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This is the published version of a paper presented at *The fourteenth research seminar of the Swedish Society for Research in Mathematics Education, MADIF14, Örebro, Sweden, March 19–20, 2024.*

Citation for the original published paper:

Olsson, J., Larsson, M., Granberg, C., Persberg, A-S., Salomonsson, L. et al. (2024) Designing teaching for creative mathematical reasoning combined with retrieval practice
In: Johan Häggström; Cecilia Kilhamn; Linda Mattsson; Hanna Palmér; Miguel Perez; Kerstin Pettersson; Ann-Sofi Röj-Lindberg; Anna Teledahl (ed.), *Mediating mathematics: proceedings of MADIF14* (pp. 25-35). Göteborg: Svensk förening för MatematikDidaktisk Forskning - SMDF
Skrifter från svensk förening för matematikdidaktisk forskning

N.B. When citing this work, cite the original published paper.

Permanent link to this version:
http://urn.kb.se/resolve?urn=urn:nbn:se:umu:diva-225735

Designing teaching for creative mathematical reasoning combined with retrieval practice

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The aim of the study was to explore teacher support to students' mathematical reasoning when they retrieve and use prior knowledge to learn new mathematical content. Two lessons were designed, one with the purpose to engage students in retrieving prior knowledge of angles and shares, and the following to learn how to construct pie-charts. The teacher's interactions with students were recorded and the result showed that the teacher could support students' reasoning by showing interest in their thinking and asking challenging questions regarding clarification and justification of their mathematical reasoning.

During the last decades, research in mathematics education have approached an agreement of that students, in substantial parts of teaching, should be active in solving and justifying solutions of mathematical problems (see e.g. Schoenfeld & Sloane, 2016; Lester & Cai, 2016). However, there are several issues to consider if such a teaching approach would be successful in terms of sustainable learning. Criticism directed to approaches of learning through problem solving often concerns students' ability to handle a large amount of information in working memory (see e.g. Kirschner et al., 2006; Sweller, 2020). Solving mathematical problems is complex and demands the solver to use prior knowledge to construct and assess the plausibility of the solution. If working memory is overloaded no learning will take place (Sweller, 2020). On the other hand, there is research showing that students who practice tasks that require construction of the solution learn better compared to those who solve the same tasks, but are provided with methods (see e.g. Jonsson et al., 2014). However, students who are not successful in constructing useful methods during practice do not learn what was intended (Olsson & Granberg, 2019). Hence, there is an opportunity to increase learning outcomes by appropriate teacher assistance to students' problem solving, provided that the teacher does not reveal a solution method.

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Häggström, J., Kilhamn, C., Mattsson, L., Palmér, H., Perez, M., Pettersson, K., Röj-Lindberg, A.-S. & Teledahl, A. (Eds.) (2024), *Mediating mathematics. Proceedings of Madif 14*. SMDF.

Creative mathematical reasoning and retrieval practice

A way to understand what support students need is to pay attention to their reasoning. Lithner (2008) has defined reasoning in which students construct their own solution method and formulate arguments for the solutions as *Creative mathematical reasoning* (CMR). We have for several years investigated how a teacher can support students' CMR (see e.g. Olsson & D'Arcy, 2022; Olsson & Granberg, 2022) and recurrently noticed that students often need assistance to retrieve the prior knowledge required to solve the current task. A concept of retrieving knowledge from long-term memory is *Retrieval practice* (RP) which in studies has shown to strengthen learning and memory (Dunlosky et al., 2013). Theoretically, the retrieval activity creates stronger associations among the individual's prior knowledge (ibid.). Hence, since learning mathematics by CMR has shown creating strong memory, CMR may enhance RP. On the other hand, RP may strengthen the ability to retrieve necessary knowledge to engage in CMR. Therefore, a combination of CMR and RP is of interest when designing mathematical teaching.

In this study, problems are defined as tasks for which students do not have a pre-existing method for solving the problem, i.e., they need to construct (parts of) the method themselves. Lithner (2008) describes the cognitive process of constructing methods and providing mathematical founded arguments supporting that method as CMR. A series of studies have shown that students who during interventions engaged in CMR to solve problems outperform, on post-tests, students who were using provided methods. (e.g. Jonsson et al., 2014; Norqvist et al., 2019). Recent studies have looked further into learning outcome depending on practicing on CMR-tasks or on tasks that provide a solution method. Jonsson et al. (2020) conducted a series of experiments, which showed that students who solved CMR-tasks (constructing methods) gained a better mathematical understanding, which they could use during post-tests to solve transfer tasks. Moreover, Norqvist et al. (2019) found that students who engaged in CMR tasks were more attentive to the mathematics embedded in the problem compared to those who were provided with a method. That is, constructing a method is likely to promote a deeper mathematical understanding compared to merely implementing given methods. Furthermore, Olsson and Granberg (2019) found in their study that knowledge gained through CMR resulted in better memory consolidation and a more comprehensive mathematical understanding. During a post-test, these students could apply what they practiced to solve problems more effectively compared to them who practiced using provided methods. The aim of this study is to explore teacher support to combine CMR and RP. The context is two planned lessons, one where students retrieve usable prior knowledge and the following where students with aid of that prior knowledge learn new mathematical content for them. The research questions are: (1) *How can a teacher assist students to retrieve prior knowledge?* and

(2) *In what ways can a teacher support students to utilize the prior knowledge they retrieved in a preceding lesson when learning new mathematical content?*

Framework

Theoretically, the teaching design is based on the definitions of CMR and RP. These concepts are operationalized by principles of teaching for CMR developed in our earlier studies (Olsson & D'Arcy, 2022; Olsson & Granberg, 2022).

