

On a generalised Lambert W branch transition function arising from p, q -binomial coefficients

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ARTICLE INFO

MSC:

primary 33B99, 33F05, 82B20

secondary 05A10, 05A30

Keywords:

Generalization of Lambert W function

Lenz-Ising model

Magnetization distribution

p, q -binomial coefficients

Special functions

ABSTRACT

With only a complete solution in dimension one and partially solved in dimension two, the Lenz-Ising model of magnetism is one of the most studied models in theoretical physics. An approach to solving this model in the high-dimensional case ($d > 4$) is by modelling the magnetisation distribution with p, q -binomial coefficients. The connection between the parameters p, q and the distribution peaks is obtained with a transition function ω which generalises the mapping of Lambert W function branches W_0 and W_{-1} to each other. We give explicit formulas for the branches for special cases. Furthermore, we find derivatives, integrals, parametrizations, series expansions, and asymptotic behaviours.

1. Introduction and background

We will study the transition function ω that arises when using p, q -binomial coefficients to attack the model of magnetism in statistical mechanics for higher dimensions. To get an overall view of our study, we start in Section 1.1 with a physical background on the Lenz-Ising model. In Section 1.2 we give an introduction to p, q -binomial coefficients, and how it relates to the given physical problem. Section 1.2 will also reveal how the transition function arises and its relation to the Lambert W function. Then, in Section 1.3, we introduce the functions ψ and ω which are the objects under study for the rest of this article.

In Section 2 we will state some properties of ψ and ω that can be obtained from elementary techniques, for example a basic parametrization of one branch of ψ and simple explicit formulas for both ψ and ω in a special case. After this, in Section 3, we give derivatives and the primitive function of ψ (Propositions 3.1 and 3.3), including a definite integral. The subject of explicit formulas of ψ and ω is continued in Section 4 where we use more powerful methods than in Section 2 to solve for some other special cases. In Section 5 we state Theorem 5.1 which gives a parametrization of both branches of ψ . This also allows us to give a definite integral of ω stated in Theorem 5.4. In Sections 6 and 7 we give Taylor series expansions for ψ around some special points by using Lagrange's Inversion Theorem and the elementary Proposition 7.1. Finally, in the remaining Sections 8 and 9, we use our results to give asymptotic formulas for both branches of ψ in (8.3) and (9.4).

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<https://doi.org/10.1016/j.amc.2023.128347>

Received 9 May 2023; Received in revised form 21 July 2023; Accepted 14 September 2023

Available online 27 September 2023

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1.1. Physical origins

The Lenz-Ising model, usually referred to as the Ising model, was introduced by Wilhelm Lenz [16] in 1920 as a simple model for magnetism. The time was ripe for such a model after the discoveries by Pierre Curie¹ that magnetic materials (iron, cobalt, nickel, etc.) undergo a phase transition at a critical temperature T_c , the Curie temperature, above which they lose their permanent magnetic properties. In the Ising model without an external field, the material is described as a system of interacting particles, each having spin ± 1 , and is governed by the temperature [7,11,20]. The system in question can be any finite graph, with the vertices corresponding to the particles and the edges indicating which particles interact. In the model’s most famous version, the system is the (infinite) d -dimensional integer lattice. The 1-dimensional model (an infinite chain of particles) was solved in 1925 by Lenz’s student, Ernst Ising, in his thesis [14]. Unfortunately, the result was disappointing since he showed that this system does not have a phase transition at any positive temperature, the critical temperature being $T_c = 0$.

It took until 1944 when the Norwegian chemist Lars Onsager² solved the model for 2-dimensional systems [21]. This solution was considered a major breakthrough, rendering a positive critical temperature, $T_c = 2/\log(1 + \sqrt{2})$ (assuming unit interaction between nearest-neighbour particles in the lattice). Unfortunately, again, the techniques used by Onsager did not point to a solution for the 3-dimensional model. In fact, to this day, very little is known rigorously about the 3-dimensional model, and the 2-dimensional model with an external field is still unsolved. For dimensions $d \geq 4$ the critical exponents, which govern the behaviour of various quantities near the critical point, are known exactly but the critical temperature is still not known exactly for any $d > 2$.

This has not made the Ising model any less attractive, instead, it has generated a staggering number of papers studying many variants of the model in different dimensions [23,5,22,12]. Also it has become a testing bed for various Monte Carlo simulation algorithms [26,25,13].

Let us focus a little closer on an interesting aspect of the Ising model without an external field, namely its magnetization distribution. The magnetization M of the Ising model is the sum of all the spins $S_i = \pm 1$ in the system, $M = \sum_i S_i$, so that for a system with n spins we have $-n \leq M \leq n$. For a finite d -dimensional system (we assume $d \geq 2$), at any given temperature T , the magnetization M obeys a symmetric distribution with peaks at $\pm an$, for some $0 \leq a \leq 1$. Now, for temperatures above the Curie temperature ($T > T_c$), the distribution of magnetizations is unimodal with its peak at $M = 0$ so that $a = 0$. When we lower the temperature below the Curie temperature ($T < T_c$) the distribution becomes bimodal with peaks at $\pm an$ for $0 < a \leq 1$, with $a \rightarrow 1$ as $T \rightarrow 0$. The parameter a here corresponds to the so-called spontaneous magnetization of the Ising model, but its relation to T is only known exactly for 2-dimensional (infinite) systems. It was conjectured by Onsager in 1948 and proved by Yang in 1952 (see [19] for details) that in this case

$$a(T) = \begin{cases} (1 - \sinh(2/T)^{-4})^{1/8} & \text{if } T < T_c \\ 0 & \text{if } T \geq T_c \end{cases}$$

where $T_c = 2/\log(1 + \sqrt{2})$. Thus a phase transition occurs at $T = T_c$ when a becomes positive. In physical terms, the unimodal distribution corresponds to losing the magnetic properties, whereas the bimodal distribution corresponds to retaining them. Finding a corresponding formula for dimensions $d \geq 3$ would be a major breakthrough.

1.2. The p, q -binomial coefficients

It was suggested in [17] that the magnetization distribution is well described by p, q -binomial coefficients for finite high-dimensional systems ($d \geq 5$). In fact, in a special case (mean-field), they are equivalent, and for $d \geq 5$ they have the same asymptotic shape when $n \rightarrow \infty$. In principle, one could then model the magnetization distribution for a finite system at temperature T with a p, q -binomial distribution where p and q depend on T according to some functions $p(T)$ and $q(T)$. It would be a more realistic project to find these functions for temperatures very close to T_c and one such attempt was made in [18].

However, it should be mentioned that the p, q -binomial coefficients have received attention also for other interesting purely mathematical properties [2,3,9].

Let us here provide some more detail on these coefficients. The p, q -binomial coefficient [8] is defined for $p \neq q$ as

$$\binom{n}{k}_{p,q} = \prod_{j=1}^k \frac{p^{n-k+j} - q^{n-k+j}}{p^j - q^j}, \quad 0 \leq k \leq n$$

from which a symmetric p, q -binomial distribution [17,18] is defined to have a probability mass function proportional to the sequence of these coefficients. Note that the sum of the p, q -binomial coefficients do not seem to have a simple expression, as opposed to standard binomial coefficients for which the sum is simply 2^n . For $p, q > 0$ it has been shown [24] that the coefficients either form a unimodal sequence with maximum at $k = \lfloor n/2 \rfloor$, or, a bimodal sequence with maxima at k and $n - k$ for some $0 \leq k \leq n/2$. We let the parameter a control the location of the sequence maximum by defining

$$k = \frac{n}{2}(1 - a), \quad 0 \leq a \leq 1.$$

¹ In his thesis 1895. He and his wife Marie later shared the 1903 Nobel prize in physics with Becquerel for their work on radioactivity.

² Nobel prize in chemistry 1968 for his work on the thermodynamics of irreversible processes.

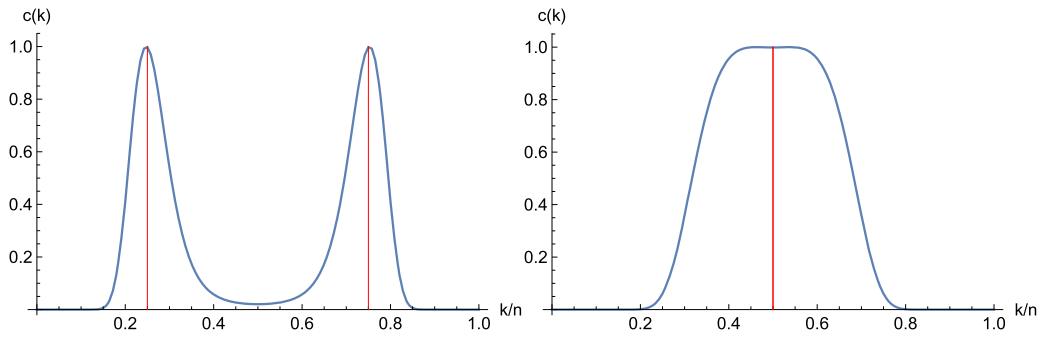


Fig. 1. Plots of $c(k) = \frac{1}{C} \binom{n}{k}_{p,q}$ vs k/n for $n = 2^{10}$, $z = -5$ and $y = \omega(a, z)$, where C is a normalizing factor. Red vertical lines at $k/n = (1 \pm a)/2$. Panels show the cases $a = 1/2$ with $y = -0.0891004$ (left) and $a = 0$ with $y = -0.0348858$ (right). As $n \rightarrow \infty$ the peak locations will approach the position of the vertical lines. Also, the coefficient mass will become more concentrated around these lines.

Having two consecutive p, q -binomial coefficients, indexed $k - 1$ and k , being equal leads, after simplification, to the equation

$$\frac{p^{n-k+1} - p^k}{q^{n-k+1} - q^k} = 1, \quad 1 \leq k \leq \lfloor n/2 \rfloor. \tag{1.1}$$

We here need to introduce the p, q -parameterizations

$$p = 1 + \frac{2y}{n}$$

$$q = 1 + \frac{2z}{n}, \quad z < 0$$

though other parameterizations are also of interest. With a and z fixed we now solve for y .

1.3. Introducing the functions ω and ψ

First the special case $a = 0$. The asymptotic form of the ratio in (1.1) now becomes simply

$$\frac{ye^y}{ze^z} + \mathcal{O}(1/n) = 1$$

and to receive a leading term of 1 we must have $ze^z = ye^y$. This defines a special case of the transition function ω :

$$y = \omega(0, z) = \begin{cases} W_0(ze^z) & \text{if } z < -1 \\ W_{-1}(ze^z) & \text{if } -1 \leq z < 0 \end{cases}. \tag{1.2}$$

Here W denotes the famous Lambert W function, which returns one of two real solutions $w = W_i(x)$ of the equation $x = we^w$. The principal solution, $w = W_0(x)$ defined for $x \geq -1/e$, gives $w \geq -1$ and the other branch, $w = W_{-1}(x)$ defined for $-1/e \leq x < 0$, gives $w < -1$. The transition function $\omega(0, z)$ thus maps solutions between the two branches of the Lambert W function. The coefficient sequence then will have its two maxima at points $k = n/2 \pm \ell$ where $\ell = o(n)$. In Fig. 1 the right panel shows an example of this case and the left panel shows the case $a = 1/2$.

