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Exploring the rigidity of planar configurations of points and rods

Signe Lundqvist *, Klara Stokes, Lars-Daniel Öhman

Umeå Universitet, 90187 Umeå, Sweden

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ABSTRACT

In this article we explore the rigidity of realizations of incidence geometries consisting of points and rigid rods: rod configurations. We survey previous results on the rigidity of structures that are related to rod configurations, discuss how to find realizations of incidence geometries as rod configurations, and how this relates to the 2-plane matroid. We also derive further sufficient conditions for the minimal rigidity of k-uniform rod configurations and give an example of an infinite family of minimally rigid 3-uniform rod configurations failing the same conditions. Finally, we construct v_3-configurations that are flexible in the plane, and show that there are flexible v_3-configurations for all sufficiently large values of v.

1. Introduction

We are interested in the rigidity of structures built from points and rigid rods in the Euclidean plane. By the rods being rigid, we mean that the rods cannot bend, and the distance between points on the same rod cannot change. We call such structures rod configurations.

Rod configurations can be seen as realizations of combinatorial structures, namely incidence geometries, as points and straight lines in the plane. If there are only two points on each rod, then the problem reduces to the well studied rigidity theory of graphs in the plane.

Another type of geometric realization of incidence geometries are body and joint frameworks. In a body and joint framework, the combinatorial lines are realized as rigid bodies; i.e. motions preserve the distance between any two points incident to the same combinatorial line. Body and joint frameworks such that all points incident to each combinatorial line are collinear provide a model for rod configurations.

Whiteley made early important contributions to the understanding and the combinatorial characterization of the infinitesimal rigidity of rod configurations [31,32], using body and joint frameworks with collinear points to model the rods. He characterized the minimally infinitesimally rigid rod configurations, using the notion of minimality for which the rod configuration is minimally rigid if there are no redundant distance constraints.

In this article we will say that a rod configuration is minimally rigid if no rod can be removed without the rod configuration becoming flexible. This is a natural generalization of the notion of minimal rigidity for graph frameworks. Our notion of minimality is not the same as the one used by Whiteley. For example, with Whiteley's notion of minimality, a configuration in which three points are covered in pairs by two rods may be smaller than if the two rods are replaced with one rod covering all three points. Our notion deems the configuration with the smallest number of rods to be the

* Corresponding author.
E-mail address: signe.lundqvist@umu.se (S. Lundqvist).

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smallest one. However, the rod configurations that are minimally rigid with the notion of minimality used by Whiteley turn out to be minimally rigid also with our notion of minimality.

In 1984, Tay and Whiteley gave a conjecture on a combinatorial characterization of the rigidity of body and joint frameworks with all joints incident to a body lying in a common hyperplane [28], in the special case where the number of bodies meeting at each joint is exactly two. This conjecture became known as the *molecular conjecture*, and has since been proven by Jackson and Jordán, and Katoh and Tanigawa [15,18]. Such body and joint frameworks are essentially rod configurations, so the molecular conjecture can be seen as a statement about the existence of rigid rod configurations. Note though that the molecular conjecture applies only when the number of bodies meeting at each joint is two.

The rigidity of frameworks of graphs that have a given set of vertices collinear has also been studied. Graphs that have rigid frameworks such that a given set of three points are placed on a line have been characterized by Jackson and Jordán [14]. Their result was extended to sets of points of arbitrary size by Eftekhari et al. [8].

In the classical literature, the study of configurations of points and lines has been mostly concerned with v_k-configurations, and more generally, (v_r, b_k)-configurations. Configurations of points and lines are important in geometry. Historically, they have fascinated many now famous mathematicians and they keep fascinating people also today.

If either of r or k equals two, then either the configuration or its dual is a graph. Therefore the literature on configurations is mostly concerned with the case when $r, k \geq 3$, a case that the available results on minimal rigidity of rod configurations do not cover.

This article was written with the aim of shedding some light on this problem, focusing mostly on k-uniform incidence geometries, motivating the reader to explore the topic of configurations within the scope of rigidity theory.

2. Background

2.1. Linear realizations of graphs, incidence geometries and configurations

An **incidence geometry** S of rank two is a triple $S = (P, L, I)$, where P and L are two sets with elements of distinct type, and $I \subseteq P \times L$ is a symmetric incidence relation. We will refer to the elements of P and L as points and lines respectively. Throughout the article, we will assume that all incidence geometries are connected, that is, that they have connected incidence graphs.

An incidence geometry is called **linear** if every pair of elements of one type is simultaneously incident with at most one element of the other type [12], since this property captures the abstract notion of the incidences of a line arrangement. In this article, we will only consider linear incidence geometries.

There is a direct correspondence between incidence geometries of rank two and hypergraphs. If all elements of one type (the 'lines') are incident to exactly two elements of the other type (the 'points'), then the incidence geometry is a graph.

The real Euclidean plane defines a rank two incidence geometry $\mathbb{E}$, by taking as types the points and the lines, and defining a point and a line to be incident if the point is on the line.

A **planar linear realization** ρ of an incidence geometry S of rank two is a function $\rho : S \to \mathbb{E}$ that preserves incidence. In other words, a linear realization of S is an assignment of a point in $\mathbb{R}^2$ to each element of P and an assignment of a line in $\mathbb{R}^2$ to each element of L such that if $(p, \ell) \in I$, then the point $\rho(p)$ lies on the line $\rho(\ell)$. In this article, we will only be concerned with planar linear realizations, although it is of course possible to define linear realizations of higher dimension.

An incidence geometry always has **trivial** linear realizations, in which all 'point' elements are mapped to the same point. We say that a linear realization is **proper** if all combinatorial points are realized with distinct coordinates.

We say that an incidence geometry $S = (P, L, I)$ is k-**uniform** if every line in L is incident to exactly k points, and r-**regular** if all points are incident to r lines. Historically, the literature has mostly been concerned with linear realizations of incidence geometries which are both k-uniform and r-regular. Such configurations are called (v_r, b_k)-**configurations**, where v is the number of points and b is the number of lines. If $r = k$, then also $v = b$, and the configuration is called balanced, in which case the name v_k-**configuration** is used. Many important examples of configurations of points and lines are balanced, such as the v_3-configurations of Pappus and Desargues. In this context, incidence geometries with proper linear realizations are also known in the literature as **geometric configurations**. Incidence geometries with the same abstract properties as geometric (v_r, b_k)-configurations or v_k-configurations are in the literature also called **combinatorial configurations** [10,22]. The combinatorial (v_r, b_2)-configurations are the (simple) r-regular graphs.

Because any two points define a line in Euclidean space, any assignment of points to the vertices of a graph give rise to a linear realization of the graph. Therefore, there are proper linear realizations of any graph. However, if the incidence geometry is not a graph, a proper linear realization can be difficult to find, and may not even exist. For example, there is no proper linear realization of the unique 7_3-configuration, also known as the Fano plane.

2.2. The 2-plane matroid, parallel redrawings and concurrence geometries

Consider an incidence geometry $S = (P, L, I)$. We want to study the set of all linear realizations of the incidence geometry in the Euclidean plane with specified line slopes. Some linear realizations will be degenerate in the sense that some points coincide. Crapo showed that the linear realizations of S with different degeneracies form a combinatorial lattice [5].

Let $S = (P, L, I)$ and suppose there is a slope f_j assigned to each $\ell_j \in L$. Consider an incidence $(p_i, \ell_j) \in I$. In any linear realization of S such that the line assigned to ℓ_j has slope f_j, the coordinates (x_i, y_i) of the point assigned to p must satisfy

$$f_j x_i + y_i + h_j = 0 \quad (1)$$

where h_j is the y-intercept of the line assigned to ℓ_j.

Define the space of **parallel redrawings** of an incidence geometry with a fixed slope f_j for every line ℓ_j to be the space of assignments of a point (x_i, y_i) to each $p_i \in P$ and an assignment of a number h_j to each $\ell_j \in L$ such that if $(p_i, \ell_j) \in I$, then the triple (x_i, y_i, h_j) satisfies Eq. (1) (see for example [31]). As defined, the space of parallel redrawings of an incidence geometry with fixed line slopes is simply the space of all linear realizations of the incidence geometry in the plane, with the given line slopes.

