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Sub-Hilbert relation for Fock–Sobolev type spaces

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ABSTRACT. In this paper, two specific sub-Hilbert spaces are studied. They arise from the action of a Toeplitz operator on Fock–Sobolev type spaces, induced by a general Gaussian type weight. The argument is based on analysing the reproducing kernel of the corresponding sub-Hilbert space.

CONTENTS

1. Introduction

This paper is concerned with sub-Hilbert functional spaces of analytic functions on planar domains. Suppose T is a bounded operator on a given Hilbert space H. We denote by $\mathcal{M}(T)$ the range of T, which is equipped with the following inner product:

$$\langle Tx, Ty\rangle_{\mathcal{M}(T)} = \langle x, y\rangle_H \qquad x, y \in H \ominus \ker T.$$

Then $\mathcal{M}(T)$ is a Hilbert space. If, in addition, T is a contraction operator, the Hilbert space

$$\mathcal{M}((I - TT^*)^{1/2})$$

is called the complemented space to $\mathcal{M}(T)$ is denoted by $\mathcal{H}(T)$ and is called a sub-Hilbert space.

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The pioneering work on sub-Hilbert spaces was done by L. de Branges, J. Rovnyak and D. Sarason [8, 9, 10, 18]. For further reading on the spaces introduced by de Branges and Rovnyak, their equivalent formulations, and their applications in function theory and operator theory, see [3]. Sarason's monograph [18] contains extensive investigation of sub-Hilbert spaces arising from Toeplitz operators T_f acting on the Hardy space on the unit circle; in this context, it is customary to agree on the notation $\mathcal{M}(T_f) = \mathcal{M}(f)$ and $\mathcal{H}(T_f) = \mathcal{H}(f)$.

Later, continuing Sarason's work, Kehe Zhu introduced sub-Bergman Hilbert spaces on the unit disk [21, 22]. To provide a brief account on this issue, we recall that the standard weighted Bergman space A^2_α, for $\alpha > -1$, consists of all analytic functions on the unit disk for which the integral

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha dxdy$$

is finite. The norm of a function in the weighted Bergman space is given by

$$\|f\|^2 = \frac{\alpha+1}{\pi} \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha dxdy.$$

We shall at times write

$$dA_\alpha(z) = \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha dxdy,$$

for normalized weighted area measure in the unit disk. Note that A^2_α is a reproducing kernel functional Hilbert space whose kernel is given by

$$K^\alpha_z(w) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+2)}{n!\,\Gamma(\alpha+2)} (\overline{z}w)^n = \frac{1}{(1-w\overline{z})^{\alpha+2}}, \quad (z,w) \in \mathbb{D} \times \mathbb{D}.$$

The Bergman projection

$$P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to A^2_\alpha(\mathbb{D})$$

is defined by

$$P_\alpha f(z) = \int_{\mathbb{D}} f(w)\overline{K^\alpha_z(w)} dA_\alpha(w).$$

Now, let φ be an analytic function in the unit disk satisfying $\|f\|_\infty \leq 1$. For $\alpha \geq 0$, we consider the Toeplitz operator

$$T^\alpha_\varphi(f) = P_\alpha(\varphi f), \quad f \in A^2_\alpha.$$

For $\alpha = 0$, the unweighted Bergman space, Kehe Zhu [21, 22] studied the sub-Bergman Hilbert spaces $\mathcal{H}_\alpha(\varphi) := \mathcal{H}(T^\alpha_\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi}) := \mathcal{H}(T^\alpha_{\overline{\varphi}})$. He proved that these sub-Bergman Hilbert spaces coincide as sets, moreover, both spaces contain the Banach space of all bounded analytic functions on the unit disk. Zhu further showed that for the symbol z^m, and more generally, for a finite Blaschke product B, we have

$$\mathcal{H}(B) = \mathcal{H}(\overline{B}) = H^2,$$

where H^2 denotes the Hardy space of analytic functions on the unit disk.