We will now generalise the transition function of W to the case of $0 < a < 1$. The ratio of (1.1) now becomes

$$\frac{\sinh(ay)e^y}{\sinh(az)e^z} + \mathcal{O}(1/n) = 1$$

so to receive the correct leading term of 1 we must now solve

$$\sinh(ay)e^y = \sinh(az)e^z. \tag{1.3}$$

We therefore define the function

$$f(a, w) = \sinh(aw)e^w$$

and introduce its inverse function $w = \psi_i(a, x)$ as one of the two real solutions of

$$x = f(a, w), \quad 0 < a < 1.$$

We note that $f(a, w)$ takes its minimum value L_a at $w = M_a$, where

$$L_a = \frac{-a}{\sqrt{1-a^2}} \left(\frac{1-a}{1+a} \right)^{\frac{1}{2a}},$$

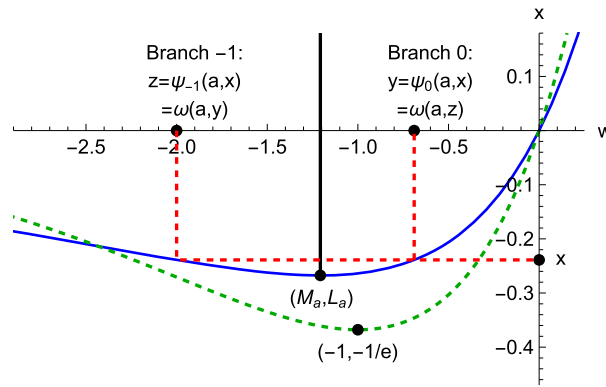


Fig. 2. Plots of $w e^{aw}$ (dashed green) and $\sinh(aw)e^{aw}$ (solid blue) for $a = 2/3$. A point z in one branch is mapped to a point $y = \omega(a, z)$ in the other branch when $z < 0$, the two branches separated by $x = M_a$. Note $x = \sinh(ay)e^y = \sinh(az)e^z$ where $L_a \leq x < 0$.

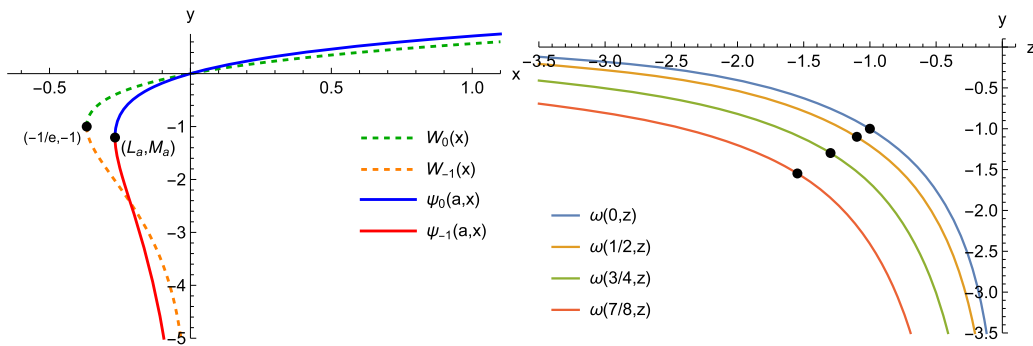


Fig. 3. Left: Plots of branch 0 (above the points) and branch -1 (below the points) of $y = W(x)$ (dashed, green and orange) and $y = \psi(2/3, x)$ (solid, blue and red). Right: Plots of $\omega(a, z)$ for $a = 0, 1/2, 3/4, 7/8$ (downwards). Points at (M_a, M_a) .

$$M_a = \frac{1}{2a} \log \left(\frac{1-a}{1+a} \right).$$

The principal branch $\psi_0(a, x) \geq M_a$ is now defined for $x \geq L_a$, while the other branch $\psi_{-1}(a, x) \leq M_a$ is defined for $L_a \leq x < 0$. Thus M_a is the branch separator with $f(a, M_a) = L_a$ and $\psi_0(a, L_a) = \psi_{-1}(a, L_a) = M_a$. In Fig. 2 we show how the equation $f(a, w) = x$ relates to the definition ψ and ω .

The transition function ω , defined above only for $a = 0$ in (1.2), can now be generalized to $0 < a < 1$

$$y = \omega(a, z) = \begin{cases} \psi_0(a, f(a, z)) & \text{if } z < M_a \\ \psi_{-1}(a, f(a, z)) & \text{if } M_a \leq z < 0 \end{cases} \tag{1.4}$$

Plots of ψ and ω are shown in Fig. 3.

Note that $\lim_{a \rightarrow 0^+} \omega(a, z) = \omega(0, z)$ so that ω for $0 < a < 1$ is indeed a natural and continuous generalisation of the transition function for Lambert W of (1.2). However, the function ψ is only related to the Lambert W for small a . In that case we have $f(a, w) \sim aw e^{aw}$ so that

$$\psi(a, x) \sim W(x/a), \quad x > L_a$$

for both branches, where the functions are defined. Let us also briefly state the following properties of ω .

- (1) $z = M_a$ is a fixed point, i.e., $\omega(a, M_a) = M_a$.
- (2) ω is an involution, i.e., $\omega(a, \omega(a, z)) = z$ for all $z < 0$.
- (3) $\omega(a, z)$ is negative, decreasing and concave for all $z < 0$.
- (4) $\lim_{z \rightarrow -\infty} \omega(a, z) = 0$ and $\lim_{z \rightarrow 0^-} \omega(a, z) = -\infty$.

The transition function ω was used in [17], though in a slightly different guise, and very little was mentioned with regard to its mathematical properties. This article aims to remedy this by first investigating the properties of ψ and apply this to ω .

To conclude this section, the p, q -binomial coefficient sequence will then have its peaks at $k = n/2(1 \pm a) \pm \ell$ where $\ell = o(n)$ and the parameter z controls the mass shape of the sequence, see Fig. 1. Note that the peaks are never at exactly $k = n/2(1 \pm a)$. To obtain this the resulting y would also depend on n and a special function $\bar{\omega}(n, a, z)$ is needed. We would then have $\bar{\omega}(n, a, z) \rightarrow \omega(a, z)$ when $n \rightarrow \infty$.

2. Results on the ψ - and ω -functions from elementary techniques

In this section we will show some basic properties of the ψ and ω -function obtained only from elementary techniques. Also we solve some interesting special cases. Later in the article we will use more powerful methods and extend these results considerably.

2.1. The boundary cases $a = 0$ and $a = 1$ and a lower bound for ψ_0

First we recall that

$$f(a, w) = \sinh(aw)e^w = \frac{1}{2}(e^{w(1+a)} - e^{w(1-a)}), \quad 0 \leq a \leq 1, w \in \mathbb{R}. \tag{2.1}$$

The limits for $a \rightarrow 0^+$ and $a \rightarrow 1^-$ when w is fixed are then

$$f(0, w) = \lim_{a \rightarrow 0^+} f(a, w) = 0,$$

$$f(1, w) = \lim_{a \rightarrow 1^-} f(a, w) = \frac{1}{2}(e^{2w} - 1).$$

For fixed w we also have:

- (1) If w is positive then $f(a, w)$ is increasing for $0 < a < 1$.
- (2) If w is negative then $f(a, w)$ is decreasing for $0 < a < 1$.

Now we can define the limit forms of ψ . Solving for w in $f(1, w) = x$ gives the limit inverse function

$$\psi_0(1, x) = \lim_{a \rightarrow 1^-} \psi_0(a, x) = \frac{1}{2} \log(1 + 2x), \quad x > -\frac{1}{2},$$

thanks to uniform convergence. In fact, since $f(a, w)$ is increasing with respect to a then $\psi_0(a, x)$ is decreasing with a so that we receive the inequality

$$\psi_0(a, x) \geq \frac{1}{2} \log(1 + 2x), \quad 0 < a < 1, x > -\frac{1}{2}.$$

In Theorem 8.1 below we will give a much sharper bound, but requiring considerably more work. Since $f(1, w) > -1/2$ is strictly increasing for all w , thus having no second branch, we obtain no limit $\psi_{-1}(1, x)$. Also, since $f(0, w) = 0$ we have no well-defined inverse function $\psi_{-1}(0, x)$.

2.2. A parametrization of ψ_0 for general a

In Section 4 we will use a powerful technique which gives a parametrization of both branches ψ_0 and ψ_{-1} for $L_a \leq x < 0$. However, here we only require an elementary technique which allows us to develop a parametrization of $\psi_0(a, x)$ for $x > 0$. First we define

$$u(x) = \frac{1}{2}x(x^a - x^{-a}) = \frac{1}{2}(x^{1+a} - x^{1-a}), \quad x > 0.$$

Then $u \circ \exp(x) = \sinh(ax)e^x = f(a, x)$, and therefore $\log \circ u^{-1}(x) = \psi(a, x)$. Hence, $e^{\psi(a, x)} = u^{-1}(x)$ so that $e^{\psi(a, u(x))} = x$. Finally, we now require $x > 0$, thus giving the branch ψ_0 , so that we can obtain

$$\psi_0\left(a, \frac{1}{2}(x^{1+a} - x^{1-a})\right) = \log x,$$

which gives the sought-after parametrization, defined for $\beta > 1$:

$$x = \frac{1}{2}(\beta^{1+a} - \beta^{1-a}),$$

$$\psi_0(a, x) = \log \beta.$$

Example 2.1. Using the parametrization we set $a = 1/2$ which then gives us $x = (1/2)\sqrt{\beta}(\beta - 1)$ and then we have $\psi_0(1/2, x) = \log \beta$. Choosing square integers for β , say $\beta = 4, 9, 16$, we then obtain $x = 3, 12, 30$ so that $\psi_0(1/2, 3) = \log 4$, $\psi_0(1/2, 12) = \log 9$, $\psi_0(1/2, 30) = \log 16$. \square

2.3. Special values of f and ψ_{-1} for general a

The ψ_{-1} -branch is more difficult to parametrize for general a . However, it is easy to give an infinite sequence of points (x_n, y_n) satisfying $\psi_{-1}(a, x_n) = y_n$ and we will do so here.