Let ρ be a linear realization of an incidence geometry $S = (P, L, I)$. Then ρ gives a set of line slopes, that we may fix. Finding the parallel redrawings of S with line slopes given by ρ requires solving $|I|$ equations of the form (1), defining a system of equations in $|L| + 2|P|$ indeterminates: one y-intercept for each line (lines are assumed to not be vertical) and two coordinates for each point.

Let $M(S, \rho)$ be the $|I| \times (|L| + 2|P|)$ coefficient matrix of this system of equations. We call $M(S, \rho)$ the **concurrence geometry matrix** of the incidence geometry with the given realization ρ, where the name is in reference to Crapo and his use of the word "concurrence geometry" [5].

Given a linear realization ρ of S, the kernel of the matrix $M(S, \rho)$ consists of the set of triples (x_i, y_i, h_j) that satisfy Eq. (1) for the set of slopes given by ρ. In other words, given a linear realization ρ of the incidence geometry, the kernel of the matrix $M(S, \rho)$ is the space of parallel redrawings with the line slopes defined by ρ.

For any set of line slopes there is a space of trivial parallel redrawings in which all points have the same coordinates. This space has dimension two, corresponding to the two coordinates, which then also determine the y-intercept.

If the incidence geometry can be realized with the given set of line slopes so that at least two points have different coordinates, then there is a three-dimensional trivial space of parallel redrawings, generated by two translations and one dilation, so the space of parallel redrawings is at least three-dimensional [31]. It follows that if we can realize the incidence geometry in such a way that at least two points have distinct coordinates, then the kernel of $M(S, \rho)$ has dimension at least three. In particular, the kernel of $M(S, \rho)$ will have dimension at least three for a proper linear realization ρ of an incidence geometry S with two or more points.

So, if S has a proper linear realization ρ such that the rows of the concurrence geometry matrix $M(S, \rho)$ are independent, then necessarily $|I| \leq |L| + 2|P| - 3$, since the kernel of $M(S, \rho)$ always has dimension at least three.

Whiteley introduced the k-plane matroid as a generalization of the so called picture matroid, as a tool in scene analysis [24,31]. The 2-plane matroid is a matroid defined on the set of incidences I of an incidence geometry $S = (P, L, I)$ in terms of independent sets as follows: I is independent if for any nonempty subset $I' \subseteq I$ the inequality $|I'| \leq |L'| + 2|P'| - 2$ holds, where $P' \times L' \subseteq P \times L$ is the support of I'.

Given an incidence geometry $S = (P, L, I)$, the rows of $M(S, \rho)$ are independent for almost all linear realizations ρ if and only if I is independent in the 2-plane matroid. In particular, the rows of $M(S, \rho)$ are independent if I is independent in the 2-plane matroid and the line slopes are chosen to be linearly independent over $\mathbb{Q}$. This is essentially Theorem 4.1 in [31].

It follows that if I is independent in the 2-plane matroid and satisfies $|I| = |L| + 2|P| - 2$, then for most choices of line slopes the concurrence geometry matrix will have a two-dimensional kernel. Hence, for most choices of line slopes, there are only trivial linear realizations of S.

Note however that incidence geometries that satisfy $|I| \geq |L| + 2|P| - 2$ can have proper linear realizations, but for proper linear realizations ρ, some of the rows in the concurrence geometry matrix must be dependent. For example, the Pappus configuration has proper linear realizations, but it satisfies $|I| = |L| + 2|P|$. If the realization is proper, then the concurrence geometry matrix has at least a three-dimensional kernel, so the rank of $M(S, \rho)$ is at most $|P| + 2|L| - 3$. Hence there are at least three dependencies among the rows of $M(S, \rho)$, for any proper linear realization of the Pappus configuration.

2.3. Notions of rigidity for rod configurations and graphs

There is some variation in the notation for linear realizations of incidence geometries in previous literature, see for example [15,21,29,31]. With this in mind, here we give our choice of notation.

Given a rank two incidence geometry $S = (P, L, I)$ and a proper planar linear realization ρ of S, we say that a **continuous motion** of ρ is a continuous curve $\tau_p(t) : [0, 1] \to \mathbb{R}^2$ for each element $p \in P$, such that $\tau_p(0) = \rho(p)$ for all $p \in P$, and $|\tau_p(t) - \tau_q(t)| = |\tau_p(0) - \tau_q(0)|$ for all $p, q \in P$ such that $(p, \ell) \in I$ and $(q, \ell) \in I$ for some line $\ell \in L$.

The continuous Euclidean motions of the plane are continuous motions of any linear realization. A linear realization is called **(continuously) flexible** if it admits a continuous motion other than the continuous Euclidean motions of the entire plane. Otherwise it is called **(continuously) rigid**. We will sometimes omit "continuously", and simply refer to continuously rigid rod configurations as rigid.

A linear realization of an incidence geometry together with this notion of rigidity is called a **rod configuration**. In the context of rod configurations, we are interested only in proper linear realizations. We will therefore assume that all rod configurations are proper linear realizations. The lines and the points are thought of as rods and pin-joints, respectively. In the classical literature a rod is a line together with a weight. The weight can be interpreted as the distance between the two end-points of the rod. Our rods are in general defined by more than two points (since the incidence geometry may have more than two points on a line), and each pair of these points defines a weight (a distance) that must be preserved in any motion of the rod configuration. In other words, in a rod configuration, points on the same rod never move in relation to each other, and the lines move as rigid bodies.

We define an **infinitesimal motion** of a rod configuration ρ realizing an incidence geometry S to be an assignment of a vector $m \in \mathbb{R}^2$ to each point $p \in P$ such that restricted to each rod the vectors define the linear part of a continuous Euclidean motion. The linear parts of the continuous Euclidean motions are infinitesimal motions of all rod configurations. We call such infinitesimal motions the **trivial infinitesimal motions** of the rod configuration. For planar rod configurations with at least two distinct points there are three independent trivial infinitesimal motions, coming from the three generators of the Euclidean planar group: one rotation and two translations. For linear realizations of graphs in the plane, there is a one-to-one correspondence between infinitesimal motions and parallel redrawings [6,7,30]; one is obtained from the other by turning all vectors $\pi/2$ radians. The trivial infinitesimal motions correspond to the trivial parallel redrawings.

We say that a rod configuration is **infinitesimally rigid** if any infinitesimal motion of the rod configuration is trivial. If there is a non-trivial infinitesimal motion, then we say that the rod configuration is **infinitesimally flexible**.

We say that a rod configuration ρ realizing an incidence geometry $S = (P, L, I)$ is **globally flexible** if there is some other rod configuration ρ' realizing S with the same pairwise distance between collinear points such that ρ and ρ' are not related by a Euclidean motion. Otherwise ρ is called **globally rigid**. Note that globally rigid rod configurations are *a fortiori* continuously rigid.

We say that a rod configuration is **minimally (continuously/infinitesimally/globally) rigid** if it is (continuously/infinitesimally/globally) rigid and no rod can be removed from the configuration without it becoming (continuously/infinitesimally/globally) flexible.

In these three definitions of minimal rigidity, we do not allow the removal of joints. If a joint belongs to only one rod, then the removal of that rod would result in a flexible configuration, since that joint would then be able to move independently of the rest of the rod configuration. This applies to continuous, and therefore also global, and infinitesimal rigidity.

2.4. Characterizing rigidity of graphs in the plane

In this section we review the special case where the incidence geometry is a graph $G = (V, E)$. A (planar) linear realization (a rod configuration) of a graph G is also called a (planar) **framework** of the graph.

In this case, an infinitesimal motion of the framework (G, ρ) is an assignment of a vector $m(v_i) \in \mathbb{R}^2$ to the point $\rho(v_i)$ for $v_i \in V$ such that $(m(v_i) - m(v_j))^T \cdot (\rho(v_i) - \rho(v_j)) = 0$ for all edges $(v_i, v_j) \in E$.

Remark 2.1. Our definition of continuous rigidity of rod configurations is the notion of continuous rigidity one obtains by replacing each rod by a complete graph on the points incident to the rod such that the vertices of the graph are placed along the rod, and then considering the continuous rigidity of that graph framework.

However, replacing a rod by a complete graph such that the vertices of the graph are placed on a line and considering the **infinitesimal** rigidity of that graph framework gives a non-trivial infinitesimal motion perpendicular to the rod. This is why we require that the infinitesimal motions of rod configurations are trivial when restricted to a single rod.

