we take a look at a lemma that will be used to prove the important Theorem 2.12.

Lemma 2.11. *Let X be a set, and let μ^* be an outer measure on X. Then each subset B of X that satisfies $\mu^*(B) = 0$ or $\mu^*(B^c) = 0$ is μ^*-measurable.*

Proof. Assume that either B or B^c has an outer measure of zero. By Corollary 2.10 we only need to show that

$$\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c).$$

By symmetry it does not matter which one has the zero measure, so let us assume that $\mu^*(B) = 0$. It follows that $\mu^*(A \cap B) = 0$ since outer measures are monotonically increasing by Definition 2.9. The only thing that is left to verify is that

$$\mu^*(A) \geq \mu^*(A \cap B^c),$$

and once again we can fall back on the definition that outer measures are monotonically increasing. □

Theorem 2.12. *Let μ^* be an outer measure on the set X, and let $\mathscr{M}_{\mu^*}$ be the collection of all μ^*-measurable subsets of X. Then*

(i) $\mathscr{M}_{\mu^*}$ is a σ-algebra, and

(ii) the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure on $\mathscr{M}_{\mu^*}$.

Proof. We begin by proving (i). First we show that X (or equivalently $\emptyset$) belongs to $\mathscr{M}_{\mu^*}$. Since μ^* is an outer measure, Lemma 2.11 implies that both X and $\emptyset$ belong to $\mathscr{M}_{\mu^*}$. To show that μ^* is closed under complementation; let B be a μ^*-measurable set, then by Definition 2.1 we have that the equation

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

holds for all $A \in \mathscr{P}(X)$. By the symmetry of this equation we can interchange B and B^c, therefore B^c is μ^*-measurable and with that μ^* is closed under complementation.

From part (i) we have left to show that $\mathscr{M}_{\mu^*}$ is closed under the formation of countable unions; but first, lets confirm that $\mathscr{M}_{\mu^*}$ is closed under the union of two sets. Suppose that B_1 and B_2 are μ^*-measurable subsets of X. We must show that $B_1 \cup B_2$ is μ^*-measurable. Let A

be an arbitrary subset of X, then $A \cap (B_1 \cup B_2)$ is also a subset of X. Hence the μ^*-measurability of B_1 implies that

$$\begin{aligned}\mu^*(A \cap (B_1 \cup B_2)) &= \mu^*(A \cap (B_1 \cup B_2) \cap B_1) + \mu^*(A \cap (B_1 \cup B_2) \cap B^c) \\ &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2). \qquad (2.2)\end{aligned}$$

For $B_1 \cup B_2$ to be μ^*-measurable the following equation must hold for all $A \in \mathscr{P}(X)$.

$$\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) = \mu^*(A). \qquad (2.3)$$

Note that $(B_1 \cup B_2)^c = B_1^c \cap B_2^c$, and that we can substitute this and equation (2.2) into equation (2.3) to get that

$$\begin{aligned}\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) = \mu^*(A \cap B_1) &+ \mu^*(A \cap B_1^c \cap B_2) \\ &+ \mu^*(A \cap B_1^c \cap B_2^c).\end{aligned}$$

Then we apply the μ^*-measurability of B_2 to the right-hand side of the above equation to get that

$$\begin{aligned}\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c) \\ &= \mu^*(A),\end{aligned}$$

where the last equality follows from the μ^*-measurability of B_1. Therefore equation (2.3) holds and $\mathscr{M}_{\mu^*}$ is closed under the formation of union of two sets.

Next we extend this result to countable unions of disjoint sets. Suppose that $\{B_i\}$ is an infinite sequence of pairwise disjoint μ^*-measurable sets. We need to show that

$$\mu^*(A) = \mu^*\left(A \cap \left[\bigcup_{i=1}^{n} B_i\right]\right) + \mu^*\left(A \cap \left[\bigcup_{i=1}^{n} B_i\right]^c\right)$$

holds for each subset A of X and each positive integer n. We prove this by induction on the following, equivalent representation of the above equation,

$$\mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \left[\bigcap_{i=1}^{n} B_i^c\right]\right). \qquad (2.4)$$

For the case $n = 1$, equation (2.4) is a restatement of the measurability of B_1 and hence true. Note that the μ^*-measurability of B_{n+1} and the pairwise disjointness of the elements in $\{B_i\}$ implies that

$$\begin{aligned}\mu^*\left(A \cap \left[\bigcap_{i=1}^{n} B_i^c\right]\right) &= \mu^*\left(A \cap \left[\bigcap_{i=1}^{n} B_i^c\right] \cap B_{n+1}\right) \\ + \mu^*\left(A \cap \left(\bigcap_{i=1}^{n} B_i^c\right) \cap B_{n+1}^c\right) &= \mu^*(A \cap B_{n+1}) + \mu^*\left(A \cap \left[\bigcap_{i=1}^{n+1} B_i^c\right]\right).\end{aligned} \qquad (2.5)$$

Substituting this into equation (2.4) yields

$$\mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*(A \cap B_{n+1}) + \mu^* \left(A \cap \left[\bigcap_{i=1}^{n+1} B_i^c \right] \right)$$
$$= \sum_{i=1}^{n+1} \mu^*(A \cap B_i) + \mu^* \left(A \cap \left[\bigcap_{i=1}^{n+1} B_i^c \right] \right).$$

Hence equality (2.4) holds for every positive integer n and the induction is complete.