Note that it is an easy exercise to show that the n th derivative of f vanishes at n th multiples of M_a , so that

$$\frac{\partial^n}{\partial w^n} f(a, w) = 0 \text{ for } w = n M_a, \quad n = 1, 2, \dots \tag{2.2}$$

If we define

$$I_a(n) = \frac{1}{2} \left(\frac{1-a}{1+a} \right)^{\frac{n(1-a)}{2a}} \left(\left(\frac{1-a}{1+a} \right)^n - 1 \right) \tag{2.3}$$

so that $I_a(1) = L_a$ and $I_a(2) = -2L_a^2/a$ is the inflection point of ψ_{-1} etc, then

$$f(a, nM_a) = I_a(n). \tag{2.4}$$

Of course, since $w = nM_a \leq M_a$ for $n \geq 1$, this only gives the solutions in branch -1 of the equation $f(a, w) = I_a(n)$, thus leading to

$$\psi_{-1}(a, I_a(n)) = n M_a, \quad n = 1, 2, \dots \tag{2.5}$$

There is also a corresponding solution for $\psi_0(a, I_a(n))$ but this value does not seem to have a simple expression. However, for special values of a we can find closed form expressions for ψ but we will return to this in Section 4.

2.4. The special case $a = 1/3$: explicit formulas of ψ and ω

In general it is difficult to find explicit formulas for ψ and ω . However, for some special cases of a we will be successful and the case $a = 1/3$ turns out to be surprisingly easy to handle. In Section 4 we will give formulas for ψ and ω at other rational values of a but they are considerably more complicated since they are based on the roots of the quartic polynomials. The formulas given here for ψ and ω seem to be the only simple exact solutions.

To compute $\psi(a, x)$ we need to solve for w in the equation $f(a, w) = x$. When $a = 1/3$ we then receive

$$f\left(\frac{1}{3}, w\right) = \frac{1}{2} \left(e^{\frac{4w}{3}} - e^{\frac{2w}{3}} \right) = \frac{1}{2} (Y^2 - Y) = x \tag{2.6}$$

where

$$Y = e^{\frac{2w}{3}}$$

Solving the second degree polynomial gives the roots

$$Y = \frac{1}{2} \pm \sqrt{2x + \frac{1}{4}} \tag{2.7}$$

where the two solutions correspond to the two branches so that

$$\psi_0\left(\frac{1}{3}, x\right) = \frac{3}{2} \log \left(\frac{1}{2} + \sqrt{2x + \frac{1}{4}} \right), \quad -\frac{1}{8} \leq x \tag{2.8}$$

$$\psi_{-1}\left(\frac{1}{3}, x\right) = \frac{3}{2} \log \left(\frac{1}{2} - \sqrt{2x + \frac{1}{4}} \right), \quad -\frac{1}{8} \leq x < 0. \tag{2.9}$$

We now combine this with the definition of ω in (1.4). The case $w < M_{1/3} = -\frac{3}{2} \log 2$ corresponds to $0 < Y < 1/2$ and we receive

$$\begin{aligned} \omega\left(\frac{1}{3}, w\right) &= \psi_0\left(\frac{1}{3}, f\left(\frac{1}{3}, w\right)\right) = \frac{3}{2} \log \left(\frac{1}{2} + \sqrt{Y^2 - Y + \frac{1}{4}} \right) \\ &= \frac{3}{2} \log \left(\frac{1}{2} + \left| Y - \frac{1}{2} \right| \right) = \frac{3}{2} \log(1 - Y) = \frac{3}{2} \log \left(1 - e^{\frac{2w}{3}} \right). \end{aligned}$$

The other case $-\frac{3}{2} \log 2 \leq w < 0$ gives $1/2 \leq Y < 1$ so that

$$\begin{aligned} \omega\left(\frac{1}{3}, w\right) &= \psi_{-1}\left(\frac{1}{3}, f\left(\frac{1}{3}, w\right)\right) = \frac{3}{2} \log \left(\frac{1}{2} - \sqrt{Y^2 - Y + \frac{1}{4}} \right) \\ &= \frac{3}{2} \log \left(\frac{1}{2} - \left| Y - \frac{1}{2} \right| \right) = \frac{3}{2} \log(1 - Y) = \frac{3}{2} \log \left(1 - e^{\frac{2w}{3}} \right). \end{aligned}$$

We thus receive the same formula in both cases so we conclude that

$$\omega\left(\frac{1}{3}, z\right) = \frac{3}{2} \log\left(1 - e^{\frac{2z}{3}}\right), \quad z < 0. \tag{2.10}$$

It is now easy to give, for example, the definite integral of ω ,

$$\int_{-\infty}^0 \omega\left(\frac{1}{3}, z\right) dz = -\frac{3\pi^2}{8}. \tag{2.11}$$

but in Section 5 we will give the general formula for a . From (2.8) and (2.9) we can now easily obtain series expansions

$$\begin{aligned} \psi_0\left(\frac{1}{3}, x\right) &\sim \frac{3}{4} \log(2x) + \frac{3}{4\sqrt{2x}}, \quad x \rightarrow \infty \\ \psi_{-1}\left(\frac{1}{3}, x\right) &\sim \frac{3}{2} \log(-2x) - 3x + 9x^2, \quad x \rightarrow 0^- \end{aligned}$$

but we will state general asymptotes in Sections 8 and 9.

2.5. $\psi(a, x)$ is transcendental number

We shall prove that if $x \neq 0$ is algebraic number ($x \in \mathcal{A}$), then $\psi(a, x)$ is transcendental number ($\psi(a, x) \in \mathcal{T}$). To prove it we need Lindemann-Weierstrass Theorem ([4, Theorem 1.4]), i.e., if $a_1, \dots, a_n \in \mathcal{A}$ and $\alpha_1, \dots, \alpha_n \in \mathcal{A}$ are distinct numbers, then the equation

$$a_1 e^{\alpha_1} + \dots + a_n e^{\alpha_n} = 0$$

has only the trivial solution $a_1 = \dots = a_n = 0$.

Proposition 2.2. *If $a, x \in \mathcal{A}$, $x \neq 0$, then $\psi(a, x) \in \mathcal{T}$.*

Proof. Assume that $a, x \in \mathcal{A}$, $x \neq 0$ and $\psi(a, x) \in \mathcal{A}$. Then $1 + a, 1 - a, 2x \in \mathcal{A}$ and therefore $(1 + a)\psi(a, x), (1 - a)\psi(a, x) \in \mathcal{A}$. By the definition of ψ we have

$$e^{(1+a)\psi(a,x)} - e^{(1-a)\psi(a,x)} - 2xe^0 = 0,$$

which is impossible by the Lindemann-Weierstrass Theorem. \square

3. Derivatives and a primitive function of ψ

In Proposition 3.1, we provide a formula for the n th derivative of ψ with respect to x . Here, a shorter notation $\psi(x) = \psi(a, x)$, is used referring to both branches, and a is treated as a fixed parameter. A primitive function of ψ is obtained later in Proposition 3.2.

Proposition 3.1. *For $n \in \mathbb{N}$*

$$\psi^{(n)}(x) = \frac{P_n(\cosh(a\psi(x)), \sinh(a\psi(x)))e^{-n\psi(x)}}{(a \cosh(a\psi(x)) + \sinh(a\psi(x)))^{2n-1}}, \tag{3.1}$$

where the polynomials P_n are given by the following recurrence formula: $P_1(x, y) = 1$ and

$$P_{n+1}(x, y) = P_n(x, y)((a - 3na)x + (a^2 - n - 2na^2)y) + \frac{\partial P_n}{\partial x}(x, y)(a^2xy + ay^2) + \frac{\partial P_n}{\partial y}(x, y)(axy + a^2x^2).$$

Proof. We proceed by induction. We shall use the shorter notation $\psi(x) = \psi(a, x)$ when referring to both branches. The case $n = 1$, can be deduced by using the implicit function theorem:

$$\psi'(x) = \frac{e^{-\psi(x)}}{(a \cosh(a\psi(x)) + \sinh(a\psi(x)))} = \frac{1}{x(a \coth(a\psi(x)) + 1)}$$

with the special value

$$\psi'_0(0) = \frac{1}{a}.$$

Assume next that (3.1) is valid for some natural number n . Let us define

$$\begin{aligned} g(x) &= a \cosh(a\psi(x)) + \sinh(a\psi(x)) \\ X &= \cosh(a\psi(x)) \\ Y &= \sinh(a\psi(x)) \end{aligned}$$

We then have

$$\begin{aligned} \psi^{(n+1)} &= \frac{e^{-n\psi} \psi' \left(-nP_n + \frac{\partial P_n}{\partial x} a \sinh(a\psi) + \frac{\partial P_n}{\partial y} a \cosh(a\psi) \right) g^{2n-1}}{g^{4n-2}} - \frac{e^{-n\psi} \psi' P_n (a^2 \sinh(a\psi) + a \cosh(a\psi)) (2n-1)g^{2n-2}}{g^{4n-2}} \\ &= e^{-n\psi} \psi' g^{-2n} \left(\left(-nP_n + aY \frac{\partial P_n}{\partial x} + aX \frac{\partial P_n}{\partial y} \right) (aX + Y) - P_n(2n-1)(a^2Y + aX) \right) \\ &= e^{-(n+1)\psi} g^{-2n-1} \left(P_n(-naX - nY - (2n-1)a^2Y - (2n-1)aX) \right) \\ &\quad + e^{-(n+1)\psi} g^{-2n-1} \left(\frac{\partial P_n}{\partial x} (a^2XY + aY^2) + \frac{\partial P_n}{\partial y} (a^2X^2 + aXY) \right) \end{aligned}$$

and we can conclude the desired result by the induction axiom. \square

Example 3.2. As an application of Proposition 3.1 we obtain the Taylor series formula of ψ_0 at $x = 0$. We then get

$$\psi_0^{(n)}(0) = \frac{P_n(1, 0)}{a^{2n-1}}.$$

In particular, the first few Taylor coefficients are

$$\psi_0(0) = 0, \quad \psi_0'(0) = \frac{1}{a}, \quad \psi_0''(0) = -\frac{2}{a^2}, \quad \text{and } \psi_0^{(3)}(0) = \frac{-a^4 + 10a^2}{a^5}.$$

In Section 6, we shall give a more efficient method for computing these coefficients. \square

Proposition 3.3. The function

$$\Psi(x) = x\psi(x) - \frac{x}{1-a^2} + \frac{ax}{1-a^2} \coth(a\psi(x)) \tag{3.2}$$

is a primitive function of $\psi(x) = \psi(a, x)$.

Proof. Using the substitution $x = \sinh(at)e^t$, $dx = (a \cosh(at) + \sinh(at))e^t dt$, we get

$$\begin{aligned} \Psi(x) &= \int \psi(x) dx = \int t(a \cosh(at) + \sinh(at))e^t dt = \frac{1}{2} \int t \left((1+a)e^{(1+a)t} - (1-a)e^{(1-a)t} \right) dt \\ &= \frac{1}{2} \left(t - \frac{1}{1+a} \right) e^{t(1+a)} - \frac{1}{2} \left(t - \frac{1}{1-a} \right) e^{t(1-a)} = t \sinh(at)e^t + \frac{1}{1-a^2} (-e^t \sinh(at) + ae^t \cosh(at)) \\ &= x\psi(x) - \frac{x}{1-a^2} + \frac{a}{1-a^2} e^{\psi(x)} \cosh(a\psi(x)) = x\psi(x) - \frac{x}{1-a^2} + \frac{ax}{1-a^2} \coth(a\psi(x)). \quad \square \end{aligned}$$

Remark. In a similar manner, using the substitution $x = \sinh(at)e^t$, one can prove that integrals of the form

$$\int x^n \psi^m(x) dx, \quad n, m \in \mathbb{N},$$

can be integrated in an elementary way.