The following lemma relates infinitesimal rigidity to rigidity.

Lemma 2.2 (*Gluck [9]*)**.** *If a framework ρ of a graph G in the plane is infinitesimally rigid, then it is rigid.*

It is well known that the algebraic dependencies among the points assigned to the vertices can affect the flexibility and the infinitesimal flexibility of a graph realized in the plane. A framework of a graph is called **generic** if its set of point coordinates is algebraically independent over $\mathbb{Q}$. The converse of Lemma 2.2 is not true in general, but it holds for generic frameworks [1]. Therefore, for generic frameworks, rigidity is equivalent to infinitesimal rigidity.

Furthermore, by the following lemma, it makes sense to talk about generic rigidity of a graph.

Lemma 2.3 (*Gluck, Lovász [9,20]*)**.** *Let $G = (V, E)$ be a graph. If there is some infinitesimally rigid framework of a graph in the plane, then any generic framework of G in the plane is rigid.*

We say that a graph is **generically rigid** in the plane if all its generic frameworks in the plane are infinitesimally rigid, or, equivalently, rigid. A graph is **generically minimally rigid** in the plane, if it is generically rigid in the plane, and the removal of any edge results in a graph that is not generically rigid in the plane. A famous result due to Geiringer, and later to Laman, gives a characterization of the generically minimally rigid graphs in the plane.

Theorem 2.4 (*Geiringer, Laman [19,23]*)**.** *Let $G = (V, E)$ be a graph. Then G is generically minimally rigid in the plane if and only if*

1. $|E| = 2|V| - 3$
2. $|E'| \leq 2|V'| - 3$ *for any nonempty subset $E' \subseteq E$, where V' is the set of vertices in the subgraph generated by E'.*

Global rigidity of graphs in the plane has also been characterized. Hendrickson gave necessary conditions for a graph framework in $\mathbb{R}^d$ to be globally rigid, in terms of the connectivity and redundant rigidity of the underlying graph [11]. Jackson and Jordán showed that Hendrickson's conditions are also sufficient in the plane [13]. Connelly gave a sufficient condition for global rigidity of generic graph frameworks in $\mathbb{R}^d$ in terms of stress matrices [4].

2.5. Characterizing rigidity of string configurations in the plane

One way to realize an incidence geometry $S = (P, L, I)$ in the Euclidean plane is as follows: First define a graph on the vertex set P by adding edges forming a tree for each element $\ell \in L$, so that the tree contains exactly the points incident to ℓ, and then consider a **framework** of this graph with the property that the edges in a tree corresponding to an element of L are all collinear. Following Whiteley, we call such a realization a **string configuration** [29].

It is important to note that a string configuration is a graph framework with certain edges collinear, rather than a linear realization of the incidence geometry. However, a string configuration of the incidence geometry cannot exist without there being a corresponding linear realization of the same incidence geometry. Given a linear realization of the incidence geometry, the edges (p, q) in the graph that constitutes the string configuration are placed along the line spanned by p and q in the linear realization. There may be several string configurations coming from the same linear realization of an incidence geometry; by choosing distinct tree graphs representing the lines, different string configurations are obtained.

In this section we will discuss rigidity of string configurations, and a characterization of which incidence geometries have realizations as infintiesimally rigid string configurations. First note that string configurations are in fact frameworks of graphs, so we can say that a string configuration realizing an incidence geometry $S = (P, L, I)$ is **infinitesimally rigid** if, when considered as a framework of a graph, it is infinitesimally rigid. Otherwise, we say that it is **infinitesimally flexible**.

Whiteley proved the following result, characterizing which incidence geometries have realizations as minimally infinitesimally rigid string configurations.

Theorem 2.5 (*Whiteley [31]*)**.** *An incidence geometry $S = (P, L, I)$ has a realization as a minimally infinitesimally rigid string configuration if and only if*

$$|I| = |L| + 2|P| - 3$$

and

$$|I'| \leq |L'| + 2|P'| - 3$$

for any proper subset $I' \subset I$, where $P' \times L' \subseteq P \times L$ is the support of I'.

If the incidence geometry is a graph $G = (V, E)$, with $P = V$ and $L = E$, then $|I| = 2|E|$. In this case, Theorem 2.5 is the characterization of generically minimally rigid graphs, Theorem 2.4.

Whiteley proved Theorem 2.5 using parallel redrawings and the concurrence geometry matrix. Since a string configuration is a framework of graph, the parallel redrawings of a planar linear realization of S are in one-to-one correspondence with the infinitesimal motions of its realizations as a string configuration with the same line slopes.

A key point in the proof of Theorem 2.5 is that if the incidence geometry satisfies the conditions in Theorem 2.5, then it has a proper linear realization for almost all choices of normals. As previously mentioned, not all incidence geometries admit proper linear realizations with generic normals. In fact, Whiteley proved that an incidence geometry has proper linear realizations with generic normals if and only if the second counting condition of Theorem 2.5 holds [31].

Recall that the rows of $M(S, \rho)$ are independent for almost all realizations ρ if the incidences of S are independent in the 2-plane matroid. The incidence geometries that have realizations as minimally infinitesimally rigid string configurations are certainly independent in the 2-plane matroid, but not maximally independent.

Berg and Jordán showed that it is NP-hard to find maximum size subsets of incidences that satisfy the counting condition in Theorem 2.5 [2]. In contrast, there is a polynomial time algorithm for finding maximum size subgraphs satisfying the counting condition in Theorem 2.4 [17].

2.6. Rigidity of body and joint frameworks and rod configurations in the plane

Following Whiteley [31], we define a planar body and joint framework realizing an incidence geometry $S = (P, L, I)$ to be an assignment $\rho : P \to \mathbb{R}^2$. A **body and joint framework** is said to be **infinitesimally rigid** if it is possible to replace each element $\ell \in L$ by a minimally infinitesimally rigid graph framework including the joints $\rho(p)$ such that $(p, \ell) \in I$, but excluding the joints $\rho(q)$ such that $(q, \ell) \notin I$, so that the entire graph framework is infinitesimally rigid.

Note that we allow replacing ℓ by a minimally infinitesimally rigid graph framework on more joints in $\mathbb{R}^2$ than those representing elements of P incident to ℓ, provided that those joints of $\mathbb{R}^2$ do not represent elements of P not incident to ℓ. Note also that we do not in general assume that the graph frameworks replacing the bodies are generic. Indeed, the focus of this article is on rod configurations, which is a special case of body and joint frameworks, with the property that all joints incident to a body are collinear. In this case the framework is not in generic position.

A body and joint framework is **(minimally) infinitesimally rigid** if it is a (minimally) infinitesimally rigid graph framework.

Whiteley gave a combinatorial characterization of minimal infinitesimal rigidity of body and joint frameworks of incidence geometries, thereby generalizing Theorem 2.4 to hypergraphs. He also showed that an incidence geometry that has a minimally infinitesimally rigid body and joint framework, also has a realization as a minimally infinitesimally rigid rod configuration.

Theorem 2.6 (*Whiteley [31]*)**.** *Given an incidence geometry $S = (P, L, I)$ the following are equivalent:*

1. *S has a minimally infinitesimally rigid body and joint framework in the Euclidean plane.*
2. *S satisfies $2|I| = 3|L| + 2|P| - 3$, and for every subset $L' \subset L$, the set of points P' incident to L' and the set of incidences $I' \subseteq I$ such that $L' \times P'$ is the support of I' satisfy $2|I'| \le 3|L'| + 2|P'| - 3$.*
3. *S has a minimally infinitesimally rigid body and joint framework in the Euclidean plane such that each body has all its joints collinear.*

Remark 2.7. Any incidence geometry that satisfies the inequality in point 2 of Theorem 2.6 will have proper linear realizations for almost all choices of line slopes, and is therefore realizable as a rod configuration. If equality is attained in the overall count, then Theorem 2.6 guarantees that this rod configuration is minimally infinitesimally rigid for almost all choices of line slopes. Furthermore, if an incidence geometry $S = (P, L, I)$ satisfies $2|I| < 3|L| + 2|P| - 3$, then any rod configuration realizing S will be infinitesimally flexible — even if some subset of incidences I' with support $L' \times P' \subset L \times P$ satisfies $2|I'| > 3|L'| + 2|P'| - 3$.