Later, in 2010, Abkar and Jafarzade [1] extended Zhu's results to the standard weighted Bergman spaces A^2_α where $\alpha \geq 0$. They proved that $H^\infty \subset \mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$, and for a finite Blaschke product B,

$$H^\infty \subset \mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = A^2_{\alpha-1}.$$

In 2014, this line of investigation was adapted by Nowak and Rososzczuk in [15] where the authors extended the latter result for $-1 < \alpha < 0$. They proved that

$$\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = \mathcal{D}_{\alpha+1},$$

where the Dirichlet space $\mathcal{D}_\alpha$ consists of all analytic functions f in the unit disk such that $f' \in L^2(\mathbb{D}, dA_\alpha)$. See also [20], and [16] where in the latter the authors studied similar problems in the unit ball of n-dimensional complex space $\mathbb{C}^n$.

Inspired by the aforementioned works, we will study the concept of a sub-Hilbert space in the context of Fock-type spaces $F^2_{\alpha,\beta,s}$, where the indices α and β appear in the exponential part of the weight, and s can be thought of as the order of the fractional derivative; see the next section. However, on these spaces, multiplication by an entire non-constant function is never bounded, let alone contractive. We will therefore focus our attention to the symbols of the type $f(z) = (z/|z|)^m$. We prove

Theorem 1. *Let $\alpha, \beta > 0$, $s \in \mathbb{R}$ and $m \in \mathbb{N}$, and let $T^{\alpha,\beta,s}_f$ be the Toeplitz operator on $F^2_{\alpha,\beta,s}$ induced by the symbol $f(z) = (z/|z|)^m$. We then have*

$$\mathcal{H}(f) = \mathcal{H}(\overline{f}) = F^2_{\alpha,\beta,s+\beta/2}.$$

2. Fock-Sobolev type spaces

Let $\mathbb{C}$ denote the complex plane, $H(\mathbb{C})$ the space of entire functions, and $dA(z)$ the Lebesgue area measure on $\mathbb{C}$;

$$dA(z) = \frac{1}{\pi} dx dy, \quad z = x + iy.$$

For $\alpha, \beta > 0$ and $s \in \mathbb{R}$, we consider the weight

$$d\lambda_{\alpha,\beta,s}(z) = |z|^{2s} e^{-\alpha|z|^\beta} dA(z).$$

In the literature, it is common to normalize $d\lambda_{\alpha,\beta,s}$ into a probability measure. However, when $s \leq -1$, this weight is no longer integrable, and cannot be normalized in an obvious way. We refrain from normalizing the weight altogether because of this.

2.1. Case $s > -1$. We define the generalized Fock-Sobolev type space $F^2_{\alpha,\beta,s}$ as those elements in $H(\mathbb{C})$ that are square integrable over $\mathbb{C}$ with respect to $d\lambda_{\alpha,\beta,s}$. That is,

$$F^2_{\alpha,\beta,s} = L^2_{\alpha,\beta,s} \cap H(\mathbb{C}).$$

It is easy to see that $F^2_{\alpha,\beta,s}$ is a closed subspace of $L^2_{\alpha,\beta,s} = L^2(\mathbb{C}, d\lambda_{\alpha,\beta,s})$, and a Hilbert space with the inner product.

$$\langle f, g\rangle_{\alpha,\beta,s} = \int_{\mathbb{C}} f(z)\overline{g(z)}d\lambda_{\alpha,\beta,s}(z).$$

2.2. Case $s \leq -1$. The spaces $F^2_{\alpha,\beta,s}$ also make sense for $s \leq -1$, but in that case, following the definition above would require the members $F^2_{\alpha,\beta,s}$ to have a deep enough zero at the origin. In [6] two ways to overcome this are presented. First, one could replace the term $|z|^{2s}$ in $d\lambda_{\alpha,\beta,s}$ by $(1 + |z|)^{2s}$. However, the other approach from [6] fits our calculations better. Given

$$f(z) = \sum_{k=0}^{\infty} f_k z^k,$$

let us denote by $p_N(f)$ the degree N Maclaurin polynomial of f;

$$p_N(f)(z) = \sum_{k=0}^{N} f_k z^k.$$

Then, denote by $R_N(f) = f - p_N(f)$ the remainder, which in our case is going to determine the membership in $F^2_{\alpha,\beta,s}$.