A set can never increase by intersecting it with another set, therefore we get the inequality:

$$\mu^* \left(A \cap \left[\bigcap_{i=1}^{n} B_i^c \right] \right) \geq \mu^* \left(A \cap \left[\bigcap_{i=1}^{\infty} B_i^c \right] \right).$$

By applying the identity of equality (2.5) to the right-hand side of the above inequality and substituting the result of that into equation (2.4) we find that

$$\mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^* \left(A \cap \left[\bigcap_{i=1}^{\infty} B_i^c \right] \right)$$
$$= \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^* \left(A \cap \left[\bigcup_{i=1}^{\infty} B_i \right]^c \right). \tag{2.6}$$

Finally we apply the fact that μ^* is an outer measure and hence countably subadditive to get

$$\mu^*(A) \geq \mu^* \left(A \cap \left[\bigcup_{i=1}^{\infty} B_i \right] \right) + \mu^* \left(A \cap \left[\bigcup_{i=1}^{\infty} B_i \right]^c \right) \geq \mu^*(A),$$

which of course implies that

$$\mu^*(A) = \mu^* \left(A \cap \left[\bigcup_{i=1}^{\infty} B_i \right] \right) + \mu^* \left(A \cap \left[\bigcup_{i=1}^{\infty} B_i \right]^c \right).$$

This is the definition of $\bigcup_{i=1}^{\infty} B_i$ being a μ^*-measurable set, thus we have shown that $\mathscr{M}_{\mu^*}$ is closed under the formation of countable unions of disjoint sets. It remains to show that $\mathscr{M}_{\mu^*}$ is closed under the formation of countable unions of arbitrary sets. We show this by constructing a sequence of disjoint sets using a sequence of arbitrary sets. Let $\{A_i\}$ be sequence of arbitrary members of $\mathscr{M}_{\mu^*}$. Then

$$A_1, A_2 \cap A_1^c, A_3 \cap A_1^c \cap A_2^c, \cdots, A_n \cap A_1^c, \cap A_2^c \cap \cdots \cap A_{n-1}^c, \cdots$$

is a sequence of disjoint sets, hence $\mathscr{M}_{\mu^*}$ is closed under the formation of countable unions of arbitrary sets, and we have shown that $\mathscr{M}_{\mu^*}$ is a σ-algebra.

To prove (ii); that the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure, we must show that it is countably additive ($\mu^*(\emptyset) = 0$ follows from μ^* being an outer measure). Let $\{B_i\}$ be a sequence of disjoint sets in $\mathscr{M}_{\mu^*}$. By replacing A with $\bigcup_{i=1}^{\infty} B_i$ in inequality (2.6) we get that

$$\mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) \geq \sum_{i=1}^{\infty} \mu^*(B_i) + \mu^*(\emptyset).$$

Since the reverse inequality follows from the subadditivity of μ^*, the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is countably additive and with that: a measure.

□

3. Topology

Topology is sometimes called the modern version of geometry. Some say that the first idea in the development of topology is the famous *Königsberg bridge problem* (also considered the first theorem of graph theory), which is a problem concerning relative position without regards to distance. This was solved by Euler in the 19th century, so according to the time perspective of mathematics; topology is a brand new field, but already we see ideas from topology in most areas of mathematics today. The type of topological spaces that is of interest in this essay is Hausdorff spaces, therefore we quickly move on to these types of spaces after introducing the necessary general definitions regarding topological spaces. The main theorems in this chapter are Theorem 3.17, Theorem 3.19 and Theorem 3.20, which all will be used directly in the proof of RRT.

The most instructive way (according to the writer) to read this chapter, is to complement the reading by drawing simple pictures of what is going on. One reason that we do not include pictures, is that the process of actually drawing a picture may be even more instructive than the picture itself.

Definition 3.1. Let X be a set. A *topology* on X is a collection T of subsets of X having the following properties:

(1) X and $\emptyset$ are in T,

(2) if S is an *arbitrary* collection of sets that belong to T, then $\cup S \in T$,

(3) if S is a *finite* collection of sets that belong to T, then $\cap S \in T$.

Note that conditions (2) and (3) say that a topology T is closed under arbitrary unions and finite intersections.

A *topological space* is a pair (X, T) where X is a set and T is a topology on X. The common way to refer to a topological space (X, T) is by the name of the underlying set, since it is unusual having to consider multiple topologies simultaneously. In this essay the underlying set will always be called X. In a topological space (X, T), all members of the topology T are called *open sets*, and an *open neighbourhood* of a point $x \in X$ is an open set containing x. A *closed set* is a set whose complement is an open set.

Example 3.2. The *trivial* or *indiscrete* topology T on X is the smallest possible topology, consisting only of the empty set and X itself. That is $T = \{\emptyset, X\}$.

Example 3.3. The largest topology on X is called the *discrete* topology and it consists of every subset of X.

Example 3.4. Let $X = \{a, c, b\}$ and $T = \{\emptyset, X, a\}$. Then T is a topology on X, and since X is a finite set, this is an example of what is called a *finite* topology.

Definition 3.5. A topological space X is *Hausdorff* if; for each pair (x, y) of distinct points in X, there exist disjoint open subsets U and V of X such that $x \in U$ and $y \in V$.

This definition says that points in a Hausdorff space can be *separated* by neighbourhoods. Before we start with the theorems in this chapter we remind ourselves of some results from analysis (the following three definitions and theorems are taken from [1], interested readers can also read more about this in [8]).

Definition 3.6. Let A be a set. An *open cover* $\{U_n\}$ of A, is a collection of open sets such that $A \subseteq \cup U_n$. A subset of $\{U_n\}$ that satisfies $A \subseteq \cup U_n$ is called a subcover of $\{U_n\}$.