Exploiting the Taylor series $\psi_0(x) = \frac{x}{a} - \frac{2x^2}{a^2} + \dots$, and the value $\psi(L_a) = M_a$, we have

$$\begin{aligned} \Psi_0(0) &= \lim_{x \rightarrow 0} \Psi_0(x) = \frac{a}{1-a^2}, \\ \Psi(L_a) &= L_a \left(M_a - \frac{2}{1-a^2} \right). \end{aligned}$$

Hence, we can now compute the following definite integral:

$$\int_{L_a}^0 \psi_0(a, x) dx = \Psi_0(0) - \Psi(L_a) = \frac{a + 2L_a}{1-a^2} - L_a M_a,$$

and, since ψ_{-1} is the inverse of f ,

$$\int_{L_a}^0 \psi_{-1}(a, x) dx = \int_{-\infty}^{M_a} f(a, w) dw - L_a M_a = \frac{2L_a}{1-a^2} - L_a M_a = \frac{\left(\frac{1-a}{1+a}\right)^{\frac{1}{2a}}}{2\sqrt{1-a^2}} \left(\frac{-4a}{1-a^2} + \log\left(\frac{1-a}{1+a}\right) \right).$$

4. Explicit formulas for ψ

In this section we shall, for certain rational values of a , find explicit formulas for both branches of the ψ -function and the transition function ω . We let $\text{root}(p(x))$ denote the real-valued positive solution(s) to the polynomial equation $p(x) = 0$. Recall that

$$f(a, w) = e^{aw} \sinh(aw) = \frac{1}{2} (e^{w(1+a)} - e^{w(1-a)}). \tag{4.1}$$

For natural numbers n and m such that $n > m$ we set

$$a = \frac{n-m}{n+m} \quad \text{and} \quad \alpha = \frac{2}{n+m},$$

so that

$$1+a = \frac{2n}{n+m} \quad \text{and} \quad 1-a = \frac{2m}{n+m}.$$

Using the substitution $Y = e^{aw} = e^{\frac{2w}{n+m}}$ we rewrite (4.1) as

$$f(a, w) = \frac{1}{2} (Y^n - Y^m) = x$$

and then

$$w = \frac{n+m}{2} \log(\text{root}(Y^n - Y^m - 2x)). \tag{4.2}$$

We are then able to give explicit solutions to this equation from classical methods in the following cases:

$$w = \begin{cases} 2 \log(\text{root}(Y^3 - Y - 2x)) & \text{if } a = 1/2, \\ \frac{5}{2} \log(\text{root}(Y^3 - Y^2 - 2x)) & \text{if } a = 1/5, \\ \frac{5}{2} \log(\text{root}(Y^4 - Y - 2x)) & \text{if } a = 3/5, \\ \frac{7}{2} \log(\text{root}(Y^4 - Y^3 - 2x)) & \text{if } a = 1/7. \end{cases}$$

And the case $a = 1/3$ has already been treated in Section 2.

Case $a = 1/2$: After some straightforward but rather lengthy calculations using Cardano’s method for solving cubic equations, we arrive at the following:

$$\psi_0(\frac{1}{2}, x) = \begin{cases} 2 \log \left(\sqrt[3]{x + \sqrt{x^2 - \frac{1}{27}}} + \sqrt[3]{x - \sqrt{x^2 - \frac{1}{27}}} \right), & \text{if } \frac{\sqrt{3}}{9} \leq x \\ 2 \log \left(\frac{2}{\sqrt{3}} \cos \left(\frac{\arccos(3\sqrt{3}x)}{3} \right) \right), & \text{if } -\frac{\sqrt{3}}{9} \leq x < \frac{\sqrt{3}}{9} \end{cases}$$

and

$$\psi_{-1}(\frac{1}{2}, x) = 2 \log \left(\frac{2}{\sqrt{3}} \cos \left(\frac{\arccos(3\sqrt{3}x) + 4\pi}{3} \right) \right), \quad \text{if } -\frac{\sqrt{3}}{9} \leq x < 0,$$

which then gives the transition function

$$\omega(\frac{1}{2}, z) = \begin{cases} 2 \log \left(\frac{2}{\sqrt{3}} \cos \left(\frac{1}{3} \arccos \left(\frac{3\sqrt{3}}{2} (e^{3z/2} - e^{z/2}) \right) \right) \right), & \text{if } z < -\log 3 \\ 2 \log \left(-\frac{2}{\sqrt{3}} \sin \left(\frac{\pi}{6} - \frac{1}{3} \arccos \left(\frac{3\sqrt{3}}{2} (e^{3x/2} - e^{x/2}) \right) \right) \right), & \text{if } -\log 3 \leq z < 0. \end{cases}$$

Case $a = 1/5$: Again using Cardano’s method for solving cubic equations, we get in this case:

$$\psi_0(x) = \begin{cases} \frac{5}{2} \log \left(\sqrt[3]{x + \frac{1}{27} + \sqrt{x^2 + \frac{2x}{27}}} + \sqrt[3]{x + \frac{1}{27} - \sqrt{x^2 + \frac{2x}{27}}} + \frac{1}{3} \right), & \text{if } 0 \leq x \\ \frac{5}{2} \log \left(\frac{2}{3} \cos \left(\frac{\arccos(27x+1)}{3} \right) + \frac{1}{3} \right), & \text{if } -\frac{2}{27} \leq x < 0, \end{cases}$$

and

$$\psi_{-1}(x) = \frac{5}{2} \log \left(\frac{2}{3} \cos \left(\frac{\arccos(27x+1) + 4\pi}{3} \right) + \frac{1}{3} \right), \quad \text{if } -\frac{2}{27} \leq x < 0.$$

And now we obtain the transition function

$$\omega(\frac{1}{5}, z) = \begin{cases} \frac{5}{2} \log \left(\frac{2}{3} \cos \left(\frac{1}{3} \arccos \left(\frac{27}{2} (e^{6x/5} - e^{4x/5}) + 1 \right) \right) + \frac{1}{3} \right), & \text{if } z < -\frac{5}{2} \log(3/2) \\ \frac{5}{2} \log \left(\frac{1}{3} - \frac{2}{3} \sin \left(\frac{\pi}{6} - \frac{1}{3} \arccos \left(\frac{27}{2} (e^{6x/5} - e^{4x/5}) + 1 \right) \right) \right), & \text{if } -\frac{5}{2} \log(3/2) \leq z < 0. \end{cases}$$

In the following cases we will only state the ψ -functions.

Case $a=3/5$: In this case we shall solve

$$Y^4 - Y - 2x = 0 \quad \text{for } x \geq -\frac{3}{8\sqrt[3]{4}}$$

by using Ferrari’s method. First we need the solution to the auxiliary equation:

$$v^3 + 2xv - \frac{1}{8} = 0.$$

In our case, $x \geq -\frac{3}{8\sqrt[3]{4}}$, so that

$$v = \frac{1}{\sqrt[3]{16}} \left(\sqrt[3]{1 - \sqrt{4\left(\frac{8x}{3}\right)^3 + 1}} + 1 + \sqrt[3]{1 + \sqrt{4\left(\frac{8x}{3}\right)^3 + 1}} \right),$$

and we then arrive at

$$\psi_{0\left(\frac{3}{5}, x\right)} = \frac{5}{2} \log \left(\frac{(2v)^{3/4} + \sqrt{2 - (2v)^{3/2}}}{2\sqrt[4]{2v}} \right), \quad \text{if } -\frac{3}{8\sqrt[3]{4}} \leq x$$

Furthermore,

$$\psi_{-1\left(\frac{3}{5}, x\right)} = \frac{5}{2} \log \left(\frac{(2v)^{3/4} - \sqrt{2 - (2v)^{3/2}}}{2\sqrt[4]{2v}} \right), \quad \text{if } -\frac{3}{8\sqrt[3]{4}} \leq x < 0.$$

Case $a=1/7$: In this case we shall solve

$$Y^4 - Y^3 - 2x = 0 \quad \text{for } x \geq -\left(\frac{3}{8}\right)^3$$

by using Ferrari’s method. After some lengthy but straightforward calculations we get:

$$\psi_{0\left(\frac{1}{7}, x\right)} = \frac{7}{2} \log \left(\frac{\sqrt{2v + \frac{1}{4}} + \sqrt{-2v + \frac{1}{2} + \frac{1}{2\sqrt{8v+1}}}}{2} + \frac{1}{4} \right), \quad \text{if } -\left(\frac{3}{8}\right)^3 \leq x$$

and

$$\psi_{-1\left(\frac{1}{7}, x\right)} = \frac{7}{2} \log \left(\frac{\sqrt{2v + \frac{1}{4}} - \sqrt{-2v + \frac{1}{2} + \frac{1}{2\sqrt{8v+1}}}}{2} + \frac{1}{4} \right), \quad \text{if } -\left(\frac{3}{8}\right)^3 \leq x < 0,$$

where

$$v = \sqrt[3]{-\frac{x}{8} + \sqrt{\left(\frac{2x}{3}\right)^3 + \left(\frac{x}{8}\right)^2}} + \sqrt[3]{-\frac{x}{8} - \sqrt{\left(\frac{2x}{3}\right)^3 + \left(\frac{x}{8}\right)^2}}.$$

5. A parametrization and a definite integral

The main aim of this section is to prove that

$$\int_{-\infty}^0 \omega(a, z) dz = \frac{\pi^2}{3(a^2 - 1)}.$$

We rely on an ingenious idea to simultaneously parameterize both branches of ψ for $x < 0$. This idea originates from [10], and is explained in [15, Theorem 1.3.1]. Our parametrization is presented in the following theorem:

Theorem 5.1. *Let*

$$x = \frac{1}{2} \left(\frac{\alpha^{1+a} - \alpha^{2a}}{\alpha^{1+a} - 1} \right)^{\frac{1-a}{2a}} \left(\frac{1 - \alpha^{2a}}{\alpha^{1+a} - 1} \right)$$

where $1 < \alpha$. Then

$$\begin{aligned} \psi_0(a, x) &= \log(\alpha) + \psi_{-1}(a, x) \\ \psi_{-1}(a, x) &= \frac{1}{2a} \log \left(\frac{\alpha^{1-a} - 1}{\alpha^{1+a} - 1} \right). \end{aligned}$$