By using the ceiling function, we define $N = -\lceil s\rceil - 1$ and introduce the inner product

$$\langle f, g\rangle_{\alpha,\beta,s} = \int_{\mathbb{C}} R_N(f)(z)\overline{R_N(g)(z)}d\lambda_{\alpha,\beta,s}(z) + \sum_{k=0}^{N} f_k\overline{g_k}.$$

The space $F^2_{\alpha,\beta,s}$ consists of entire functions f with

$$\|f\|^2_{\alpha,\beta,s} := \langle f, f\rangle_{\alpha,\beta,s} < \infty,$$

and by the virtue of the above definition, always contains all polynomials. In practice, we will not need to worry about this definition, as we are only interested in $R_N(f)$ for large enough N.

2.3. Relation to other Fock spaces. For particular choice of parameters, the spaces $F^2_{\alpha,\beta,s}$ reduce to more well-known spaces. The choice $(\alpha, \beta, s) = (\alpha, 2, 0)$ gives rise to classical Fock spaces, where standard references include the book of Folland [11] and the more recent book of Zhu [23].

Adding the parameter s is known to be equivalent to the membership of (fractional) derivatives in the standard Fock space. This motivates the terminology

Fock-Sobolev space, which corresponds to the choice $(\alpha, \beta, s) = (\alpha, 2, s)$ studied in [7, 6, 5]. These references do not always contain α as a parameter, but passage to this more general case is easy for most purposes of this paper.

In [4], Bommier-Hato, Engliš and Youssfi studied the so-called Fock-type spaces. These correspond to changing the Gaussian in the weight: $(\alpha, \beta, s) = (1, \beta, 0)$. Here again, slightly more general parameters do not cause much of an obstacle.

Finally, there are several generalization of the Fock-spaces to the case where the weight is non-radial; we mention [12], [14] and [19], but there are many more. These spaces are often called generalized Fock spaces, but we refrain from studying them, because having a radial weight is essential for our approach.

3. Gamma function and reproducing kernels

3.1. Gamma function. The Euler Gamma function (or simply the Gamma function) is a well-known special function that generalizes the concept of a factorial to non-integer values. As we have already seen, it appears naturally in the context of exponential weights.

The Gamma function can be defined by a convergent improper integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad R(z) > 0.$$

The Gamma function satisfies the crucial recurrence relation: $\Gamma(z+1) = z\Gamma(z)$, and the following standard estimate for fixed complex numbers a and b

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \asymp z^{a-b}, \qquad z \to \infty.$$

In this paper, we will need a more refined variant of the latter. The following formula can be found in [13].

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\Big[1 + \frac{(a-b)(a+b-1)}{2z} + \frac{(a-b)(a-b-1)}{24z^2} \times \{3(a+b-1)^2 - a + b - 1\}\Big][1 + O(z^{-3})], \quad z \to \infty. \quad (1)$$

Lemma 2. *The Gamma function satisfies*

$$1 - \frac{(\Gamma(z + \frac{a+b}{2}))^2}{\Gamma(z+a)\Gamma(z+b)} = \frac{(a-b)^2}{4z} + O(z^{-2}), \qquad z \to \infty.$$

Proof. By using the equation (1) we can obtain estimates for $\frac{\Gamma(z+\frac{a+b}{2})}{\Gamma(z+a)}$ and $\frac{\Gamma(z+\frac{a+b}{2})}{\Gamma(z+b)}$, so that we have

$$\frac{\Gamma(z + \frac{a+b}{2})}{\Gamma(z+a)} = z^{\frac{b-a}{2}} \left[1 + \frac{(b-a)(3a+b-2)}{8z} + O(z^{-2})\right]$$

and

$$\frac{\Gamma(z+\frac{a+b}{2})}{\Gamma(z+b)} = z^{\frac{a-b}{2}}\left[1+\frac{(a-b)(a+3b-2)}{8z}+O(z^{-2})\right]$$

as $z \to \infty$. Multiplying these completes the proof. □

The equation (1) can also be used to partially refine the recurrence relation:

Lemma 3. *For any complex number δ the Gamma function satisfies*

$$\Gamma(z+\delta) \asymp z^{\delta}\Gamma(z), \qquad z \to \infty.$$

Proof. The proof is easy and we omit the details. □

3.2. Reproducing kernels and projections. The approach of this paper is based on identifying the sub-Hilbert space by calculating its reproducing kernel. This is a well-known approach, see [1, 18, 21, 22]. The theory of reproducing kernels is a fascinating field in its own right, extending far beyond what is needed here. Some classical references include [2] and [17].

Since the weight $d\lambda_{\alpha,\beta,s}$ is radial, the Fock-type space $F^2_{\alpha,\beta,s}$ possesses a monomial Schauder basis. If $s \le -1$ and $n \le -\lceil s\rceil - 1$, we set $e_n^{\alpha,\beta,s}(z) = z^n$ and observe that $\|e_n^{\alpha,\beta,s}\|_{\alpha,\beta,s} = 1$. Otherwise, we compute in polar coordinates and using the change of variables $t = \alpha r^{\beta}$:

$$\begin{aligned}\|z^n\|^2_{\alpha,\beta,s} &= \int_{\mathbb{C}} |z|^{2n}|z|^{2s}e^{-\alpha|z|^{\beta}}\,dA(z)\\ &= \frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi} r^{2n+2s+1}e^{-\alpha r^{\beta}}\,d\theta\,dr\\ &= \frac{2}{\beta\alpha^{\frac{2n+2s+2}{\beta}}}\int_0^{\infty} t^{\frac{2n+2s+2}{\beta}-1}e^{-t}\,dt\\ &= \frac{2}{\beta\alpha^{\frac{2n+2s+2}{\beta}}}\Gamma\left(\frac{2n+2s+2}{\beta}\right).\end{aligned}$$

So, for $n > -\lceil s\rceil - 1$, we observe that then the functions

$$e_n^{\alpha,\beta,s} = \sqrt{\frac{\beta\alpha^{\frac{2n+2s+2}{\beta}}}{2\Gamma\left(\frac{2n+2s+2}{\beta}\right)}}\,z^n$$

are unit vectors, and $(e_n^{\alpha,\beta,s})_{n=0}^{\infty}$ forms the basis of $F^2_{\alpha,\beta,s}$.

Let $K_z^{\alpha,\beta,s}$ denote the reproducing kernel of $F^2_{\alpha,\beta,s}$ – that is, the unique function in $F^2_{\alpha,\beta,s}$ with the property

$$f(z) = \langle f, K_z^{\alpha,\beta,s}\rangle_{\alpha,\beta,s}, \quad f \in F^2_{\alpha,\beta,s}.$$

By a well-known identity, we obtain

$$K_z^{\alpha,\beta,s}(\xi) = \sum_{n=0}^{\infty} \frac{\beta\alpha^{\frac{2n+2s+2}{\beta}}}{2\Gamma\left(\frac{2n+2s+2}{\beta}\right)}(\xi\overline{z})^n,$$

when $s > -1$. When $s \leq -1$, we obtain

$$\begin{aligned} K_z^{\alpha,\beta,s}(\xi) &= \sum_{n=0}^{\infty} e_n^{\alpha,\beta,s}(\xi)\overline{e_n^{\alpha,\beta,s}(z)} \\ &= \frac{1-(\xi\overline{z})^{-\lceil s\rceil}}{1-\xi\overline{z}} + \sum_{n=-\lceil s\rceil}^{\infty} \frac{\beta\alpha^{\frac{2n+2s+2}{\beta}}}{2\Gamma\left(\frac{2n+2s+2}{\beta}\right)}(\xi\overline{z})^n. \end{aligned}$$