Definition 3.7. Let X be a topological space. A subset K of X is called *compact* if every open cover of K has a finite subcover.

Note that this definition says that if we have a compact set K and an open cover $\{U_n\}$ of K, then there exists a natural number N such that $K \subseteq \cup_{n=1}^{N} U_n$.

Theorem 3.8. *If X is a compact space, and if $f : X \to \mathbb{R}$ is a continuous function, then f takes a maximum and a minimum value on X.*

This means that continuous functions are bounded on compact sets. This comes to great use since we deal with continuous functions with compact support in this essay.

Now we are ready to begin with the theorems regarding Hausdorff spaces. The first one says that disjoint compact sets can be separated. So in some sense, points and compact sets are similar in Hausdorff spaces.

Theorem 3.9. *Let X be a Hausdorff space, and let K and L be disjoint compact subsets of X. Then there are disjoint open subsets U and V of X such that $K \subseteq U$ and $L \subseteq V$.*

Proof. We prove this by considering multiple cases regarding the number of elements in the compact sets K and L.

Case 1: Assume that at least one of K and L are empty, lets say L is empty. Then we can simply let $U = X$ and $V = \emptyset$ because X and $\emptyset$ are both open and closed. Thus, we have $K \subseteq U = X$ and $L \subseteq V = \emptyset$ as desired.

Case 2: Let L be nonempty and let K consist of a single point, say $K = \{k\}$. By X being Hausdorff we know that there exists disjoint open sets U_y and V_y such that $y \in V_y$ and $k \in U_y$ for all $y \in L$. Note that $\bigcup_{y \in L} V_y$ is an open cover of L. Since L is a compact set there exist a finite open cover $V = \bigcup_{i=1}^{n} V_{y_i}$ of L. Now, for every $y \in L$ we have that U_y and V_y are disjoint, hence $\bigcap_{i=1}^{n} U_{y_i}$ is disjoint to V.

Case 3: Suppose both K and L consist of more than one element. For every x in K we can use the same construction as in Case 2. From this we get open and disjoint sets U_x and V_x such that $x \in U_x$ and $L \subseteq V_x$. The union of U_x for all $x \in K$ is obviously an open cover of K. Since K is a compact set there is a finite subcover of K of the form $U = \bigcup_{i=1}^{m} U_{x_i}$. Since every V_x is an open cover of L, their intersection must be as well. That is $V = \bigcap_{i=1}^{m} V_{x_i}$ is an open cover of L. Only thing left to check is that U and V are disjoint. We know that $U_{x_i} \cap V_{x_i} = \emptyset$ from which it follows that U and V are disjoint. □

The *closure of a set* A of a topology X, which we denote by $\overline{A}$ is the union of A and the boundary of A; and we define the operation $\overline{A} - A$ as the points in $\overline{A}$ that are not in A.

For the rest of this essay we will only be concerned about locally compact Hausdorff spaces, which of course is the kind of topological space regarded in RRT.

Definition 3.10. A topological space is said to be *locally compact* if each of its points has an open neighbourhood whose closure is compact.

Remark. As the name gives prominence to; a locally compact Hausdorff space is a Hausdorff space that is also locally compact.

Theorem 3.11. *Let X be a locally compact Hausdorff space, let x be a point in X, and let U be an open neighbourhood of x. Then x has an open neighbourhood whose closure is compact and included in U.*

Proof. By assumption X is locally compact, hence there exist an open neighbourhood $N(x)$ of x with compact closure. Let W be the intersection of $N(x)$ and U. Then W is included in U. What we do not know about W is whether its closure is included in U, or if some of it is outside of U. If the closure of W is included in U we are done since the closure of W is a closed subset of $\overline{N(x)}$ and therefore compact since $\overline{N(x)}$ is compact.

Suppose that some of the closure of W is outside of U. That is $\overline{W} - W$ is nonempty. Now note that $\{x\}$ and $\overline{W} - W$ are two disjoint and compact subsets of X. Thus Theorem 3.9 gives us two open and disjoint subsets of X; one which covers $\{x\}$, and one which covers $\overline{W} - W$. We only need the one that covers $\{x\}$, because if we intersect that one with W, its closure will be completely in U. It will also be compact, since it is a closed subset of the compact set $\overline{W}$. □

Theorem 3.12. *Let X be a locally compact Hausdorff space, let K be a compact subset of X, and let U be an open subset of X that includes*

K. Then there is an open subset V of X that has a compact closure and satisfies $K \subseteq V \subseteq \overline{V} \subseteq U$.

Proof. Theorem 3.11 tells us that every $x \in K$ have an open neighbourhood $N(x)$ such that $\overline{N(x)}$ is a compact set included in U. Now since K is a compact set it suffices to grab a finite number of neighbourhoods to cover K. Let us call the union of these neighbourhoods V. Since V is a union of finitely many open sets which all have compact closures, we know that V has a compact closure, and since every one of them is included in U, their union must be as well. □

Why we care about these theorems is not yet apparent, but note that the previous theorems have been about compact sets and how we can separate them. The most recent one does not state it explicitly; but K and $\overline{V} - V$ are disjoint compact sets. These are great tools for our next, very famous theorem (Theorem 3.16). But first we need some additional definitions.

Definition 3.13. Let f be a continuous function on a topological space X. The *support* of f, denoted by $\text{supp}(f)$, is the closure of $\{x \in X : f(x) \neq 0\}$.