Our notion of minimal infinitesimal rigidity of rod configurations is not the same as the notion of minimal infinitesimal rigidity that appears in statement 3 of Theorem 2.6, but the two notions are related. In short, there are incidence geometries that have realizations as minimally infinitesimally rigid rod configurations in our context, but for which there is no minimally infinitesimally rigid body and joint framework such that each body has all its joint collinear. However, Theorem 2.6 shows that the converse is true; any incidence geometry that can be realized as a minimally infinitesimally rigid body and joint framework can also be realized as a minimally infinitesimally rigid rod configuration.

If S has a realization as a rod configuration, then it is possible to construct a body and joint framework, by replacing each rod with a cone on the points incident to the rod, such that each body has all its joints collinear, with the same infinitesimal and continuous rigidity properties as the rod configuration (see for example [21]). In fact, Whiteley constructs this body and joint framework in [31] to prove the implication 2 ⇒ 3 of Theorem 2.6. Hence, if S has a realization as an infinitesimally rigid rod configuration, then S has a realization as an infinitesimally rigid body and joint framework. If that realization is minimally infinitesimally rigid, which is the case in Theorem 2.6, then so is the rod configuration.

As an example, the incidence geometry realized as a rod configuration in Fig. 1 is minimally rigid as a rod configuration. However, it does not satisfy condition 2 of Theorem 2.6. The realizations of the incidence geometry as a body and joint framework are infinitesimally rigid, but not minimally infinitesimally rigid.

Tay and Whiteley independently characterized which incidence geometries have realizations as rigid body and hinge frameworks in $\mathbb{R}^d$, where a body and hinge framework is a body and joint framework such that each joint is incident to at most two bodies [27,30]. Tay and Whiteley jointly conjectured in [28] that any incidence geometry that can be realized as a rigid body and hinge framework in $\mathbb{R}^d$ can be realized as a rigid body and hinge framework in $\mathbb{R}^d$ such that all hinges incident to a body lie in a common hyperplane. This conjecture is known as the molecular conjecture.

Remark 2.8. Our model of body and joint frameworks is the same as in [31]. This model is different to the typical description of body and joint frameworks (see for example [28], or [16]), although the two models are equivalent.

A special case of the molecular conjecture in the plane follows from Theorem 2.6; namely the special case where the rigidity of the body and hinge framework is minimal. However Theorem 2.6 holds for general body and joint frameworks, not only body and hinge frameworks, allowing more than two bodies to meet at a point.

Jackson and Jordán proved in [15] that the molecular conjecture holds in the plane, and Katoh and Tanigawa proved in [18] that the molecular conjecture holds in general. Jackson and Jordán therefore solved the question about rigidity for planar rod configurations in the special case when each point is incident to two lines only. Body and hinge structures are further studied in [16,26]. None of these results solve the question of minimal infinitesimal rigidity for rod configurations in its generality.

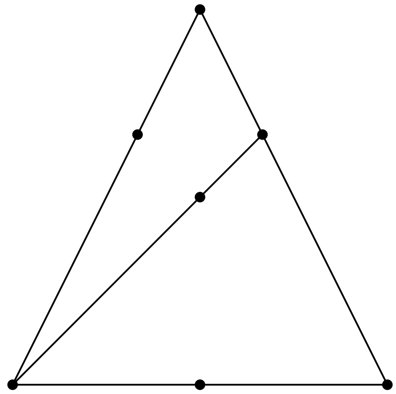

Fig. 1. A minimally infinitesimally rigid rod configuration with $3|L| = 2|P| - 2$.

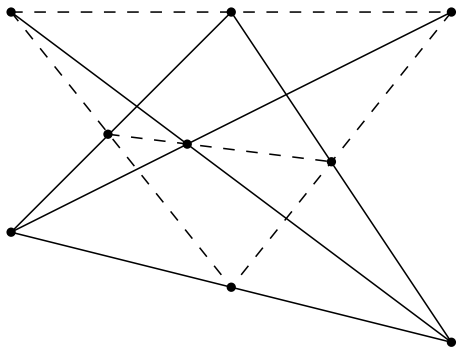

Fig. 2. A minimally rigid subconfiguration of the Pappus configuration.

3. Minimal rigidity of rod configurations in the plane

In this section, we will focus on k-uniform incidence geometries. In a k-uniform incidence geometry $|I| = k|L|$, in which case the implication $2 \Rightarrow 3$ of Theorem 2.6 can be restated as follows:

Corollary 3.1. *Let $S = (P, L, I)$ be a k-uniform incidence geometry. If*

1. $(2k - 3)|L| = 2|P| - 3$ *and*
2. $(2k - 3)|L'| \leq 2|P'| - 3$ *for every subset $L' \subseteq L$, where P' is the set of points generated by L'.*

then S has a realization as a minimally infinitesimally rigid rod configuration.

For body and joint frameworks, the converse of Corollary 3.1 also holds. However, there are examples of rod configurations that are minimally infinitesimally rigid, but that do not satisfy the count of Theorem 2.6 and Corollary 3.1. See for example Fig. 1.

An interesting special case of k-uniform incidence geometries are v_k-configurations. Note that $|L| = |P|$ for a v_k-configuration, so $(2k - 3)|L| = 2|P| - 3$ is never true for v_k-configurations with $k \geq 3$. If $k = 2$, then $|P| = |L| = 3$ is the only solution, so the only v_2-configuration satisfying Corollary 3.1 is a triangle. However, v_k-configurations may have spanning subconfigurations that satisfy the count given in Corollary 3.1.

Example 3.2 (*Pappus Configuration*)**.** The points and lines in Fig. 2 form the Pappus configuration. By removing the dotted lines we obtain a spanning subconfiguration that clearly is minimally infinitesimally rigid (but not globally rigid), as it consists of two triangles sharing a common edge. Indeed, all lines in the spanning subconfiguration have a point incident only to that line. Therefore, we cannot remove any line without the rod configuration becoming infinitesimally flexible. The existence of this minimally infinitesimally rigid spanning subconfiguration tells us that the Pappus configuration must be infinitesimally rigid in the position shown in Fig. 2.

The same spanning subconfiguration satisfies the count in Corollary 3.1. It follows that it has at least one realization as a minimally infinitesimally rigid rod configuration.

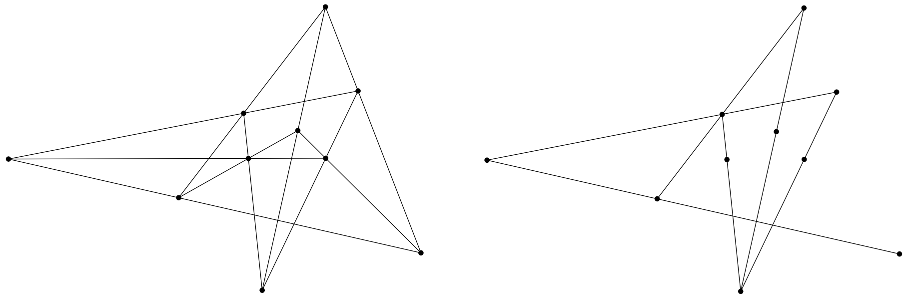

Fig. 3. The Desargues configuration and a minimally rigid subconfiguration.

As we can see in Fig. 2, at least one of those realizations extends to a geometric realization of the Pappus configuration. We can therefore conclude that Pappus configuration has at least one realization as an infinitesimally rigid rod configuration.

It is not necessarily true that a rod configuration realizing a spanning subconfiguration extends to a rod configuration realizing the whole configuration. As already pointed out, the Fano plane has a spanning subconfiguration that can be realized as a minimally infinitesimally rigid rod configuration (see Example 3.5). But the Fano plane has no proper linear realization, so any linear realization of this spanning subconfiguration that extends to a linear realization of the Fano plane must have all points in the same position, or all lines placed along the same line, a position in which it is not rigid.

By the definition of minimal rigidity of rod configurations, it is clear that any rigid rod configuration has at least one minimally rigid spanning subconfiguration. A minimally rigid spanning subconfiguration of a k-uniform incidence geometry can only satisfy the counting condition in Corollary 3.1 in certain cases, namely when $2|P|-3$ is divisible by $2k-3$.