In either case, we are only interested the asymptotics of the general term in the sum as n is large; that is

$$\frac{\beta\alpha^{\frac{2n+2s+2}{\beta}}}{2\Gamma\left(\frac{2n+2s+2}{\beta}\right)} \asymp \frac{\alpha^{\frac{2n}{\beta}}}{\Gamma\left(\frac{2n+2s+2}{\beta}\right)}. \tag{2}$$

In general, these power series can be understood in terms of the generalized Mittag-Leffler functions; see [4]. Of course, it is well-known (there are many references, see for instance [23]) that

$$K_z^{\alpha,2,0}(\xi) = \alpha e^{\alpha\xi\overline{z}}.$$

Finally, we are now able to write the orthogonal projection (the Bergman projection) $P^{\alpha,\beta,s} : L^2_{\alpha,\beta,s} \to F^2_{\alpha,\beta,s}$ as

$$P^{\alpha,\beta,s}f(z) = \int_{\mathbb{C}} f(\xi)\overline{K_z^{\alpha,\beta,s}(\xi)}\,d\lambda_{\alpha,\beta,s}(\xi).$$

By the standard theory of orthogonal projections, $P^{\alpha,\beta,s}$ is bounded; in fact the norm of $P^{\alpha,\beta,s}$ is one.

4. The main results

4.1. Toeplitz operators. Before proving the main result, a short discussion on Toeplitz operators in in order. Given an essentially bounded function f on the complex plane, let M_f denote the multiplication induced by f. It is clearly bounded from $F^2_{\alpha,\beta,s} \to L^2_{\alpha,\beta,s}$. The Toeplitz operator

$$T_f^{\alpha,\beta,s} : F^2_{\alpha,\beta,s} \to F^2_{\alpha,\beta,s}$$

is then defined as

$$T_f^{\alpha,\beta,s} = P^{\alpha,\beta,s}M_f.$$

Observe that $T_f^{\alpha,\beta,s}$ is contractive, whenever $\|f\|_\infty \le 1$. Since orthogonal projections are self-adjoint, it is easy to see that the adjoint of $T_f^{\alpha,\beta,s}$ equals $T_{\overline{f}}^{\alpha,\beta,s}$. In particular, if $\|f\|_\infty \le 1$, the operators

$$I - T_f^{\alpha,\beta,s} T_{\overline{f}}^{\alpha,\beta,s} \quad \text{and} \quad I - T_{\overline{f}}^{\alpha,\beta,s} T_f^{\alpha,\beta,s}$$

are positive.

In [1, 18, 21, 22] the authors study function spaces on the unit disk, and problem of sub-Hilbert spaces induced by a Toeplitz operator (given by the orthogonal projection of the respective space). Special focus is given to symbols $f(z) = z^m$ and f being a finite Blaschke product. Neither option seems to work directly for our setting. Instead we take:

$$f(z) = \left(\frac{z}{|z|}\right)^m \quad \text{and} \quad \overline{f}(z) = \left(\frac{\overline{z}}{|z|}\right)^m,$$

where the contractivity requirement is automatically satisfied.

4.2. Proof of the main result. We will make use of the following result, which can be found in [18] (it is proven for the unit disk, but the exact same argument works for any reproducing kernel Hilbert space).

Lemma 4. *Let H be a reproducing kernel Hilbert space over a domain Ω, K_z its reproducing kernel and $T : H \to H$ a contraction. Then the reproducing kernel of $\mathcal{H}(T)$ is given by $(I - TT^*)K_z$.*

Note that every $F^2_{\alpha,\beta,s}$ is isometrically isomorphic to a weighted ℓ^2 space, with the weight coming from the moments of the weight $d\lambda_{\alpha,\beta,s}(z)$. On the other hand, also the reproducing kernel is related to these moments. Therefore, in order to determine $\mathcal{H}(T)$ and $\mathcal{H}(T^*)$, it suffices to study the asymptotic of the power series expansion of the reproducing kernel.