Remark. In RRT we want to construct a measure that gives the integral a specific value. Because $f \equiv 0$ on $X - \text{supp}(f)$ we can regard $\text{supp}(f)$ as the relevant part of X, since independently of the measure, the integral of the zero function will always be zero.

The following two definitions are only about notation which will be used frequently. They are labelled here as definitions just to make it easy for the reader to find them whenever needed.

Definition 3.14. Let X be a locally compact Hausdorff space. We denote the set of all continuous functions $f : X \to \mathbb{R}$ with compact support by $\mathscr{K}(X)$.

Given a subset A of X, recall that the *indicator function* $\chi_A : X \to \{0, 1\}$ is defined by setting $\chi_A(x) = 1$, for $x \in A$ and $\chi_A(x) = 0$, for $x \notin A$.

Definition 3.15. Let $U \subseteq X$ be an open set and f be a function in $\mathscr{K}(X)$. If f satisfies $0 \leq f \leq \chi_U$ and $\text{supp}(f) \subseteq U$ we use the notation $f \prec U$.

Remark. At first glance it may not be apparent what $\text{supp}(f) \subseteq U$ adds to the requirement $0 \leq f \leq \chi_U$. This can easily be seen however by considering the special case $f = \chi_U$.

Theorem 3.16. *Let X be a Hausdorff space and let E and F be disjoint compact subsets of X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ holds at each x in E and $f(x) = 1$ holds at each x in F.*

Remark. Theorem 3.16 is a special case of a famous theorem known as Urysohn's lemma. The difference is that we have made it slightly less general by restricting it to Hausdorff spaces and compact sets instead of normal spaces and closed sets. See [2] for more details on this.

Note that Theorem 3.16 can be used to approximate indicator functions by continuous ones. Ideas related to this are used in later proofs.

Theorem 3.17. *Let X be a locally compact Hausdorff space, let $K \subseteq X$ be a compact set and let U be an open set such that $K \subseteq U \subseteq X$. Then there is a function f that belongs to $\mathscr{K}(X)$, satisfies $\chi_K \leq f \leq \chi_U$, and is such that* $\text{supp}(f) \subseteq U$.

Proof. Let us take an open set V with compact closure satisfying $K \subseteq V \subseteq \overline{V} \subseteq U$, which we can do because of Theorem 3.12. To use Theorem 3.16 we need two disjoint compact sets. We already have K, and we find another one that is separated to K by taking $\overline{V} - V$.

Theorem 3.16 now say that there exists a continuous function $g : \overline{V} \rightarrow [0, 1]$ such that $g(x) = 1$ for all $x \in K$ and $g(x) = 0$ for all $x \in \overline{V} - V$. What we need is a function f that is continuous not only on $\overline{V}$, but on the whole space X, that agrees with g on $\overline{V}$ and satisfies $f \leq \chi_U$. Note that $\text{supp}(g) \subseteq \overline{V}$ and $g = 0$ on the boundary of $\overline{V}$. This means that we can simply let f be the extension of g to X defined by

$$f(x) = \begin{cases} g(x) & \text{if } x \in \overline{V}, \\ 0 & \text{elsewhere.} \end{cases}$$

This function is continuous on X, and it satisfies $\chi_K \leq f \leq \chi_U$ since $f = 1$ in K, $f = 0$ outside U and the range of f is $[0, 1]$. □

Theorem 3.18. *Let X be a Hausdorff space, let K be a compact subset of X,and let U_1 and U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then there are compact sets K_1 and K_2 such that $K = K_1 \cup K_2, K_1 \subseteq U_1$ and $K_2 \subseteq U_2$.*

Proof. Let $L_1 = K - U_1$ and $L_2 = K - U_2$. Then L_1 and L_2 are both disjoint and compact. By Theorem 3.9 we can separate them with disjoint open sets that covers L_1 and L_2 respectively. Let V_1 be the open set that covers L_1 and let V_2 be the open set that covers L_2. Note that V_1 is included in U_2. Therefore $K - V_1$ is included in U_1 and it is compact, call it K_1. By similar fashion we create K_2 and we now have the two sets we were looking for since $K_1 \cup K_2 = K$, $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$ and they are both compact. □

Theorem 3.19. *Let X be a locally compact Hausdorff space, let f belong to $\mathscr{K}(X)$, and let $U_1, ..., U_n$ be open subsets of X such that* $\text{supp}(f) \subseteq \bigcup_{i=1}^{n} U_i$. *Then there are functions $f_1, ..., f_n$ in $\mathscr{K}(X)$ such that $f = f_1 + ... + f_n$ and such that, for each i, the support of f_i is included in U_i. Furthermore, if the function f is nonnegative, then the functions $f_1, ..., f_n$ can be chosen so that all are nonnegative.*

Proof. To prove this, we will construct functions g_i satisfying $g_1 + ... + g_n = 1$ for all x in the support of f. Then we complete the proof by defining the $f_i's$ by $f_i = f \cdot g_i$. This is a proof by induction which has a trivial base case, because if $n = 1$ we just let $f = f_1$. We begin by proving that the theorem is true for $n = 2$, which we do as a tool for the induction step.