If $k=2$, so that the incidence geometry is a graph, then $2|P|-3$ is always divisible by $2k-3=1$, and Corollary 3.1 is the Geiringer–Laman Theorem, Theorem 2.4. If $k=3$, then $2k-3=3$, so the first condition in Corollary 3.1 becomes $3|L|=2|P|$. For any given P, exactly one of $2|P|-1$, $2|P|-2$ and $2|P|-3$ is divisible by 3. This gives a lower bound on how many lines a 3-uniform incidence geometry needs in order to have a realization as an infinitesimally rigid rod configuration, as the next proposition shows.

Proposition 3.3. *Let $S=(P,L,I)$ be a k-uniform incidence geometry such that S has a realization as an infinitesimally rigid rod configuration. Assume also that*

$$2|P|-3 \leq (2k-3)|L| \leq 2|P|-3+(2k-3-1).$$

Then S has a realization as a minimally infinitesimally rigid rod configuration.

Proof. Assume for a contradiction that it is possible to remove some line ℓ from S to obtain an infinitesimally rigid rod configuration. Let $L'=L\setminus\{\ell\}$ and consider the incidence geometry $S'=(P',L',I')$ generated by L'. If ℓ can be removed without the rod configuration becoming infinitesimally flexible, then it must hold that $|P'|=|P|$. Therefore

$$(2k-3)|L'|=(2k-3)(|L|-1)=(2k-3)|L|-(2k-3)\leq 2|P|-4=2|P'|-4,$$

where the inequality holds, since $(2k-3)|L|\leq 2|P|-(2k-3)-4$.

Then by Theorem 2.6, S' does not have a realization as an infinitesimally rigid body and joint framework. Therefore S' also cannot have a realization as an infinitesimally rigid rod configuration. Hence the infinitesimally rigid rod configuration realizing S is minimally infinitesimally rigid. □

Example 3.4 (*Desargues Configuration*)**.** The Desargues configuration, to the left in Fig. 3, is a 10_3-configuration. As $2|P|-3=17$ is not divisible by $2k-3=3$, a minimally infinitesimally rigid spanning subconfiguration of the Desargues configuration cannot satisfy the count in Corollary 3.1.

Fig. 3 also shows, to the right, a minimally infinitesimally rigid spanning subconfiguration of the Desargues configuration. The spanning subconfiguration satisfies $3|L|=2|P|-2$, but any strict subset $L'\subset L$ satisfies $3|L'|\leq 2|P'|-3$, where P' is the set of points generated by L'.

Example 3.5. Consider the rod configuration in Fig. 1. Clearly, it is minimally infinitesimally rigid, as removing a line would either leave a point not incident to any line, or leave a line with two points that are not incident to any other line, making the configuration flexible in either case. In this rod configuration, $P=7$ and $L=4$, so $3L=2P-2$.

For the incidence geometries in Figs. 1 and 3, we can consider $L' = L \setminus \{\ell\}$, where ℓ is any line with exactly one point incident only to ℓ, and the points P' generated by L'. In both cases we obtain an incidence geometry satisfying the condition of Corollary 3.1. This is the idea behind the next proposition.

Proposition 3.6. *Let $S = (P, L, I)$ be a 3-uniform incidence geometry.*

1. *If $3|L| = 2|P| - 2$ and $3|L'| \leq 2|P'| - 3$ for all $L' \subsetneq L$, with P' the point set covered by L', then S has a realization as a minimally infinitesimally rigid rod configuration.*
2. *If $3|L| = 2|P| - 1$, $3|L'| \leq 2|P'| - 2$ for all $L' \subsetneq L$ and $3|L'| \leq 2|P'| - 3$ for all $L' \subset L$ with $|L'| \leq |L| - 2$ then S has a realization as a minimally infinitesimally rigid rod configuration.*

Proof. Let $S = (P, L, I)$ be an incidence geometry.

Claim. *Suppose that $3|L| \leq 2|P| - 1$, and for any strict subset $L' \subset L$, $3|L'| \leq 2|P'| - 2$. Then there is a line $\ell \in L$ such that if $L' = L \setminus \{\ell\}$, then $|P'| = |P| - 1$. Note that this holds for both cases in the statement of the proposition.*

Proof of claim. Suppose no such line exists. Then for any line $\ell \in L$ and $L' = L \setminus \{\ell\}$, $|P'| = |P|$ or $|P'| = |P| - 2$, since we assume that S is connected. Suppose L contains a line ℓ so that $|P'| = |P| - 2$. Then $3|L'| = 3|L| - 3 \leq 2(|P| - 1) - 3 = 2|P| - 5 = 2|P'| - 1$, which contradicts our assumption that $3|L'| \leq 2|P'| - 2$. Hence L cannot contain any such line, and all lines $\ell \in L$ must be such that $|P| = |P'|$. In that case, any point is incident to at least two lines, so $|I| \geq 2|P|$. As $|I| = 3|L|$ for any 3-uniform incidence geometry, it follows that $3|L| \geq 2|P|$ which, again, contradicts our assumptions. Hence there must be a line ℓ so that $|P'| = |P| - 1$.

1. By the claim, there is some line $\ell \in L$ such that if $L' = L \setminus \{\ell\}$, then $|P'| = |P| - 1$. Take such a line ℓ and let $L' = L \setminus \{\ell\}$. Consider the subgeometry $S' = (P', L', I')$ generated by L'. It will satisfy $3|L'| = 3|L| - 3 = 2|P| - 2 - 3 = 2|P'| - 3$, and for any subset $L'' \subseteq L'$, the incidence geometry $S'' = (P'', L'', I'')$ generated by L'' will satisfy the inequality $3|L''| \leq 2|P''| - 3$, as L'' is a strict subset of L. It follows from Corollary 3.1 that S' has a proper linear realization, which is a minimally infinitesimally rigid rod configuration.

Since S is 3-uniform, the line ℓ must be incident to exactly two points in P'. It is therefore possible to add back the line ℓ between the two points in P' incident to ℓ in S. The two points incident to ℓ will have distinct coordinates in the proper linear realization of S' which is guaranteed to exist. It is possible to then add a point on ℓ. The addition of a line meeting two existing points cannot make the configuration infinitesimally flexible, so the result is an infinitesimally rigid rod configuration realizing S.

By Proposition 3.3 this infinitesimally rigid rod configuration realizing S is also minimally infinitesimally rigid.

2. Again it follows from the claim that there is some line $\ell \in L$ such that if $L' = L \setminus \{\ell\}$, then $|P'| = |P| - 1$. Take such a line ℓ. Let $L' = L \setminus \{\ell\}$ and consider $S' = (P', L', I')$. Then $3|L'| = 3|L| - 3 = 2|P| - 1 - 3 = 2|P'| - 2$. Consider a subset $L'' \subsetneq L'$ and its generated incidence geometry $S'' = (P'', L'', I'')$. Then $3|L''| \leq 2|P''| - 3$, as L'' is a subset of L with $|L''| \leq |L| - 2$.

By case 1, S' has a realization as a minimally infinitesimally rigid rod configuration. It is again always possible to add back the line ℓ between the two appropriate points in P' and to add the remaining point on ℓ. The result is an infinitesimally rigid rod configuration realizing S.

By Proposition 3.3 this infinitesimally rigid rod configuration realizing S is also minimally infinitesimally rigid. □

The incidence geometries that satisfy either set of conditions given in Proposition 3.6 can all be constructed from an incidence geometry that has a realization as a minimally infinitesimally rigid rod configuration satisfying the conditions of Corollary 3.1 by adding a line incident to two existing points, and one new point incident only to the added line. Those incidence geometries that satisfy the first set of conditions can be constructed by adding one line in this way to an incidence geometry satisfying the conditions in Corollary 3.1, and those that satisfy the second set of conditions can be constructed by adding two lines.

Example 3.7. The rod configuration (P, L, I) in Fig. 4 has $|L| = 6$ and $|P| = 9$, so $3|L| = 2|P|$. Any rod that does not have a point incident only to that rod can be removed without the rod configuration becoming infinitesimally flexible.

Furthermore, removing either of the two lines with a point incident only to that line results in a rod configuration with $|L| = 5$ and $|P| = 8$ that satisfies the second set of conditions given in Proposition 3.6.