We are now in position to prove the main theorem. Let $\alpha, \beta > 0$ and $s \in \mathbb{R}$, and let $T_f^{\alpha,\beta,s}$ be the Toeplitz operator on $F^2_{\alpha,\beta,s}$ induced by the symbol $f(z) = (z/|z|)^m$. We then have

$$\mathcal{H}(f) = \mathcal{H}(\overline{f}) = F^2_{\alpha,\beta,s+\beta/2}.$$

We now prove this.

Proof. Suppose m is a natural number. We will calculate the reproducing kernels of sub-Fock-Sobolev Hilbert spaces. The formula

$$(I - T_{(\frac{\overline{z}}{|z|})^m} T_{(\frac{z}{|z|})^m}) K_z^{\alpha,\beta,s}$$

gives the reproducing kernels of these spaces. So, we consider the Toeplitz operator induced by $(\frac{z}{|z|})^m$ in Fock-Sobolev spaces $F^2_{\alpha,\beta,s}$. By using the definition of Toeplitz operator and the formula (2), we have

$$\begin{aligned}
\frac{2}{\beta}\alpha^{\frac{-2s-2}{\beta}} T_{(\frac{z}{|z|})^m} z^n &= \int_{\mathbb{C}} (\frac{\xi}{|\xi|})^m \xi^n \sum_{k\geq 0} \frac{1}{\Gamma\left(\dfrac{2k+2s+2}{\beta}\right)} (\alpha^{2/\beta} z\bar{\xi})^k |\xi|^{2s} e^{-\alpha|\xi|^\beta} dA(\xi) \\
&= \frac{1}{\Gamma(\frac{2}{\beta}(n+m+s+1))} \alpha^{\frac{2}{\beta}(m+n)} z^{m+n} \int_{\mathbb{C}} |\xi|^{2n+2s+m} e^{-\alpha|\xi|^\beta} dA(\xi) \\
&= \frac{\alpha^{\frac{2}{\beta}(m+n)}}{\Gamma(\frac{2}{\beta}(n+m+s+1))} z^{m+n} 2\int_0^\infty r^{2n+2s+m+1} e^{-\alpha r^\beta} dr \\
&= \frac{\alpha^{\frac{2}{\beta}(m+n)}}{\Gamma(\frac{2}{\beta}(n+m+s+1))} z^{m+n} \frac{2}{\alpha\beta} \int_0^\infty (\frac{t}{\alpha})^{\frac{1}{\beta}(2n+m+2s+2)-1} e^{-t} dt \\
&= \frac{\alpha^{\frac{1}{\beta}(m-2s-2)}}{\Gamma(\frac{2}{\beta}(n+m+s+1))} z^{m+n} \frac{2}{\beta}\ \Gamma(\frac{2}{\beta}(n+s+1)+\frac{m}{\beta}) \\
&= \frac{\Gamma(\frac{2}{\beta}(n+s+1)+\frac{m}{\beta})}{\Gamma(\frac{2}{\beta}(n+m+s+1))} \frac{2}{\beta} z^{m+n} \alpha^{\frac{1}{\beta}(m-2s-2)}.
\end{aligned}$$

By a similar calculation for $T_{(\frac{\bar{z}}{|z|})^m} z^n$, using (2) we have