Proof. We are looking for α such that

$$x(\alpha) = e^{\psi_0(x(\alpha))} \sinh(a\psi_0(x(\alpha))) = e^{\psi_{-1}(x(\alpha))} \sinh(a\psi_{-1}(x(\alpha))).$$

Let $\psi_0 = \psi_{-1} + p/a$, where $p > 0$ is a parameter. Then,

$$e^{\psi_0} \sinh(a\psi_0) = e^{(\psi_{-1} + \frac{p}{a})} \sinh(a(\psi_{-1} + p/a)) = e^{\psi_{-1}} \sinh(a\psi_{-1}),$$

which implies

$$e^{-\frac{p}{a}} = \frac{\sinh(a\psi_{-1} + p)}{\sinh(a\psi_{-1})} = \cosh(p) + \coth(a\psi_{-1}) \sinh(p).$$

Hence,

$$\psi_{-1} = \frac{1}{a} \operatorname{arccoth} \left(\frac{e^{-\frac{p}{a}} - \cosh(p)}{\sinh(p)} \right) = \frac{1}{2a} \log \left(\frac{e^{-p} - e^{-\frac{p}{a}}}{e^p - e^{-\frac{p}{a}}} \right).$$

Continuing in a similar manner, we set $\psi_{-1} = \psi_0 - p/a$, where $p > 0$ is a parameter, which implies

$$e^{\psi_0 - p/a} \sinh(a\psi_0 - p) = e^{\psi_0} \sinh(a\psi_0).$$

Hence,

$$e^{\frac{p}{a}} = \frac{\sinh(a\psi_0 - p)}{\sinh(a\psi_0)} = \cosh(p) - \coth(a\psi_0) \sinh(p),$$

which yields

$$\psi_0 = \frac{1}{a} \operatorname{arccoth} \left(\frac{-e^{\frac{p}{a}} + \cosh(p)}{\sinh(p)} \right) = \frac{1}{2a} \log \left(\frac{-e^p + e^{\frac{p}{a}}}{-e^{-p} + e^{\frac{p}{a}}} \right).$$

Thus, for $p \in (0, \infty)$ it holds

$$x = e^{\psi_0} \sinh(a\psi_0) = \frac{1}{2} \left(\frac{-e^p + e^{\frac{p}{a}}}{-e^{-p} + e^{\frac{p}{a}}} \right)^{\frac{1-a}{2a}} \left(\frac{e^{-p} - e^p}{e^{\frac{p}{a}} - e^{-p}} \right).$$

Set $p = a \log(\alpha)$ where $\alpha \in (1, \infty)$. Then

$$\begin{aligned} x &= \frac{1}{2} \left(\frac{\alpha^{1+a} - \alpha^{2a}}{\alpha^{1+a} - 1} \right)^{\frac{1-a}{2a}} \left(\frac{1 - \alpha^{2a}}{\alpha^{1+a} - 1} \right) \\ \psi_0(x) &= \log(\alpha) + \psi_{-1}(x) \\ \psi_{-1}(x) &= \frac{1}{2a} \log \left(\frac{\alpha^{1-a} - 1}{\alpha^{1+a} - 1} \right). \quad \square \end{aligned}$$

Thanks to the parametrization in Theorem 5.1 we get an alternative way to find explicit formulas for ψ_0 , and ψ_{-1} . The below example is for the case $a = 1/3$.

Example 5.2. We return to the special case of $a = 1/3$. For $-1/8 \leq x < 0$ and $\alpha \geq 1$ we let

$$x = \frac{-\alpha^{2/3}}{2(\alpha^{2/3} + 1)^2}, \tag{5.1}$$

and

$$\psi_0 = -\frac{3}{2} \log(1 + \alpha^{-2/3}) \quad \text{and} \quad \psi_{-1} = -\frac{3}{2} \log(1 + \alpha^{2/3}).$$

Thus, by solving (5.1) in $\alpha^{2/3}$, and using Theorem 5.1 we get

$$\psi_0 = \frac{3}{2} \log \left(\frac{1}{2} + \sqrt{2x + \frac{1}{4}} \right) \quad \text{and} \quad \psi_{-1} = \frac{3}{2} \log \left(\frac{1}{2} - \sqrt{2x + \frac{1}{4}} \right). \quad \square$$

Example 5.3. Setting $a = 1/2$ and $\alpha = \beta^2$ we obtain the following nice parametrisation of x and ψ

$$x = -\frac{\beta(1 + \beta)}{2(1 + \beta + \beta^2)^{3/2}}$$

$$\psi_0(1/2, x) = 2\log(\beta) - \log(1 + \beta + \beta^2)$$

$$\psi_{-1}(1/2, x) = -\log(1 + \beta + \beta^2)$$

Thus, for example, from $\beta = 2$ we get $x = -3/\sqrt{343} \sim -0.1620$ and $\psi_0(1/2, x) = \log(4) - \log(7)$ and $\psi_{-1}(1/2, x) = -\log(7)$. \square

Employing Theorem 5.1, we can now state the following theorem:

Theorem 5.4.

$$\int_{-\infty}^0 \omega(a, z) dz = \frac{\pi^2}{3(a^2 - 1)}.$$

Proof. The definition of the transition function states:

$$\omega(a, z) = \begin{cases} \psi_0(a, f(a, z)) & \text{if } z < M_a \\ \psi_{-1}(a, f(a, z)) & \text{if } M_a \leq z < 0. \end{cases}$$

For $y \geq L_a$ in the relation $y = f(a, z)$ we note that one inverse branch is $z = \psi_0(a, y) \geq M_a$, and the other branch is $z = \psi_{-1}(a, y) \leq M_a$. Hence,

$$\int_{-\infty}^0 \omega(a, z) dz = \int_0^{L_a} (\psi_0(y)\psi'_{-1}(y) - \psi_{-1}(y)\psi'_0(y)) dy.$$

From Theorem 5.1 it follows

$$\psi_0(y(\alpha)) = \log(\alpha) + \psi_{-1}(y(\alpha)) \quad \text{and} \quad \psi_{-1}(y(\alpha)) = \frac{1}{2a} \log\left(\frac{\alpha^{1-a} - 1}{\alpha^{1+a} - 1}\right),$$

and therefore

$$\int_{-\infty}^0 \omega(a, z) dz = \int_1^\infty \left(\frac{1}{\alpha} \Psi(\alpha) - \log(\alpha) \Psi'(\alpha)\right) d\alpha,$$

where

$$\Psi(\alpha) = \frac{1}{2a} \log\left(\frac{\alpha^{1-a} - 1}{\alpha^{1+a} - 1}\right).$$

Finally, from the substitution $\alpha = \frac{1}{\beta}$ it follows

$$\int_{-\infty}^0 \omega(a, z) dz = \int_0^1 \left(\frac{1}{2a\beta} \sum_{n=1}^\infty \frac{-((\beta^{1-a})^n - (\beta^{1+a})^n)}{n} + \frac{\log(\beta)}{2a\beta} \left((1-a) \sum_{n=1}^\infty (\beta^{1-a})^n - (1+a) \sum_{n=1}^\infty (\beta^{1+a})^n\right)\right) d\beta$$

$$= \frac{1}{1-a^2} \left(-\frac{\pi^2}{6} - \frac{\pi^2}{6}\right) = \frac{\pi^2}{3(a^2 - 1)}. \quad \square$$

6. Taylor series of ψ_0 at zero

In this section and the next, we will focus on series expansions. Here we use Lagrange’s inversion theorem to determine the Taylor series for ψ_0 about $x = 0$ (see (6.1)). First let us recall Lagrange’s inversion theorem [1,6]:

Theorem 6.1 (Lagrange’s inversion theorem). *If $x = f(w)$, $x_0 = f(w_0)$, for some real-analytic function in a neighbourhood of w_0 with $f'(w_0) \neq 0$, then*

$$w = g(x) = w_0 + \sum_{k=1}^\infty \frac{x - x_0}{k!} \left(\frac{d^{k-1}}{dw^{k-1}} \left(\frac{w - w_0}{f(w) - x_0} \right)^k \right) \Big|_{w=w_0},$$

where the convergence radius is strictly positive. Furthermore, if

$$f(w) = \sum_{k=0}^{\infty} f_k \frac{w^k}{k!} \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} g_k \frac{x^k}{k!},$$

then it holds

$$\begin{aligned} g_n &= \frac{1}{f_1^n} \sum_{k=1}^{n-1} (-1)^k n^{\bar{k}} B_{n-1,k} \left(\frac{f_2}{2f_1}, \frac{f_3}{3f_1}, \dots, \frac{f_{n-k+1}}{(n-k+1)f_1} \right) \\ &= \frac{1}{a^n} \sum_{k=1}^{n-1} (-1)^k a^k n^{\bar{k}} B_{n-1,k} \left(\frac{f_2}{2}, \frac{f_3}{3}, \dots, \frac{f_{n-k+1}}{(n-k+1)} \right), \quad n \geq 2, \end{aligned}$$

where $g_1 = \frac{1}{f_1} = \frac{1}{a}$, $n^{\bar{k}} = n(n+1) \cdots (n+k-1)$ is the rising factorial and $B_{n,k}$ is the partial exponential Bell polynomial.

We apply this theorem to the function

$$f(w) = f(a, w) = \frac{1}{2} (e^{(1+a)w} - e^{(1-a)w}) = \sum_{n=1}^{\infty} \frac{(1+a)^n - (1-a)^n}{2n!} w^n = \sum_{n=1}^{\infty} \frac{f_n}{n!} w^n,$$

and then we arrive at

$$\psi_0(a, x) = \sum_{n=1}^{\infty} \frac{g_n}{n!} x^n,$$

where

$$g_n = \frac{1}{f_1^n} \sum_{k=1}^{n-1} (-1)^k n^{\bar{k}} B_{n-1,k} \left(\frac{f_2}{2f_1}, \frac{f_3}{3f_1}, \dots, \frac{f_{n-k+1}}{(n-k+1)f_1} \right).$$

Thus, the first few terms of the Taylor series of ψ_0 around $x = 0$ are given by:

$$\begin{aligned} \psi_0(a, x) &= \frac{x}{a} - \frac{1}{a^2} x^2 - \frac{(a^2 - 9)}{6a^3} x^3 + \frac{2(a^2 - 4)}{3a^4} x^4 + \frac{(9a^4 - 250a^2 + 625)}{120a^5} x^5 - \frac{2(4a^4 - 45a^2 + 81)}{15a^6} x^6 \\ &\quad - \frac{(225a^6 - 12691a^4 + 84035a^2 - 117649)}{5040a^7} x^7 + \frac{16(9a^6 - 196a^4 + 896a^2 - 1024)}{315a^8} x^8 \\ &\quad + \frac{(1225a^8 - 116244a^6 + 1439046a^4 - 4960116a^2 + 4782969)}{40320a^9} x^9 \\ &\quad - \frac{2(576a^8 - 20500a^6 + 170625a^4 - 468750a^2 + 390625)}{2835a^{10}} x^{10} + \mathcal{O}(x^{11}) \end{aligned} \tag{6.1}$$

Furthermore, the radius of converges of the Taylor series can not exceed $|L_a|$.