Theorem 3.18 allows us to construct compact sets K_1 and K_2 satisfying $K_1 \subseteq U_1$, $K_2 \subseteq U_2$ and $\text{supp}(f) = K_1 \cup K_2$. Note that K_1 and K_2 are not necessarily disjoint and therefore we can not just take the normal indicator functions and be done here. Now that we have compact sets included in every open set, Theorem 3.17 allows us to construct functions $h_i \in \mathscr{K}(X)$ satisfying $\text{supp}(h_i) \subseteq U_i$ and $\chi_{K_i} \leq h_i \leq \chi_{U_i}$. We are now ready to construct the functions g_i mentioned in the beginning of this proof. Let $g_1 = h_1$ and $g_2 = h_2 - \min\{h_1, h_2\}$. It is clear that g_1 satisfies the following:

(a) $g_1(x) = 1$, if $x \in K_1$,

(b) $g_1(x) = 0$, if $x \notin U_1$,

(c) $g_1(x) \in [0, 1]$, if $x \in \{U_1 - K_1\}$.

The behaviour of the function g_2 is not as clear however. We must consider the two possibilities (i): that $h_1 \geq h_2$, and (ii): that $h_1 < h_2$. If (i) is true we have that $g_2 = h_2 - h_2 = 0$, and if (ii) is true we have that $g_2 = h_2 - h_1 > 0$. Note that we have shown that g_1 and g_2 are both nonnegative functions. This is important because it means that we will have no problem making the f_i's nonnegative when f is nonnegative. Next we express $g_1 + g_2$ in terms of h_1 and h_2 to prove that $g_1 + g_2 = 1$ holds for all x in the support of f (recall that this was the goal of this construction),

$$g_1 + g_2 = h_1 + h_2 - \min\{h_1, h_2\}. \tag{3.1}$$

Since $K_1 \cup K_2 =\text{supp}(f)$ we know that either $h_1 = 1$ or $h_2 = 1$ holds for all x in the support of f. Using this, it is easy to verify that equation (3.1) is true for all x in the support of f by considering the following three cases:

(1) $h_1 = h_2 = 1$.

(2) $h_1 = 1$ and $h_2 \neq 1$.

(3) $h_1 \neq 1$ and $h_2 = 1$.

By defining $f_i = f \cdot g_i$ we have shown that the theorem is true for $n = 2$. We shall now put this to use in the induction step.

Suppose that the theorem is true for $n = m$. The induction step is to show that this implies that the theorem is true for $n = m + 1$. Let $V_1 = \bigcup_{i=1}^{m} U_i$ and $V_2 = U_{m+1}$. This allows us to use the case $n = 2$ that we just proved to split f into two functions, f_1 and f_2, satisfying

$f = f_1 + f_2$ and $\mathrm{supp}(f_i) \subseteq V_i$. We can now use the induction hypothesis to find functions f_i' satisfying $f_1 = f_1' + ... f_m'$ and $\mathrm{supp}(f_i') \subseteq U_i$ and we are done. $\square$

Theorem 3.20. *Let X be a locally compact Hausdorff space and let μ be a regular Borel measure on X. If U is an open subset of X, then*

$$\begin{aligned}\mu(U) &= \sup\left\{\int f d\mu : f \in \mathscr{K}(X) \textit{ and } 0 \leq f \leq \chi_U\right\} \\ &= \sup\left\{\int f d\mu : f \in \mathscr{K}(X) \textit{ and } f \prec U\right\}.\end{aligned}$$

Proof. From the definition of $f \prec U$ we know that the second supremum is definitely not smaller than the first supremum, and because $f \in [0, 1]$ it follows that the integral can not be greater than the measure of U. That is

$$\begin{aligned}\mu(U) &\geq \sup\left\{\int f d\mu : f \in \mathscr{K}(X) \text{ and } 0 \leq f \leq \chi_U\right\} \\ &\geq \sup\left\{\int f d\mu : f \in \mathscr{K}(X) \text{ and } f \prec U\right\},\end{aligned}$$

and therefore we only need to prove that

$$\mu(U) \leq \sup\left\{\int f d\mu : f \in \mathscr{K}(X) \text{ and } f \prec U\right\}. \tag{3.2}$$

Let α be a nonnegative number satisfying $\alpha < \mu(U)$. Then we can, because of the regularity of μ, choose a compact subset K of U such that $\alpha < \mu(K)$. Now we can use Theorem 3.17 to find a function f in $\mathscr{K}(X)$ that satisfies $\chi_K \leq f \leq \chi_U$ and *supp*$(f) \subseteq U$. Therefore the requirements for $f \prec U$ are satisfied as well, and it follows that $\alpha < \int f d\mu$, and of course

$$\alpha < \sup\left\{\int f d\mu : f \in \mathscr{K}(X) \text{ and } f \prec U\right\}.$$

Since α was an arbitrary positive number less than $\mu(U)$; inequality (3.2) holds. $\square$

4. Riesz representation theorem

Recall that a *linear functional* is a linear map from a vector space to in our case $\mathbb{R}$. This means that a linear functional I on $\mathscr{K}(X)$ must satisfy $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for all $f, g \in \mathscr{K}(X)$ and all $\alpha, \beta \in \mathbb{R}$. For more on linear maps and linear functionals see e.g. [9]. We are now finally ready to prove the Riesz representation theorem.

Theorem 4.1. *Let X be a locally compact Hausdorff space, and let I be a positive linear functional on $\mathscr{K}(X)$. Then there is a unique regular Borel measure μ on X such that*

$$I(f) = \int f d\mu \tag{4.1}$$

holds for each f in $\mathscr{K}(X)$.

Proof. We want to construct a measure that together with the integral can represent the functional I. We do this by creating an outer measure that we mathematically mould into a regular Borel measure. The proof has been divided into seven parts in an attempt to make it as apprehendable as possible. In Part 1 we prove that the measure must be unique if it exists, in Part 2 to Part 6 we construct a function and prove that this function indeed is a regular Borel measure and finally; in Part 7 we prove that equation (4.1) holds for the measure that we have created.