$$\begin{aligned}
\frac{2}{\beta}\alpha^{\frac{-2s-2}{\beta}} T_{(\frac{\bar{z}}{|z|})^m} z^n &= \int_{\mathbb{C}} (\frac{\bar{\xi}}{|\xi|})^m \xi^n \sum_{k\geq 0} \frac{1}{\Gamma(\frac{2}{\beta}(k+s+1))} (\alpha^{2/\beta} z\bar{\xi})^k |\xi|^{2s} e^{-\alpha|\xi|^\beta} dA(\xi) \\
&= \frac{\alpha^{\frac{2}{\beta}(n-m)}}{\Gamma(\frac{2}{\beta}(n-m+s+1))} z^{n-m} \int_{\mathbb{C}} |\xi|^{2n+2s-m} e^{-\alpha|\xi|^\beta} dA(\xi) \\
&= \frac{\alpha^{\frac{2}{\beta}(n-m)}}{\Gamma(\frac{2}{\beta}(n+m+s+1))} z^{n-m} 2\int_0^\infty r^{2n+2s-m+1} e^{-\alpha r^\beta} dr \\
&= \frac{\alpha^{\frac{2}{\beta}(n-m)}}{\Gamma(\frac{2}{\beta}(n-m+s+1))} z^{n-m} \frac{2}{\alpha\beta} \int_0^\infty (\frac{t}{\alpha})^{\frac{1}{\beta}(2n-m+2s+2)-1} e^{-t} dt \\
&= \frac{\alpha^{\frac{1}{\beta}(-m-2s-2)}}{\Gamma(\frac{2}{\beta}(n-m+s+1))} z^{n-m} \frac{2}{\beta}\ \Gamma(\frac{2}{\beta}(n+s+1)-\frac{m}{\beta})
\end{aligned}$$

$$= \frac{\Gamma(\frac{2}{\beta}(n+s+1)-\frac{m}{\beta})}{\Gamma(\frac{2}{\beta}(n-m+s+1))}\frac{2}{\beta}z^{n-m}\alpha^{\frac{1}{\beta}(-m-2s-2)}.$$

It follows that

$$\left(I - T_{(\frac{\bar{z}}{|z|})^m} T_{(\frac{z}{|z|})^m}\right) z^n = \left(1 - \frac{\Gamma(\frac{2}{\beta}(n+s+1)+\frac{m}{\beta})\Gamma(\frac{2}{\beta}(n+s+1)+\frac{m}{\beta})}{\Gamma(\frac{2}{\beta}(s+m+n+1))\Gamma(\frac{2}{\beta}(s+n+1))}\right) z^n.$$

From Lemma (2), we conclude that

$$1 - \frac{\Gamma\left(\frac{2}{\beta}(n+s+1)+\frac{m}{\beta}\right)\Gamma\left(\frac{2}{\beta}(n+s+1)+\frac{m}{\beta}\right)}{\Gamma\left(\frac{2}{\beta}(s+m+n+1)\right)\Gamma\left(\frac{2}{\beta}(s+n+1)\right)} \asymp \frac{1}{n},$$

therefore

$$\left(I - T_{(\frac{\bar{z}}{|z|})^m} T_{(\frac{z}{|z|})^m}\right) K_z^{\alpha,\beta,s}(\xi) = \sum_{n=0}^{\infty} \frac{1}{n} \frac{1}{\Gamma\left(\frac{2n+2s+2}{\beta}\right)} (\alpha^{2/\beta}\xi\overline{z})^n. \tag{3}$$

For large enough n, we have

$$n\,\Gamma\left(\frac{2n+2s+2}{\beta}\right) \asymp \left(\frac{2n+2s+2}{\beta}\right)\Gamma\left(\frac{2n+2s+2}{\beta}\right) = \Gamma\left(\frac{2n+2(s+\frac{\beta}{2})+2}{\beta}\right), \tag{4}$$

Substituting (4) into (3), we get the reproducing kernel of Fock-Sobolev space of order $(s+\frac{\beta}{2})$, which completes the proof. □

As a consequence of the main theorem, we obtain the following corollary for the Fock-Sobolev space $F^2_{\alpha,2,s}$.

Corollary 5. *Let $f(z) = (z/|z|)^m$, and let us consider Toeplitz operators acting on $F^2_{\alpha,2,s}$. Then*

$$\mathcal{H}(f) = \mathcal{H}(\overline{f}) = F^2_{\alpha,2,s+1}.$$

Note that this is in line with the well-known Bergman space results of Zhu [21, 22] and Abkar-Jafarzadeh [1].

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