7. Series expansions of ψ at $x = L_a$ and of ω at $z = M_a$

This section aims to determine the series expansions of ψ at $x = L_a$ and of ω at $z = M_a$ with the help of Proposition 7.1 as a tool. The series are stated in (7.1), (7.2), and (7.3), respectively.

Proposition 7.1. *Let f be a smooth function in the neighbourhood of 0 such that $f(0) = f'(0) = 0$ and $f''(0) > 0$. Then f has two smooth inverse functions (in some neighbourhood of 0), a right inverse $h(x) > 0$ for $x > 0$, and a left inverse $g(x) < 0$ for $x > 0$, which can be expressed as*

$$h(x) = h_1 \sqrt{x} + h_2 x + h_3 x^{3/2} + \dots,$$

$$g(x) = g_1 \sqrt{x} + g_2 x + g_3 x^{3/2} + \dots,$$

where $g_{2k} = h_{2k}$ and $g_{2k-1} = -h_{2k-1}$ for $k = 1, 2, \dots$

Proof. Without loss of generality, we may assume that f is convex in a possible smaller neighbourhood of zero. The Taylor expansion at zero can be presented as

$$f(x) = x^2(f_2 + f_3x + \dots), \quad \text{with } f_2 > 0.$$

Therefore, $f(x) = x^2 \tilde{f}(x)$ for some smooth function \tilde{f} with $\tilde{f}(x) > 0$. Thus, $\sqrt{\tilde{f}(x)}$ is smooth in the neighbourhood of 0. Set

$$F(x) = \begin{cases} \sqrt{x^2 \tilde{f}(x)}, & \text{if } x \geq 0 \\ -\sqrt{x^2 \tilde{f}(x)}, & \text{if } x \leq 0. \end{cases}$$

The function F is invertible and smooth in a possible punctured neighbourhood of 0. Furthermore, for all $n \in \mathbb{N}$ the following limits exist:

$$\lim_{x \rightarrow 0^+} f^{(n)}(x) \quad \text{and} \quad \lim_{x \rightarrow 0^-} f^{(n)}(x)$$

and $F^2 = f$. From this it follows that f has two smooth inverse functions, $f^{-1} = F^{-1} \circ (x^2)^{-1}$, which can be expressed as $F^{-1}(\sqrt{y})$ and $F^{-1}(-\sqrt{y})$, $y \geq 0$. Finally, by applying the Taylor series expansion to F^{-1} we get the desired conclusion. \square

Let $K_a = L_a(a^2 - 1)$ and recall that $f(a, M_a) = L_a$. Then we receive the following series expansion around $y = 0$

$$\begin{aligned} \frac{1}{K_a} (f(a, y + M_a) - L_a) &= \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{a^2 + 3}{24}y^4 + \frac{a^2 + 1}{30}y^5 + \frac{a^4 + 10a^2 + 5}{720}y^6 + \frac{3a^4 + 10a^2 + 3}{2520}y^7 \\ &\quad + \frac{a^6 + 21a^4 + 35a^2 + 7}{40320}y^8 + \mathcal{O}(y^9) \end{aligned}$$

The inverse series expansion gives us ψ_0 can be found by using the series ansatz of f and h in Proposition 7.1 and solve $f(h(x)) = x$ term by term.

$$\begin{aligned} \psi_0(a, x K_a + L_a) - M_a &= \sqrt{2}\sqrt{x} - \frac{2}{3}x + \frac{11 - 3a^2}{18\sqrt{2}}x^{3/2} - \frac{43 - 27a^2}{135}x^2 + \frac{81a^4 - 786a^2 + 769}{2160\sqrt{2}}x^{5/2} - \frac{8(81a^4 - 318a^2 + 221)}{8505}x^3 \\ &\quad + \frac{680863 - 1273509a^2 + 551853a^4 - 30375a^6}{2721600\sqrt{2}}x^{7/2} + \mathcal{O}(x^4) \end{aligned} \tag{7.1}$$

which converges for $0 \leq x < -L_a/K_a = 1/(1 - a^2)$. Note that the coefficients of x^3 and $x^{7/2}$ can change sign for $0 < a < 1$.

Thanks to Proposition 7.1 we can deduce the other branch

$$\begin{aligned} \psi_{-1}(a, x K_a + L_a) - M_a &= -\sqrt{2}\sqrt{x} - \frac{2}{3}x - \frac{11 - 3a^2}{18\sqrt{2}}x^{3/2} - \frac{43 - 27a^2}{135}x^2 - \frac{81a^4 - 786a^2 + 769}{2160\sqrt{2}}x^{5/2} - \frac{8(81a^4 - 318a^2 + 221)}{8505}x^3 \\ &\quad - \frac{680863 - 1273509a^2 + 551853a^4 - 30375a^6}{2721600\sqrt{2}}x^{7/2} + \mathcal{O}(x^4) \end{aligned} \tag{7.2}$$

and again this series converges for $0 \leq x < 1/(1 - a^2)$. Note that the second argument in both series is $x K_a + L_a \in [L_a, 0)$.

Now we find the composition of the series of $\psi_{-1}(a, x K_a + L_a) - M_a$ with that of $(f(a, x + M_a) - L_a)/K_a$, which gives us the series of $\omega(a, x + M_a) - M_a$, and note that the K_a -factor cancels out,

$$\begin{aligned} \omega(a, x + M_a) - M_a &= -x - \frac{2}{3}x^2 - \frac{4}{9}x^3 + \frac{2(3a^2 - 22)}{135}x^4 + \frac{4(9a^2 - 26)}{405}x^5 - \frac{4(3a^4 - 88a^2 + 150)}{2835}x^6 - \frac{8(81a^4 - 808a^2 + 956)}{42525}x^7 \\ &\quad + \frac{2(27a^6 - 2106a^4 + 11124a^2 - 9968)}{127575}x^8 + \frac{4(135a^6 - 3258a^4 + 11064a^2 - 7928)}{229635}x^9 \\ &\quad - \frac{4(2025a^8 - 341604a^6 + 4049838a^4 - 9850752a^2 + 5857336)}{189448875}x^{10} + \mathcal{O}(x^{11}). \end{aligned} \tag{7.3}$$

Note that the coefficient of x^4 is the first which depends on a . Note also that the coefficient of x^9 is the first which can become zero for some $0 < a < 1$. This happens at $a = 0.998689$. The coefficient for x^{10} becomes zero at $a = 0.952489$.

Example 7.2. In Fig. 4 we show $\psi_0, \psi_{-1}, \omega$ for $a = 1/2$ and their respective series expansion as just stated. In this case the first few terms of the respective expansions are

$$\begin{aligned} \psi_0\left(\frac{1}{2}, \frac{x}{4\sqrt{3}} - \frac{1}{3\sqrt{3}}\right) + \log(3) &= \sqrt{2}\sqrt{x} - \frac{2}{3}x + \frac{41}{72\sqrt{2}}x^{3/2} - \frac{29}{108}x^2 + \frac{9241}{34560\sqrt{2}}x^{5/2} + \dots \\ \psi_{-1}\left(\frac{1}{2}, \frac{x}{4\sqrt{3}} - \frac{1}{3\sqrt{3}}\right) + \log(3) &= -\sqrt{2}\sqrt{x} - \frac{2}{3}x - \frac{41}{72\sqrt{2}}x^{3/2} - \frac{29}{108}x^2 - \frac{9241}{34560\sqrt{2}}x^{5/2} + \dots \\ \omega\left(\frac{1}{2}, x - \log(3)\right) + \log(3) &= -x - \frac{2}{3}x^2 - \frac{4}{9}x^3 - \frac{17}{54}x^4 - \frac{19}{81}x^5 + \dots \quad \square \end{aligned}$$

8. Asymptotics of the branch ψ_0 as $x \rightarrow \infty$

Here we determine the asymptotic behaviour of $\psi_0(a, x)$ as $x \rightarrow \infty$. An important tool in doing so is the following theorem.

Theorem 8.1. Let

$$x \geq \frac{1}{2} \left(1 + \frac{1}{e^{\frac{1}{3} - a} - 1} \right)^{\frac{1+a}{2a}}.$$

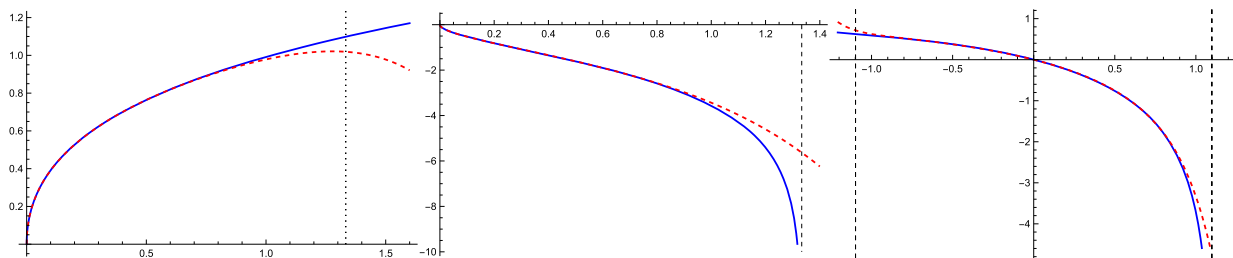


Fig. 4. Left: Plots of $\psi_0(a, x/K_a + L_a) + M_a$ (solid blue curve) and the corresponding 6th order series expansion (dashed red curve). Middle: Plots of $\psi_{-1}(a, x/K_a + L_a) + M_a$ (solid blue curve) and the corresponding 6th order series expansion (dashed red curve). Right: Plots of $\omega(a, x - M_a) + M_a$ (solid blue curve) and the corresponding 11th order series expansion. Plots show $a = 1/2$. The dotted vertical lines in the plots indicate the end-points of the convergence interval. These are at $x = -L_a/K_a$ for the left&middle plots and at $x = \pm M_a$ in the right plot.

Then the following holds

(1) if $0 < a < \frac{1}{3}$, then

$$\frac{1}{1+a}(2x)^{-\frac{2a}{1+a}} + \frac{\log(2x)}{1+a} \leq \psi_0(a, x) \leq \frac{\log(2x)}{1+a} + \frac{2}{1+a}(2x)^{-\frac{2a}{1+a}}.$$

(2) if $\frac{1}{3} < a < 1$, then

$$\frac{1}{3(1+a)}(2x)^{-\frac{2a}{1+a}} + \frac{\log(2x)}{1+a} \leq \psi_0(a, x) \leq \frac{\log(2x)}{1+a} + \frac{1}{1+a}(2x)^{-\frac{2a}{1+a}};$$

Remark. Let x_0 be the point such that the bounds hold when $x \geq x_0$. Numerical experimentation suggests that for the lower bounds we have

(i) If $0 < a < 1/3$ then

$$x_0 = \frac{e^{-29/4}}{(\frac{1}{3} - a)^2}$$

This seems to be the exact optimal point when $a \rightarrow (1/3)^-$. It is not optimal when $a \rightarrow 0^+$ but it still works then.