Hence, the rod configuration in Fig. 4 can be constructed from a rod configuration that satisfies the conditions of Corollary 3.1 by adding lines between existing points and points incident only to those lines; similarly to how those incidence geometries that satisfy either set of conditions in Proposition 3.6 were constructed, only in this case, three lines are added. However, unlike the rod configurations that satisfy the conditions given in Proposition 3.6, it is not minimally rigid. Therefore, the method of constructing minimally rigid rod configurations by adding lines with a single point only incident to that line is not guaranteed to work if three or more lines are added.

Corollary 3.1 and Proposition 3.6 do not characterize the minimally infinitesimally rigid rod configurations; i.e. it is not true that a rod configuration is minimally infinitesimally rigid if and only if it satisfies the counts of either Corollary 3.1 or Proposition 3.6, as seen in the following example.

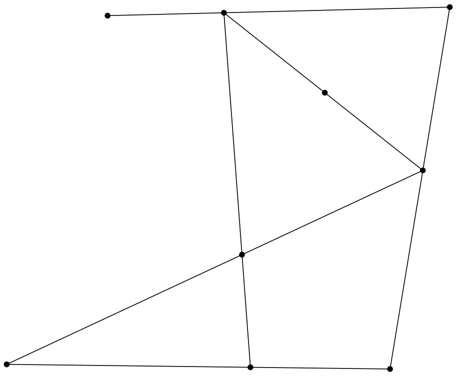

Fig. 4. A rod configuration with $3L = 2P$ that is rigid but not minimally rigid.

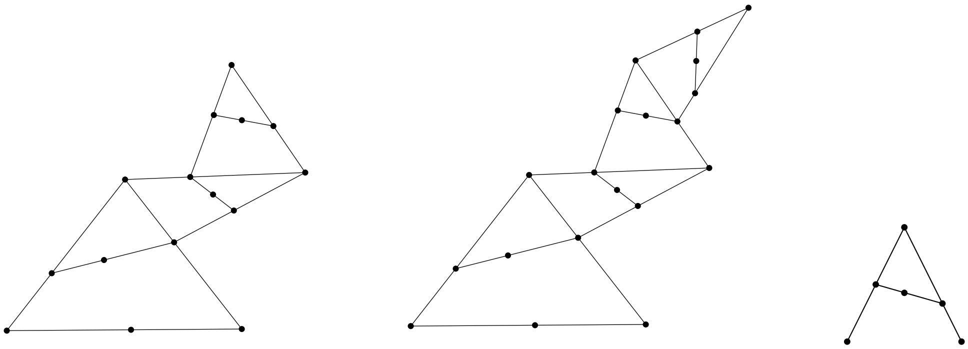

Fig. 5. a. A minimally rigid rod configuration satisfying $3|L| = 2|P|$. b. A minimally rigid rod configuration which satisfies $3|L| = 2|P| + 1$. c. The substructure by which the rod configuration in b. can be constructed from the rod configuration in a.

Example 3.8. Consider the rod configuration on the left in Fig. 5. In this rod configuration, $3|L| = 2|P|$. Yet it is easy to see that it is minimally infinitesimally rigid.

An infinite sequence of minimally infinitesimally rigid rod configurations can be constructed by extending the rod configuration in Fig. 5 a. by the structure in Fig. 5 c. The rightmost rod configuration in Fig. 5 b. is the next rod configuration in this sequence. Any rod configuration in this sequence has four more points and three more lines than the previous one.

Suppose that a rod configuration in this sequence satisfies $3|L| = 2|P| + k$. Then the next rod configuration in the sequence, realizing an incidence geometry $S' = (P', L', I')$, satisfies $3|L'| = 3|L| + 9 = 2|P| + k + 9 = 2|P'| + k + 1$, where the first equality holds as $|L'| = |L| + 3$, and the last equality holds since $|P'| = |P| + 4$. All rod configurations in this sequence are minimally infinitesimally rigid, so by induction there is a minimally infinitesimally rigid rod configuration that satisfies $3|L| = 2|P| + k$ for any $k \geq 0$.

Fig. 6 shows a minimally infinitesimally rigid rod configuration with 15 points and 9 lines. Recall that the leftmost rod configuration in Fig. 5 is minimally infinitesimally rigid with 15 points and 10 lines. Rigidity is a much more complex question for rod configurations than it is for graphs; any minimally rigid graph with a specified number of vertices will have the same number of edges, but it is not true that all minimally rigid rod configurations with some fixed number of points will have the same number of lines.

4. Flexible balanced rod configurations in the plane

In this section we focus on v_3-configurations, and give examples of v_3-configurations that are flexible in the plane.

4.1. Infinite families of flexible v_3-configurations with the motions of polygons

By a theorem of Steinitz, every combinatorial v_3-configuration can be realized as a rod configuration if one incidence is removed [10,25]. The two smallest v_3-configurations have 7 and 8 points, and they are known as the Fano plane and the Möbius–Kantor configuration, respectively. Neither of them can be realized as rod configurations. Fig. 7 shows a proper linear realization of the Möbius–Kantor configuration with one line removed.

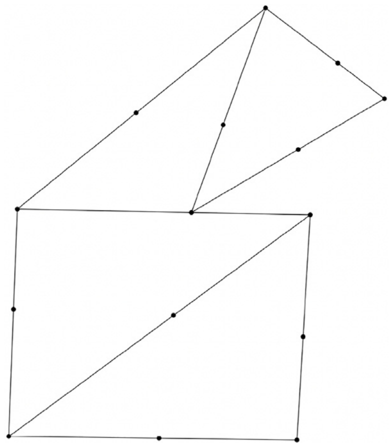

Fig. 6. A minimally infinitesimally rigid rod configuration with 9 lines and 15 points.

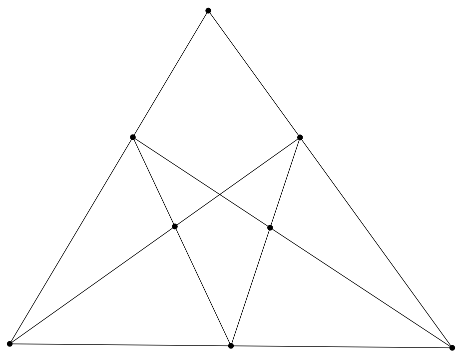

Fig. 7. A proper linear realization of the Möbius–Kantor configuration with a line removed.

For $v = 9$ there are 3 distinct combinatorial v_3-configurations, all of which have proper linear realizations. For $v = 10$ there are 10 distinct combinatorial v_3-configurations, one of which does not have a proper linear realization. For $v = 11$ and $v = 12$ all combinatorial v_3-configurations have proper linear realizations. It is however generally believed that, asymptotically in v, the combinatorial configurations that cannot be realized as rod configurations form a large portion of the set of all combinatorial configurations.

There are, however, not only combinatorial v_3-configurations that have proper linear realizations for all v large enough, but even flexible geometric v_3-configurations, as we prove in the following theorem.

Theorem 4.1. *There are flexible geometric v_3-configurations for all $v \geq 28$.*

Proof. Take $n \geq 2$ disjoint combinatorial v_3-configurations $C_0, \ldots, C_{n-1}$ (possibly $C_i \sim C_j$ with $i \neq j$) that do not have proper linear realizations.

Use Steinitz's Theorem [25] to find disjoint proper linear realizations of the n disjoint copies of combinatorial configurations with one incidence removed in each configuration. Each configuration C_i then contains a line ℓ_i with two incidences and a point p_i with two incidences. Move each configuration independently using the rigid motions of the plane so that the geometric realization of the line ℓ_i passes through the geometric realization of the point $p_{i+1 \pmod n}$.

Combinatorially, this is a variant of repeated use of the "incidence switch" [3,10] on the pairs $(C_i, C_{i+1 \pmod n})$, which constructs a connected combinatorial configuration from the n disjoint copies.