Part 1. *Uniqueness.*

Assume that μ and v are regular Borel measures on X such that $\int f d\mu = \int f dv = I(f)$ holds for each f in $\mathscr{K}(X)$. These assumptions together with Theorem 3.20 says that $\mu(U)$ and $v(U)$ are both the supremum of the same set, and therefore obviously the same. That is $\mu(U) = v(U)$ for all open sets $U \subseteq X$.

What we really need is for $\mu(A) = v(A)$ to hold for all the Borel subsets of X. The outer regularity (Definition 2.8) of μ and v say that

$$\mu(A) = \inf\{\mu(U) : A \subseteq U \text{ and } U \text{ is open}\},$$

and that

$$v(A) = \inf\{v(U) : A \subseteq U \text{ and } U \text{ is open}\}.$$

Hence $\mu(A) = v(A)$ holds since we have already shown that it holds for open sets.

Part 2. *Construction of an outer measure μ^*.*

We define a function μ^* on the open subsets of X by

$$\mu^*(U) = \sup\{I(f) : f \in \mathscr{K}(X) \quad \text{and} \quad f \prec U\}, \tag{4.2}$$

and extend it to all subsets A of X by

$$\mu^*(A) = \inf\{\mu^*(U) : U \quad \text{is open and} \quad A \subseteq U\}. \tag{4.3}$$

(This construction is inspired by Theorem 3.20 and the outer regularity property from Definition 2.8). Because the function defined by (4.3) is an extension of the function in (4.2) we have to verify that they agree on the domain of the function in (4.2). Let A in (4.3) be any element in the domain of the function in (4.2). This forces A to be an open set. The infimum of $\mu^*(U)$ for all open sets $U \subseteq X$ such that $A \subseteq U$ is clearly $\mu^*(A)$, which means that the extension of (4.2) into (4.3) makes sense.

Part 3. *μ^* is an outer measure on X.*

To show that μ^* is an outer measure we have to verify that $\mu^*(\emptyset) = 0$, that μ^* is monotonically increasing and that μ^* is countably subadditive. The first two requirements are quite easily seen so we do them a bit informally. Note that the empty set is an open set, which means that we can use (4.2) to calculate this measure. Note also that $f \prec \emptyset$ implies that f has to be the zero function, and hence $\mu^*(\emptyset) = I(0) = 0$, since I is a linear functional and must satisfy $I(\alpha \cdot 0) = \alpha I(0)$ for all $\alpha \in \mathbb{R}$ and especially $\alpha = 0$.

For the monotonicity we look at (4.2) and (4.3) and see that if $A_1 \subseteq A_2 \subseteq X$, then $\mu^*(A_2)$ is the supremum of a set which includes the set that $\mu^*(A_1)$ is the supremum of. Thus $\mu^*(A_1) \leq \mu^*(A_2)$. We take a closer look at why the countable subadditivity requirement holds. First we consider the open sets like we did when we defined the function, and then we extend it to cover all the subsets of X.

Suppose that $\{U_n\}$ is a sequence of open subsets of X. We need to show that

$$\mu^*\left(\bigcup U_n\right) \leq \sum \mu^*(U_n).$$

Let f be a function that belongs to $\mathscr{K}(X)$ and satisfies $f \prec \cup U_n$. Then the support of f is a compact subset of $\cup U_n$, and hence there is a positive integer N such that

$$\operatorname{supp}(f) \subseteq \bigcup_{n=1}^{N} U_n.$$

Theorem 3.19 implies that f is the sum of functions $f_1, ..., f_N$ that belongs to $\mathscr{K}(X)$ and satisfy $f_n \prec U_n$ for $n = 1, ..., N$. It follows that

$$I(f) = \sum_{n=1}^{N} I(f_n) \leq \sum_{n=1}^{N} \mu^*(U_n) \leq \sum_{n=1}^{\infty} \mu^*(U_n).$$

Note that the first inequality above holds because $\mu^*(U_n)$ is the supremum of a set of functions where one of them is the given function f_n. Now 4.2 yields the subadditivity of μ^* regarding opens sets because the above inequality holds for all f and thus for

$$\sup\{I(f)\} = \mu^*\left(\bigcup U_n\right).$$

Next we extend this result to show that μ^* is subadditive for all sets and not just the open ones. Suppose that $\{A_n\}$ is an arbitrary sequence of subsets of X. The inequality

$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n) \tag{4.4}$$

is trivially true if $\sum_n \mu^*(A_n) = \infty$. Therefore we can assume that $\sum_n \mu^*(A_n) < \infty$ without any loss of generality. Now we shall use a common "epsilon trick" based on the convergent series $\sum_{n=1}^{\infty} 2^{-n} = 1$. Let ε be a positive number, and for each n use (4.3) to choose an open set U_n that includes A_n and satisfies

$$\mu^*(U_n) \leq \mu^*(A_n) + \varepsilon/2^n. \tag{4.5}$$

We can do this because $\mu^*(A_n)$ is an infimum over open sets that all covers A_n, and we just recently showed that μ^* is monotonic. Because of this; we can choose U_n's that satisfies (4.5). It follows from the subadditivity of the open sets and the epsilon trick that

$$\mu^*\left(\bigcup_n A_n\right) \leq \mu^*\left(\bigcup_n U_n\right) \leq \sum_{n=1}^{\infty} \mu^*(U_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

We can pick ε to be as small as we want, hence inequality (4.4) holds, which of course is the definition of subadditivity. With that we have shown that μ^* satisfies all the requirements necessary to be called an outer measure.