(ii) If $1/3 < a < 1$ then $x_0 \approx 0.1$ works. The difference between ψ_0 and the lower bound is decreasing when $x \geq 1/4$ (this is the limit when $a \rightarrow 1^-$).

For the upper bounds we get

(i) If $0 < a \leq 0.102$ then $x_0 \sim \exp(-3.8 + \frac{0.117}{a})$. The difference between the upper bound and ψ_0 is decreasing for $x > \exp(-3.1 + \frac{0.35}{a})$. Both of these formulas are rough estimates but they are much smaller than the x -bound in the theorem. In any case, the upper bounds require very large x if they are to hold for small a .

(ii) When $0.102 \lesssim a < 1$ the upper bound holds for all $x > 0$.

(iii) When $0.1255 \lesssim a < 1$ the difference between the upper bound and ψ_0 is decreasing for all $x > 0$.

Let us start with observing that if $y = \psi_0(a, x)$, then

$$x = \sinh(ay)e^y = \frac{e^{(1+a)y} - e^{(1-a)y}}{2} < \frac{e^{(1+a)y}}{2},$$

and therefore we get

$$\psi_0(a, x) = y > \frac{1}{1+a} \log(2x).$$

We shall next continue with proving two auxiliary lemmas.

Lemma 8.2. For $0 < a < 1$ it holds

$$0 \leq \psi_0(a, x) - \frac{\log(2x)}{1+a} \leq \log\left(\frac{(2x)^{\frac{a}{1+a}}}{(2x)^{\frac{a}{1+a}} - (2x)^{-\frac{a}{1+a}}}\right),$$

for all $x > 0$. In particular,

$$\lim_{x \rightarrow \infty} \left(\psi_0(a, x) - \frac{1}{1+a} \log(2x) \right) = 0.$$

Proof. Using the definition of ψ_0 and the fact that function \sinh is increasing we have

$$1 \leq e^{\psi_0(a,x) - \frac{\log(2x)}{1+a}} = e^{\psi_0(a,x)} (2x)^{-\frac{1}{1+a}} = \frac{x(2x)^{-\frac{1}{1+a}}}{\sinh(a\psi_0(a,x))} \leq \frac{x(2x)^{-\frac{1}{1+a}}}{\sinh\left(a \frac{\log(2x)}{1+a}\right)} = \frac{(2x)^{\frac{a}{1+a}}}{(2x)^{\frac{a}{1+a}} - (2x)^{-\frac{a}{1+a}}}.$$

The last term above tends to 1 as $x \rightarrow \infty$, which proves the final claim. \square

The following lemma is a straight-forward exercise obtained from taking the Taylor expansion of e^y about 0.

Lemma 8.3. For any $0 < a < \frac{1}{3}$ there exists a point $b(a) > 0$ such that

- (i) $e^{2ay} - e^{(a-1)y} - (a+1)y \leq 0$, and
- (ii) $e^{2ay} - e^{(a-1)y} - (a+1)y + \frac{1}{2}(-3a^2 - 2a + 1)y^2 \geq 0$,

for all $0 \leq y \leq b(a)$. Furthermore, for $a > \frac{1}{3}$ there exists a point $b(a) > 0$ such that

$$(1+a)y \leq e^{2ay} - e^{(a-1)y} \leq 3(1+a)y$$

for all $0 \leq y \leq b(a)$. In fact in both cases one can take $b(a) = |a - \frac{1}{3}|$.

We are now ready to present the proof of the main result of this section.

Proof of Theorem 8.1. From Lemma 8.2 it follows that $\varphi(x) = \psi_0(a, x) - \frac{\log(2x)}{1+a}$ satisfies: $\varphi > 0$, $\varphi(x) \rightarrow 0$, as $x \rightarrow \infty$ and $\varphi(x) \leq h(x)$, where

$$h(x) = \log \left(\frac{(2x)^{\frac{a}{1+a}}}{(2x)^{\frac{a}{1+a}} - (2x)^{-\frac{a}{1+a}}} \right).$$

Next, substitute $x = \varphi(x)$ in the implicit definition of ψ_0 , and we arrive at

$$e^{(1+a)\left(\varphi(x) + \frac{\log(2x)}{1+a}\right)} - e^{(1-a)\left(\varphi(x) + \frac{\log(2x)}{1+a}\right)} = 2x,$$

and after a rearrangement we obtain

$$e^{2a\varphi(x)} - e^{(a-1)\varphi(x)} = (2x)^{-\frac{2a}{1+a}}. \tag{8.1}$$

We have that $0 < \varphi(x) \leq h(x)$. Furthermore, by inspection, for any $\epsilon > 0$ we have that $h(x) \leq \epsilon$ if, and only if,

$$x \geq \frac{1}{2} \left(1 + \frac{1}{e^\epsilon - 1} \right)^{\frac{1+a}{2a}}. \tag{8.2}$$

Choose $\epsilon = b(a) = |\frac{1}{3} - a|$, where $b(a)$ is from Lemma 8.3. To conclude this proof we apply Lemma 8.3 with $y = \varphi(x)$ to (8.1). To summarize and write out certain details:

(1) By Lemma 8.3

$$\begin{aligned} \varphi(x) &\geq \frac{1}{1+a} (2x)^{-\frac{2a}{1+a}}, \text{ for } a < \frac{1}{3}; \\ \frac{1}{3(1+a)} (2x)^{-\frac{2a}{1+a}} &\leq \varphi(x) \leq \frac{1}{1+a} (2x)^{-\frac{2a}{1+a}}, \text{ for } a \geq \frac{1}{3}. \end{aligned}$$

(2) For $a < \frac{1}{3}$, again by using Lemma 8.3

$$\beta_1(a)\varphi(x) + \frac{1}{2}\beta_2(a)\varphi^2(x) \leq (2x)^{-\frac{2a}{1+a}},$$

where $\beta_1(a) = 1+a$ and $\beta_2(a) = 3a^2 + 2a - 1$. Note that by (8.2)

$$\Delta = \beta_1(a)^2 + 2\beta_2(a)(2x)^{-\frac{2a}{1+a}} = (1+a)^2 \left(1 + \frac{6a-2}{1+a} (2x)^{-\frac{2a}{1+a}} \right) \geq 0,$$

since

$$1 + \frac{1}{e^{|a-\frac{1}{3}|} - 1} \geq \left| \frac{1+a}{2-6a} \right|.$$

In the case $a < \frac{1}{3}$, we have that $\beta_2(a) = (3a - 1)(a + 1) < 0$ and

$$\varphi_1 = \frac{\beta_1(a) + \sqrt{\Delta}}{-\beta_2(a)} > \varphi_2 = \frac{\beta_1(a) - \sqrt{\Delta}}{-\beta_2(a)} > 0.$$

Therefore, $\varphi \leq \varphi_2$. Thus,

$$\varphi(x) \leq \frac{1 - \sqrt{1 - \frac{2-6a}{1+a}(2x)^{-\frac{2a}{1+a}}}}{1 - 3a} \leq \frac{2}{1+a}(2x)^{-\frac{2a}{1+a}}. \quad \square$$

As a corollary of the previous proof, we can get a series expansion in the variable $Y = (2x)^{-\frac{2a}{1+a}}$. First we re-state (8.1):

$$e^{2a\varphi(x)} - e^{(a-1)\varphi(x)} = (2x)^{-\frac{2a}{1+a}},$$

and then we use the Taylor expansion of the exponential function about 0:

$$\sum_{n=1}^{\infty} \frac{(2a)^n - (a-1)^n}{n!} \varphi^n(x) = \sum_{n=1}^{\infty} \frac{\beta_n(a)}{n!} \varphi^n(x) = (2x)^{-\frac{2a}{1+a}},$$

where $\beta_k(a) = (2a)^k - (a-1)^k$, $\beta_1(a) = 1+a$, $\beta_2(a) = 3a^2 + 2a - 1$, $\beta_3(a) = 7a^3 + 3a^2 - 3a + 1$. Then by using Theorem 6.1 we arrive at

$$\psi(a, x) = \sum_{n=1}^{\infty} \frac{g_n(a)}{n!} Y^n,$$

where

$$g_n(a) = \frac{1}{\beta_1^n} \sum_{k=1}^{n-1} (-1)^k n^{(k)} B_{n-1,k} \left(\frac{\beta_2}{2\beta_1}, \frac{\beta_3}{3\beta_1}, \dots, \frac{\beta_{n-k+1}}{(n-k+1)\beta_1} \right), \quad n \geq 2,$$

and $g_1(a) = \frac{1}{\beta_1} = \frac{1}{a+1}$, $n^{(k)} = n(n+1)\dots(n+k-1)$ is the rising factorial and $B_{n,k}$ denote the partial or incomplete exponential Bell polynomials.

Let us compute a few coefficients:

$$g_1(a) = \frac{1}{\beta_1} = \frac{1}{a+1}$$

$$g_2(a) = \frac{1}{\beta_1^2} (-1) 2 B_{1,1}(\hat{\beta}_1) = -\frac{\beta_2(a)}{\beta_1^3(a)} = \frac{1-3a}{(a+1)^2}$$

$$g_3(a) = \frac{1}{\beta_1^3} (-3 B_{2,1}(\hat{\beta}_1, \hat{\beta}_2) + 12 B_{2,2}(\hat{\beta}_1)) = \frac{1}{\beta_1^3(a)} (-3\hat{\beta}_2 + 12\hat{\beta}_1^2) = \frac{-\beta_3\beta_1 + 3\beta_2^2}{\beta_1^5} = \frac{2(10a^2 - 7a + 1)}{(a+1)^3}.$$

Thus,

$$\varphi(x) \sim g_1(a)Y + \frac{g_2(a)}{2}Y^2 + \frac{g_3(a)}{6}Y^3 = \frac{1}{a+1}Y + \frac{1-3a}{2(a+1)^2}Y^2 + \frac{10a^2 - 7a + 1}{3(a+1)^3}Y^3,$$

which implies

$$\psi_0(a, x) \sim \frac{\log(2x)}{1+a} + \frac{1}{a+1}(2x)^{-\frac{2a}{1+a}} + \frac{1-3a}{2(a+1)^2}(2x)^{-\frac{4a}{1+a}} + \frac{10a^2 - 7a + 1}{3(a+1)^3}(2x)^{-\frac{6a}{1+a}}. \quad (8.3)$$

Example 8.4. We compare $\psi_0(a, x)$ and its asymptote in (8.3), here denoted $A_0(a, x)$, over $x > 1$. Let $\Delta_0(a, x) = \psi_0(a, x) - A_0(a, x)$ denote the difference. The series expansion clearly predicts the difference to be on the order of $x^{\frac{-8a}{a+1}}$. Choosing $a = 1/2$ then gives $\Delta_0(1/2, x) = O(x^{-8/3})$.