Applying this construction to a copies of the Fano plane and b copies of the Möbius–Kantor configuration gives us a geometric v_3-configuration for all parameters v of the form $v = 7a + 8b$, $a, b \in \mathbb{Z}_{\geq 0}$. This is a numerical semigroup generated by the two coprime natural numbers 7 and 8. The largest natural number not on the form $7a + 8b$ is 41 a.k.a. the Frobenius number of the numerical semigroup $\langle 7, 8 \rangle$). This bound can be improved by using configurations with other parameters. For example, the Frobenius number of $\langle 7, 8, 10 \rangle$ is 19, giving the bound $v \geq 20$. The rod configurations constructed in this way from n rod configurations, isomorphic to either the Fano plane or the Möbius–Kantor configuration, have the motions of the n-gon. Therefore they are flexible if $n \geq 4$ (with $n - 3$ degrees of freedom)

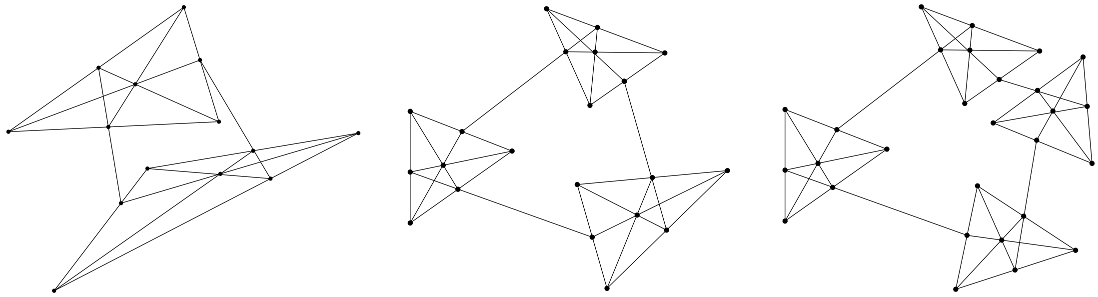

Fig. 8. The first three configurations in the infinite sequence of $(7n)_3$-configurations with the motions of the n-gon.

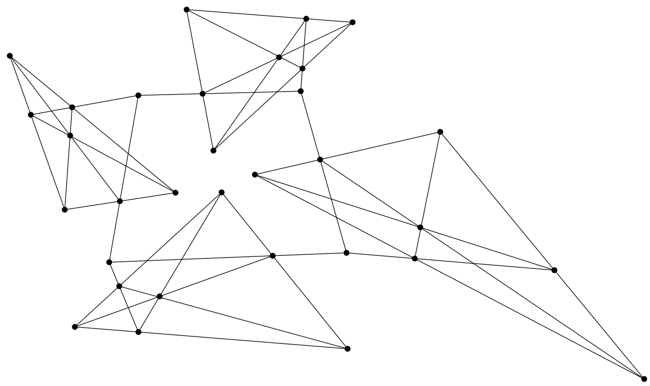

Fig. 9. Another flexible v_3-configuration with the motions of a square.

and rigid if $n \leq 3$. It can easily be checked that all integers larger than or equal to 42 are of the form $7a + 8b$ with $a+b \geq 4$. If we instead consider $\langle 7, 8, 10\rangle$ the bound becomes 28, but it is possible that this bound can be improved. □

For an illustration of the construction used in the proof of Theorem 4.1, see Fig. 8. It is possible that the bound $v \geq 28$ could be improved further, for example by also using geometrically realizable v_3-configurations with one incidence removed in the construction.

The infinite family of flexible rod configurations constructed in the proof of Theorem 4.1 can be modified slightly to give another infinite family of flexible rod v_3-configurations. This family is illustrated in Fig. 9, which shows a 32_3 point-line incidence geometry constructed from a quadrilateral by adding a copy of the Fano plane with one line removed to each of its edges.

4.2. Configurations that are flexible in special positions

The point coordinates of the configurations in Figs. 8 and 9 are not algebraically independent, so the configurations are not in generic position (according to the usual definition for graphs). The vertices of the square are in generic position, however, and more importantly, the motion will not disappear if the vertices of the square are placed elsewhere.

The 45_3-configuration in Fig. 10, on the contrary, features a motion that only exists because certain lines are parallel. It is constructed from six Fano planes, each with one line removed (the line that is realized as a circle in the common representation of the Fano plane in the real plane). These are joined together with two parallel grid structures that move independently. If the lines that are parallel in each of the two grid structures were not parallel, the configuration would not move.

As another example, we consider the Gray configuration, which is a 27_3-configuration that has a geometric realization with the points on a square lattice in dimension three. A planar point-line configuration is obtained by projecting it on a plane, see Fig. 11. The configuration in dimension three is flexible, as is its planar projection. Again, the flexibility is due to the parallel positions of the lines; if some of the lines were not in parallel position, the configuration would be rigid. It is easy to see that the generalized Gray configuration, obtained through the same construction but starting with a square lattice in dimension n, has the same property.

5. Conclusions and open problems

In this article, we have surveyed realizations of incidence geometries as rod configurations in the plane and their rigidity properties.

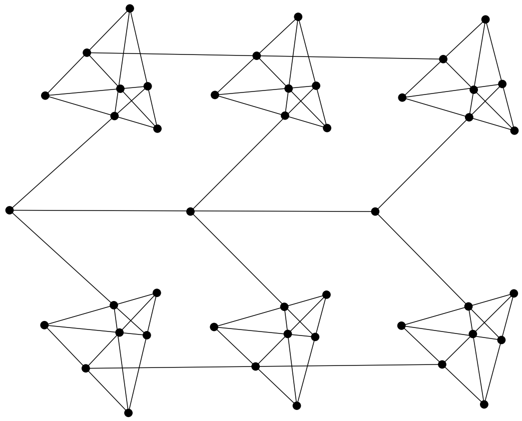

Fig. 10. A flexible 45_3-configuration with a motion in special position.

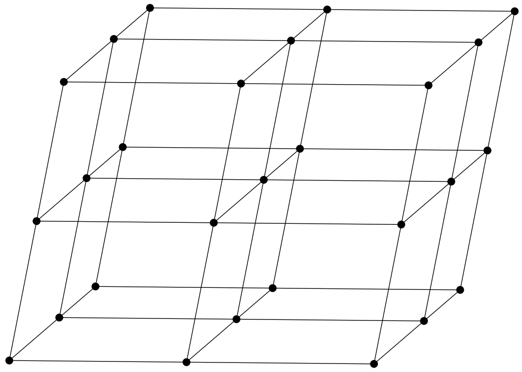

Fig. 11. A flexible realization of the Gray configuration.

Theorem 2.6 gives a combinatorial condition for an incidence geometry to have a realization as a minimally infinitesimally rigid rod configuration in the plane. By adding certain rods to configurations that satisfy that condition, we derived further combinatorial conditions that are sufficient for an incidence geometry to have a realization as a minimally infinitesimally rigid rod configuration in the plane. We have also provided examples that show that these conditions are not necessary; in fact, minimally infinitesimally rigid rod configurations can be arbitrarily far from satisfying the combinatorial conditions.

Further, we provide examples of flexible geometric v_3-configurations. There are infinitely many such v_3-configurations, and if v is sufficiently large, there is at least one flexible v_3-configuration.

We conclude with some open problems.

1. We know that any infinitesimally rigid graph $G = (V, E)$ has (at least one) spanning minimally infinitesimally rigid subgraph $G' = (V, E')$. All minimally infinitesimally rigid graphs on a given number of vertices have the same number of edges.
   Similarly, any infinitesimally rigid rod configuration has at least one spanning minimally infinitesimally rigid subconfiguration. As we have seen, not all minimally infinitesimally rigid subconfigurations on a given number of points have the same number of lines. It would be interesting to see if there are rigid k-uniform rod configurations with $k > 2$ with more than one spanning minimally infinitesimally rigid subconfiguration, in particular if two of them have a different number of lines.
2. In Section 3 we saw that the minimally rigid rod configurations that satisfy either set of conditions in Proposition 3.6 can be constructed from a rod configuration satisfying Corollary 3.1 by adding lines between two existing points and a point incident only to each of those lines. We have also seen that if we add more than two lines in this way, then we can no longer guarantee that the result is minimally infinitesimally rigid.
   However, it is possible to add more than two lines in this way to a given minimally infinitesimally rigid rod configuration and obtain a rod configuration that is minimally rigid. In fact, the minimally rigid rod configurations in the family constructed in Example 3.8 can be constructed by adding lines in this way to a rod configuration satisfying Corollary 3.1; the rod configuration satisfying $3|L| = 2|P| - 3 + k$ can be constructed by adding k lines.