Part 4. *Every Borel subset of X is μ^*-measurable.*

Since the family of μ^*-measurable sets is a σ-algebra, see Theorem 2.12, we can show that every Borel subset of X is μ^*-measurable by verifying that each open subset of X is μ^*-measurable.

Let U be an open subset of X. Recall from Corollary 2.10 that we prove the μ^*-measurability of U by showing that the inequality

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c) \tag{4.6}$$

holds for each subset $A \subseteq X$ that satisfies $\mu^*(A) < \infty$.

Let A be such a set and let ε be a positive number. Then we can use (4.3) to choose an open set V that includes A and satisfies $\mu^*(V) < \mu^*(A) + \varepsilon$ (like we did in (4.5)). If

$$\mu^*(V) > \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon, \tag{4.7}$$

holds, it follows that

$$\mu^*(A) + \varepsilon > \mu^*(A \cap U) + \mu^*(A \cap U^c) - 2\varepsilon,$$

and since ε is an arbitrary positive number this would imply inequality (4.6), and with that the μ^*-measurability of U. To verify inequality (4.7), we start by choosing a function f_1 in $\mathscr{K}(X)$ that satisfies

$$f_1 \prec V \cap U \text{ and } I(f_1) > \mu^*(V \cap U) - \varepsilon.$$

Then we let $K = \text{supp}(f_1)$ and choose a function f_2 in $\mathscr{K}(X)$ such that

$$f_2 \prec V \cap K^c \text{ and } I(f_2) > \mu^*(V \cap K^c) - \varepsilon.$$

Since $f_1 + f_2 \prec V$ and $V \cap U^c \subseteq V \cap K^c$ we have that

$$\mu^*(V) \geq I(f_1 + f_2) > \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon,$$

and therefore inequality (4.7) holds and every Borel subset of X is μ^*-measurable indeed.

Part 5. *Creating a measure from an outer measure.*

Now we want to use the outer measure μ^*, to create a regular Borel measure. Recall from Theorem 2.12 that $\mathscr{M}_{\mu^*}$ is the collection of all the μ^*-measurable subsets of X. From the same theorem we also know that the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure on $\mathscr{M}_{\mu^*}$.

We recently showed that every Borel subset of X is μ^*-measurable. So $\mathscr{B}(X)$ is a subset of $\mathscr{M}_{\mu^*}$ and therefore the restriction of μ^* to $\mathscr{B}(X)$ is a measure on $\mathscr{B}(X)$. Recall that a Borel measure is a measure that has $\mathscr{B}(X)$ as domain, where X is Hausdorff. The restriction of μ^* to $\mathscr{B}(X)$ is therefore a Borel measure. Let us call this measure μ.

Before we go into part 6 we include a lemma here that will make the rest of the proof easier.

Lemma 4.2. *Let X and I be as in RRT and let μ^* be defined by* (4.2) *and* (4.3). *Suppose that A is a subset of X and that f belongs to $\mathscr{K}(X)$. If $\chi_A \leq f$, then $\mu^*(A) \leq I(f)$, while if $0 \leq f \leq \chi_A$ and if A is compact, then $I(f) \leq \mu^*(A)$.*

Proof. Assume that $\chi_A \leq f$ and let $\varepsilon \in (0, 1)$, and define U_ε by $U_\varepsilon = \{x \in X : f(x) > 1 - \varepsilon\}$. Then U_ε is open and covers A, and each g in $\mathscr{K}(X)$ that satisfies $g \leq \chi_{U_\varepsilon}$ also satisfies $g \leq \frac{1}{1-\varepsilon} f$ (because $f > 1 - \varepsilon$) and since I is linear it follows that

$$I(g) \leq \frac{1}{1-\varepsilon} I(f).$$

Because U_ε is open it follows from (4.2) and the inequality above that

$$\mu^*(U_\varepsilon) = \sup\{I(g)\} \leq \frac{1}{1-\varepsilon} I(f).$$

We noted earlier that A is covered by U_ε and we have already proven that μ^* is monotonic. It follows from this together with the fact that we can let ε be arbitrarily close to zero that

$$\mu^*(A) \leq I(f)$$

as desired.

Now suppose that $0 \leq f \leq \chi_A$ and that A is compact. Let U be an open set that includes A. Then $f \prec U$ and so (4.2) implies that

$$I(f) \leq \mu^*(U).$$

Since U was an arbitrary open set that includes A, (4.3) implies that

$$I(f) \leq \mu^*(A).$$

□

Now we are ready to continue with the main proof.

Part 6. *Proof of regularity.*

Next we want to show that μ is a regular measure (remember from part 5 that μ is the restriction of μ^* to $\mathscr{B}(X)$). First we show that $\mu(K) < \infty$ for all compact subsets of X. Note first that all functions in $\mathscr{K}(X)$ are bounded (Theorem 3.8), hence $I(f) < \infty$ for all $f \in \mathscr{K}(X)$, and secondly that every compact $K \subseteq X$ is contained by an open subset U of X (we can always set $U = X$). These two properties let us use Theorem 3.17 to find a function f satisfying $f(x) \geq \chi_K(x)$ for all $x \in X$. Now the first part of Lemma 4.2, let us say that $\mu^*(K) \leq I(f) < \infty$, and since μ is a restriction of μ^* this also holds for μ.

Because the outer regularity of μ follows directly from construction (4.3) it only remains to show that μ is inner regular. Recall from Definition 2.8 that an inner regular measure satisfies

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \text{ is compact}\}$$

for all open subsets U of X.