Indeed, plotting $\log \Delta_0(1/2, x)$ versus $\log x$ ($1 \leq x \leq 10^5$) is very well fitted by a line with slope $-8/3$, giving $\log \Delta_0(1/2, x) \sim -6.428 - 2.666 \log x$, so that the difference decreases as $\Delta_0(1/2, x) \sim 0.00162/x^{-8/3}$ for large x . This also works for x near 1. At $x = 1$ the actual difference is 0.00126 so the asymptote is in fact useful for $x \geq 1$.

Repeating this for $a = 3/4$ the prediction is $\Delta_0(3/4, x) = O(x^{24/7})$ and indeed the log-log plot gives a line with the correct slope. We estimate $\Delta_0(3/4, x) \sim -0.00329/x^{-24/7}$ and the actual difference at $x = 1$ is now -0.0027 . Note, by the way, the change of sign which occurs around $a \sim 3/5$.

Repeating the process for $a = 1/4$ should give $\Delta_0(1/4, x) = O(x^{8/5})$. We estimate $\Delta_0(1/4, x) \sim -0.00369/x^{8/5}$ for large x and at $x = 1$ the actual difference is -0.00337 . Again the sign has become negative and this happens just below $a \sim 1/3$ and there is another change of sign around $a \sim 1/8$, but it is not clear how the sign depends on a .

Note that for smaller values of a , say $a \leq 1/10$, the asymptote requires larger x to give a good estimate of ψ_0 . \square

9. Asymptotics of the branch ψ_{-1} as $x \rightarrow 0^-$

Using the same techniques as in the previous sections, we shall study the branch $\psi_{-1}(a, x)$'s behaviour near zero. We shall prove the following theorem. Recall that the branch $\psi_{-1}(a, x)$ is defined for $L_a \leq x < 0$.

Theorem 9.1. *Let $0 < a < 1$. Then*

- (1) $\lim_{x \rightarrow 0^-} \left(\psi_{-1}(a, x) - \frac{1}{1-a} \log(-2x) \right) = 0.$
- (2) $\psi_{-1}(a, x) \geq \frac{\log(-2x)}{1-a} - \log \left(1 - (-2x)^{\frac{2a}{1-a}} \right).$
- (3) For $-\frac{1}{2} \left(\frac{1-a}{6a+2} \right)^{\frac{1-a}{2a}} \leq x \leq 0$ it holds

$$\frac{\log(-2x)}{1-a} + \frac{1}{1-a} (-2x)^{\frac{2a}{1-a}} \leq \psi_{-1}(a, x) \leq \frac{\log(-2x)}{1-a} + \frac{2}{1-a} (-2x)^{\frac{2a}{1-a}}.$$

In the proof of Theorem 9.1 we shall need the following elementary facts.

Lemma 9.2. *For $x > 0$ it holds*

- (i) $e^{-2ax} - e^{-(a-1)x} - (1-a)x \leq 0$ and
- (ii) $e^{-2ax} - e^{-(a-1)x} - (1-a)x - \frac{1}{2}(3a^2 - 2a - 1)x^2 \geq 0.$

Proof of Theorem 9.1. (1) If $y = \psi_{-1}(x)$, then

$$x = \sinh(ay)e^y = \frac{e^{(1+a)y} - e^{(1-a)y}}{2} > -\frac{e^{(1-a)y}}{2}, \tag{9.1}$$

and therefore we get

$$\psi_{-1}(a, x) = y > \frac{1}{1-a} \log(-2x). \tag{9.2}$$

Then, let $\varphi(x) = \psi_{-1}(a, x) - \frac{\log(-2x)}{1-a}$. We know that $\varphi > 0$. From (9.1) it follows

$$e^{(1+a)\left(\varphi(x) + \frac{\log(-2x)}{1-a}\right)} - e^{(1-a)\left(\varphi(x) + \frac{\log(-2x)}{1-a}\right)} = 2x,$$

and after rearranging we have

$$e^{-2a\varphi(x)} - e^{-(a-1)\varphi(x)} = (-2x)^{\frac{2a}{1-a}}. \tag{9.3}$$

Each term of the left side is bounded and since the right hand side tends to zero, when $x \rightarrow 0^-$, then so does $\varphi(x)$.

(2) Using the definition of ψ_{-1} , and the fact that the function \sinh is increasing and by (9.2) we have

$$\begin{aligned} e^{-\psi_{-1}(a,x) + \frac{\log(-2x)}{1-a}} &= e^{-\psi_{-1}(x)} (-2x)^{\frac{1}{1-a}} = \frac{(-2x)^{\frac{1}{1-a}} \sinh(a\psi_{-1}(a, x))}{x} \\ &\leq \frac{(-2x)^{\frac{1}{1-a}} \sinh\left(a \frac{\log(-2x)}{1-a}\right)}{x} = \frac{(-2x)^{\frac{1+a}{1-a}} - (-2x)^{\frac{1-a}{1-a}}}{2x} = 1 - (-2x)^{\frac{2a}{1-a}}. \end{aligned}$$

(3) Let $\varphi(x) = \psi_{-1}(x) - \frac{\log(-2x)}{1-a}$. This proof follows closely that of Theorem 8.1, but instead of relation (8.1) we use (9.3), and Lemma 8.3 is replaced by Lemma 9.2. \square

Proceeding in a similar manner as in Section 8, we get an expansion of ψ_{-1} in terms of $Z = (-2x)^{\frac{2a}{1-a}}$:

$$\psi_{-1}(a, x) \sim \frac{\log(-2x)}{1-a} + \frac{1}{1-a} (-2x)^{\frac{2a}{1-a}} + \frac{1+3a}{2(1-a)^2} (-2x)^{\frac{4a}{1-a}} + \frac{10a^2+7a+1}{3(1-a)^3} (-2x)^{\frac{6a}{1-a}}. \tag{9.4}$$

Example 9.3. In the spirit of Example 8.4 we let $A_{-1}(a, x)$ denote the asymptote in (9.4) and define the difference $\Delta_{-1}(a, x) = \psi_{-1}(a, x) - A_{-1}(a, x)$. The next term of A_{-1} is $O(x^{\frac{8a}{1-a}})$ which then predicts the order of Δ . Choosing $a = 1/2$ should then give $\Delta_{-1}(1/2, x) = O(x^8)$. Indeed, a log-log plot of Δ versus x for $L_a < x < L_a/1000$ gives an line with slope 8 and we estimate from the fitted line that $\Delta_{-1}(1/2, x) \sim 21124 x^8$ when $x \rightarrow 0^-$. In fact we have $\Delta_{-1}(1/2, -0.1) \sim 0.00026$, $\Delta_{-1}(1/2, -0.01) \sim 2.12 \times 10^{-12}$ and $\Delta_{-1}(1/2, -0.001) \sim 2.11 \times 10^{-20}$.

Choosing $a = 3/4$ we should have $\Delta_{-1}(3/4, x) = O(x^{24})$ and we get the estimate $\Delta_{-1}(3/4, x) \sim 4.91 \times 10^{10} x^{24}$. This matches very well with the actual values, for example, $\Delta_{-1}(3/4, -0.1) \sim 4.91 \times 10^{-14}$ and $\Delta_{-1}(3/4, -0.01) \sim 4.91 \times 10^{-38}$.

Decreasing a to $1/4$ we analogously estimate $\Delta_{-1}(1/4, x) \sim 34.5 x^{8/3}$, which matches the actual difference at $x = -0.01$ which is $\Delta_{-1}(1/4, -0.001) \sim 3.55 \times 10^{-7}$. \square

10. Conclusions and future work

Building on the connection between the Ising model and p, q -binomials [17] we have defined functions ψ and ω where the two branches of ψ are inverse functions of $f(a, w) = \sinh(aw)e^{aw}$. This function turns up naturally when controlling the mode a of the p, q -binomial distribution, corresponding to the spontaneous magnetisation of the Ising model. The transition function ω then maps between the two branches of ψ and is a natural generalisation of a corresponding function for mapping between branches of the Lambert W function.

Elementary techniques allowed us to give, for example, simple explicit formulas for ψ and ω when $a = 1/3$. Later, using techniques for solving cubic and quartic equations, we obtained this also for a few rational values, $a = 1/2, 1/5, 3/5$ and $1/7$. A general explicit formula for ψ or ω appears difficult to obtain. An immediate interesting question is whether explicit formulas for ψ or ω can be found for other, larger, families of a .

From Propositions 3.1 and 3.3 we can express derivatives and the primitive function of $\psi(a, x)$ with respect to x . Then Theorem 5.1 provided us with a useful parametrization of both branches of ψ which ultimately allowed for a definite integral of ω stated in Theorem 5.4.

The Lagrange Inversion Theorem allowed us to compute the series expansion of $\psi_0(a, x)$ around $x = 0$. However, to obtain the series expansions of both branches of ψ around the minimum of $f(a, w)$ we used the elementary but convenient Proposition 7.1.

The remaining sections were spent on obtaining bounds and asymptotes of both branches of ψ . Thus the asymptote for $\psi_0(a, x)$ as $x \rightarrow \infty$ is given by (8.3) while the asymptote of $\psi_{-1}(a, x)$ as $x \rightarrow 0^-$ is given by (9.4). Extending these series to more terms is now a matter of computing Bell polynomials and the recipe should be clear. However, the upper and lower bounds on $\psi_0(a, x)$ stated in Theorem 8.1 require a big enough x to hold. This lower bound on x could certainly be improved as numerical experiments mentioned in the Remark.

The results stated in this article treat a as a fixed parameter in, e.g., $\omega(a, z)$. However, since a controls the mode of the p, q -binomial distribution, it would be of great interest if properties of ψ and ω could be obtained for fixed z and small a , i.e., right at the onset of the corresponding phase transition in the Ising model. For example, the first few terms of a series expansion of $\omega(a, z)$ for small a and fixed z can be computed as

$$\omega(a, z) = w + \frac{wz^2}{6(1+w)} a^2 + \frac{(3-4w-2w^2)wz^4}{360(1+w)^3} a^4 + \dots$$

where $w = \omega(0, z)$. Can explicit formulas for ω in a be found? Also, what is the asymptotic behaviour of $\psi_0(a, 1)$ (or for any fixed x) when $a \rightarrow 0^+$?

We have not mentioned the subject of numerical computation of ψ or ω but rather focused on their analytical properties. Using builtin routines of Mathematica to numerically solve, e.g., $f(a, w) = x$ for $w = \psi(a, x)$, has thus been sufficient for our purposes. However, for extreme values of a or x the situation could be improved, especially regarding ψ_{-1} . It would be valuable to obtain efficient routines for high-precision computation of ψ and ω . The techniques mentioned in [10] should be quite applicable, including that of employing Padé approximants.

Data availability

No data was used for the research described in the article.

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