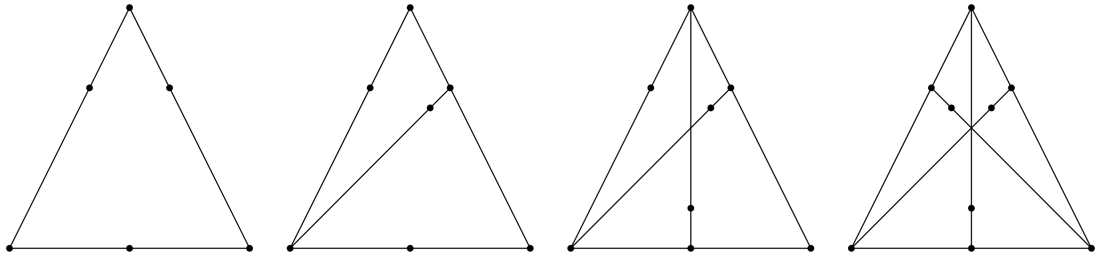

Fig. 12. Adding lines to a minimally rigid rod configuration sometimes preserves minimal rigidity.

If we start from a given minimally rigid 3-uniform rod configuration satisfying Corollary 3.1, then it seems that it is only possible to add a finite number of lines in this way, while still preserving minimal rigidity. For example, it seems that it is possible to add at most three lines to the leftmost rod configuration in Fig. 12, before the rod configuration stops being minimally rigid. It would be interesting to find criteria, perhaps in terms of local density of incidences, for when this construction creates a new minimally rigid 3-uniform rod configuration from an old one.

3. We have only started to explore the topic of inductive constructions of minimally rigid 3-uniform rod configurations. In general, it would be very interesting to give a set of inductive constructions capable of producing all minimally rigid 3-uniform rod configurations.
4. In Section 3, we saw that there are infinitesimally rigid geometric v_3-configurations, and in Section 4 we saw that there are infinitesimally flexible v_3-configurations. However, we still do not have any examples of minimally infinitesimally rigid geometric v_3-configurations. It would be interesting to find an example of a minimally rigid v_3-configuration.
5. The flexible v_3-configurations that we construct in Section 4 have the same motions as graph frameworks of cycles. Can we construct flexible v_3-configurations with the same motions as other graph frameworks? Is it possible to construct a v_k-configuration with the same motions as any given flexible graph framework?
6. Determining conditions on incidence geometries that have realizations as globally rigid rod configurations in the plane is a possible line for further research.
Note that some of the continuously rigid v_3 configurations we have presented, such as the two joined Fano planes, and the triangle with a reduced Fano plane on each side (left and center in Fig. 8), are globally flexible. The triangle with a reduced Fano plane on each side admits a reflection of one of the reduced Fano planes in the line it is attached to. In contrast, for example, the Pappus configuration (Fig. 2) is globally rigid.

Data availability

No data was used for the research described in the article.

Acknowledgments

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References

[1] L. Asimow, B. Roth, The rigidity of graphs, Trans. Amer. Math. Soc. 245 (1978) 279–289.
[2] Alex R. Berg, Tibor Jordán, Algorithms for graph rigidity and scene analysis, in: Algorithms—ESA 2003, in: Lecture Notes in Comput. Sci., vol. 2832, Springer, Berlin, 2003, pp. 78–89.
[3] Maria Bras-Amorós, Klara Stokes, The semigroup of combinatorial configurations, Semigroup Forum 84 (1) (2012) 91–96.
[4] Robert Connelly, Generic global rigidity, Discrete Comput. Geom. 33 (4) (2005) 549–563.
[5] Henry Crapo, Concurrence geometries, Adv. Math. 54 (3) (1984) 278–301.
[6] Henry Crapo, The combinatorial theory of structures, in: Matroid Theory (Szeged, 1982), in: Colloq. Math. Soc. János Bolyai, vol. 40, North-Holland, Amsterdam, 1985, pp. 107–213.
[7] Henry Crapo, Walter Whiteley, Autocontraintes planes et polyèdres projetés. I. Le motif de base, Struct. Topol. (20) (1993) 55–78, Dual French-English text.
[8] Yaser Eftekhari, Bill Jackson, Anthony Nixon, Bernd Schulze, Shin-ichi Tanigawa, Walter Whiteley, Point-hyperplane frameworks, slider joints, and rigidity preserving transformations, J. Combin. Theory Ser. B 135 (2019) 44–74.
[9] Herman Gluck, Almost all simply connected closed surfaces are rigid, in: Geometric Topology (Proc. Conf., Park City, Utah, 1974), in: Lecture Notes in Math., vol. 438, Springer, Berlin, 1975, pp. 225–239.
[10] Branko Grünbaum, Configurations of Points and Lines, in: Graduate Studies in Mathematics, vol. 103, American Mathematical Society, Providence, RI, 2009, p. xiv+399.

[11] Bruce Hendrickson, Conditions for unique graph realizations, SIAM J. Comput. 21 (1) (1992) 65–84.
[12] Milagros Izquierdo, Klara Stokes, Isometric point-circle configurations of surfaces from uniform maps, in: Symmetries in Graphs, Maps, and Polytopes, in: Springer Proc. Math. Stat., vol. 159, Springer, Cham, 2016, pp. 201–211.
[13] Bill Jackson, Tibor Jordán, Connected rigidity matroids and unique realizations of graphs, J. Combin. Theory Ser. B 94 (1) (2005) 1–29.
[14] Bill Jackson, Tibor Jordán, Rigid two-dimensional frameworks with three collinear points, Graphs Combin. 21 (4) (2005) 427–444.
[15] Bill Jackson, Tibor Jordán, Pin-collinear body-and-pin frameworks and the molecular conjecture, Discrete Comput. Geom. 40 (2) (2008) 258–278.
[16] Bill Jackson, Tibor Jordán, The generic rank of body-bar-and-hinge frameworks, European J. Combin. 31 (2) (2010) 574–588.
[17] Donald J. Jacobs, Bruce Hendrickson, An algorithm for two-dimensional rigidity percolation: the pebble game, J. Comput. Phys. 137 (2) (1997) 346–365.
[18] Naoki Katoh, Shin-ichi Tanigawa, A proof of the molecular conjecture, Discrete Comput. Geom. 45 (4) (2011) 647–700.
[19] G. Laman, On graphs and rigidity of plane skeletal structures, J. Engrg. Math. 4 (1970) 331–340.
[20] László Lovász, Graphs and Geometry, in: American Mathematical Society Colloquium Publications, vol. 65, American Mathematical Society, Providence, RI, 2019, p. x+444.
[21] Anthony Nixon, Bernd Schulze, Walter Whiteley, Rigidity through a projective lens, Appl. Sci. 11 (24) (2021).
[22] Tomaž Pisanski, Brigitte Servatius, Configurations from a Graphical Viewpoint, in: Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer, New York, 2013, p. xiv+279.
[23] Hilda Pollaczek-Geiringer, Über die gliederung ebener fachwerke, Zamm-Z. Angew. Math. Mech. 7, 58–72.
[24] Brigitte Servatius, Walter Whiteley, Constraining plane configurations in computer-aided design: combinatorics of directions and lengths, SIAM J. Discrete Math. 12 (1) (1999) 136–153.
[25] E. Steinitz, Über die construction der configurationen n_3, in: Dissertationen, R. Galle, 1894.
[26] Tiong-Seng Tay, Rigidity of multigraphs. I. Linking rigid bodies in n-space, J. Combin. Theory Ser. B 36 (1) (1984) 95–112.
[27] Tiong-Seng Tay, Linking $(n-2)$-dimensional panels in n-space. II. $(n-2,2)$-frameworks and body and hinge structures, Graphs Combin. 5 (3) (1989) 245–273.
[28] Tiong-Seng Tay, Walter Whiteley, Recent advances in the generic rigidity of structures, Struct. Topol. (9) (1984) 31–38, Dual French-English text.
[29] Walter Whiteley, Parallel redrawings, 1987, unpublished manuscript.
[30] Walter Whiteley, The union of matroids and the rigidity of frameworks, SIAM J. Discrete Math. 1 (2) (1988) 237–255.
[31] Walter Whiteley, A matroid on hypergraphs, with applications in scene analysis and geometry, Discrete Comput. Geom. 4 (1) (1989) 75–95.
[32] Walter Whiteley, Some matroids from discrete applied geometry, in: Matroid Theory (Seattle, WA, 1995), in: Contemp. Math., vol. 197, Amer. Math. Soc., Providence, RI, 1996, pp. 171–311.