From (4.2) we have that for open sets U; our function μ is defined by

$$\mu(U) = \sup\{I(f) : f \in \mathscr{K}(X) \text{ and } f \prec U\}.$$

The support of f is compact and $f \prec U$ implies that $0 \leq f \leq \chi_{\text{supp}(f)}$. Therefore we can apply the second part of Lemma 4.2 (with $A = \text{supp}(f)$) to get the inequality $I(f) \leq \mu(\text{supp}(f))$. We use this inequality to find an upper bound for $\mu(U)$ (which later turns out to also be a lower bound), which is

$$\begin{aligned} \mu(U) &= \sup\{I(f) : f \in \mathscr{K}(X) \text{ and } f \prec U\} \\ &\leq \sup\{\mu(\text{supp}(f) : f \in \mathscr{K}(X) \text{ and } f \prec U\}. \end{aligned}$$

The support of f is always a compact set that is included in U, so if we let $K = \text{supp}(f)$ we get from the above equation that

$$\mu(U) \leq \sup\{\mu(K) : K \subseteq U \text{ and } K \text{ is compact}\}.$$

Since U includes K, the reverse inequality follows from the monotonic property of outer measures. Thus the inequality above is actually an equality, which means that μ satisfy the inner regularity property and we have shown that μ is a regular Borel measure.

Part 7. *$I(f) = \int f d\mu$ holds for each f in $\mathscr{K}(X)$.*

We restrict our attention to the nonnegative functions in $\mathscr{K}(X)$. This is sufficient since every negative function in $\mathscr{K}(X)$ can be expressed as the difference of two nonnegative functions in $\mathscr{K}(X)$.

Let f be a nonnegative function and let ε be a positive number. For each positive integer n define a function $f_n(x) : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } f(x) \leq (n-1)\varepsilon, \\ f(x) - (n-1)\varepsilon & \text{if } (n-1)\varepsilon < f(x) \leq n\varepsilon, \\ \varepsilon & \text{if } n\varepsilon < f(x). \end{cases}$$

All these functions belong to $\mathscr{K}(X)$ and their sum equals f, that is $f = \sum_n f_n$. Since f is bounded, there is a natural number N such that $f_n(x) = 0$ for all $n > N$. Let $K_0 = \text{supp}(f)$ and for each positive integer n, let

$$K_n = \{x \in X : f(x) \geq n\varepsilon\}.$$

Note that K_n are compact sets that satisfy $K_{n+1} \subseteq K_n$. We get the inequality $\varepsilon\chi_{K_n} \leq f_n(x) \leq \varepsilon\chi_{K_n-1}$, that holds for each n.
From Lemma 4.2 we get that

$$\varepsilon\mu(K_n) \leq I(f_n) \leq \varepsilon\mu(K_{n-1}),$$

and since $f = \sum_n f_n$ it follows that

$$\sum_{n=1}^{N} \varepsilon\mu(K_n) \leq I(f) \leq \sum_{n=0}^{N-1} \varepsilon\mu(K_n).$$

By basic properties of the integral we also get that

$$\varepsilon\mu(K_n) \leq \int f_n d\mu \leq \varepsilon\mu(K_{n-1}),$$

and that

$$\sum_{n=1}^{N} \varepsilon\mu(K_n) \leq \int f d\mu \leq \sum_{n=0}^{N-1} \varepsilon\mu(K_n).$$

Note that both $I(f)$ and $\int f d\mu$ are contained in the same interval. If we can show that this interval has length zero this implies that they are the same and the proof is complete. This interval has the length

$$\sum_{n=0}^{N-1} \varepsilon\mu(K_n) - \sum_{n=1}^{N} \varepsilon\mu(K_n) = \varepsilon(\mu(K_0) - \mu(K_N)).$$

Recall that μ is a regular measure and hence it has a finite measure for all compact sets. This completes the proof since the interval can be made arbitrarily small. □

5. Acknowledgements

Firstly I would like to thank André Berglund for helping me with LaTeX related questions and more. Further I would like to thank my supervisor Lisa Hed for reading multiple versions of this essay and providing me with way more feedback and general advice than she probably should have. And last but not least; I would like to thank my examiner Per Åhag, not only for providing me with great advice in the last stages of the process of finishing this essay, but also for all the moral support he has given me during my years of university studies.

References

[1] A. Browder, *Mathematical analysis, an introduction.* Undergraduate texts in mathematics. Springer, 1996.

[2] D. L. Cohn, *Measure theory.* Second edition. Birkhäuser advanced texts Basler Lehrbucher. Birkhäuser, 2013.

[3] J.D. Gray, *The shaping of the Riesz representation theorem: a chapter in the history of analysis.* Archive of history of exact sciences. Volume 31, Issue 2, pp 127-187. Springer. 1984.

[4] S. Kakutani. *Concrete representation of abstract (m)-spaces. (a characterization of the space of continuous functions.).* Ann. of Math., 42(2):994–1024, 1941.

[5] A. Markov. *On mean values and exterior densities.* Rec. Math. Moscou, n. Ser., 4:165–190, 1938.

[6] A. Narkawicz, *A formal proof of the Riesz representation theorem.* Journal of formalized reasoning Vol. 4, No. 1, 2011.

[7] F. Riesz. *Sur les opérations fonctionnelles linéaires.* C. R. Acad. Sci. Paris, 149:974–977, 1909.

[8] W. Rudin, *Principles of mathematical analysis.* Third edition. Springer. 1976.

[9] W. Rudin, *Functional analysis.* Second edition. McGraw-Hill, inc. 1991.