The CMR framework emphasizes the active process of creating and constructing knowledge, particularly students' ability to create their own solutions of tasks and formulate mathematical arguments for the solution. A CMR-task does not include a method to employ. The RP framework, on the other hand, emphasizes memory consolidation through the act of actively recalling information from memory. Theoretically, through the active process of retrieving knowledge from long-term memory, RP starts a process of re-consolidation which creates stronger connections between memorized items.

Our earlier studies focus on developing principles for teaching where students learn through CMR (Olsson & D'Arcy, 2022; Olsson & Granberg, 2022). The principle for CMR-teaching is: *If the goal is that students will learn mathematics by CMR, tasks and teacher-student interactions should guide students into initiating, developing, justifying, and verifying their reasoning when constructing and formulating arguments for the solution*. This study aims at exploring possibilities for combining CMR and RP and the proposition (initial principle) for RP is: *If students are supposed to use prior knowledge when learning new mathematical content, students should, in a separate lesson before learning new mathematics, have opportunities to activate adequate prior knowledge*. These principles will guide the intervention's design and constitute a base for outcome conjectures, which in turn will guide the analysis.

Method

The method is based on the design of an intervention guided by the earlier presented principles for CMR and RP. Tasks and teacher support were prepared with the intention to engage students in CMR, and the students' prior knowledge was considered. Conjectures were made regarding how the tasks would engage students in constructing the solutions and how the teacher could support initiating, developing, justifying, and verifying reasoning. These conjectures were foci for the analysis.

Methodology

The current paper reports from an intervention within a larger project, in which a teaching design is developed and investigated. The innovative character of

the study means it is not likely that regular classrooms will meet the conditions the investigations require. Therefore, teaching aiming at promoting CMR must be designed (see Cobb et al., 2016). Design-based research methodology combines the construction of scientific knowledge and its applicability in educational contexts (Cobb et al., 2016; Schoenfeld, 2007). A fundamental issue of design research is that principles guide rather than determine a design and therefore require creative input, and development through feedback on trials (Swan, 2008). Typically, a design-based project is carried out in iterative interventions, each intervention starting in assumptions of the outcome. The design of a specific intervention (as in this paper) is based on principles developed in previous interventions within a project. Analysis of outcomes of the specific intervention guides developments and revisions of the principles (Cobb et al., 2016).

Participants

22 students aged 13–14 years agreed to participate in the study. Their parents were informed and signed agreements for the students to participate. The teacher considers the class to be on an average level in terms of mathematical ability and there are both high and low achievers. The teacher is experienced and participates in a long-term project this study is a part of. This is her first attempt to teach according to the design.

Intervention design

The intervention consisted of two lessons. The overall learning goal was to construct pie-charts and it was considered the students could use prior knowledge of angles, fractions, decimal numbers, and percent. The first lesson aimed at activating that prior knowledge. The goal was that all students would retrieve definitions of angles, how to measure angles, the sum of angles in rectangles and triangles, that a full circle is 360° and how to calculate percent and shares. The tasks both had the character of retrieving factual knowledge (e.g. how many degrees is a full circle?) and explaining concepts (e.g. explain why 1/4 is 25 %). The teacher prepared in collaboration with the researchers how to interact with the students. Conjectures were made that solving the tasks in pairs would engage students in remembering at least parts of properties such as acute angle, right angle, obtuse angle, and abilities to measure and construct angles by ruler and protractor. Furthermore, if parts were not remembered the students were anticipated to, in collaboration with peers and the teacher, retrieve those parts by CMR.

The researchers created, together with the teacher, conjectures on how teacher-student interactions that do not involve facts and procedures for how to solve the task can promote students to retrieve relevant knowledge via RP (knowledge that the student needed to engage in CMR). Once the students had retrieved prior knowledge, they would work together to formulate arguments for

how the retrieved knowledge is correct. In cases where the students do not try to retrieve prior knowledge, the teacher needs to challenge them to do so. Furthermore, the teacher should not verify students' conclusions as correct or not, but instead ask questions like *how can you be sure you are right?* and prompts like *discuss with your peers*. Considering that it was the teacher's first attempt; the conjectures were deliberately few and assessed as reasonable to implement.

The goal of the second lesson was for the students to learn how to construct a pie chart and to do that they needed to find methods to translate percentages into degrees. The students were given three sub tasks. In the first, students could draw simple pie charts without translating percentages to degrees (e.g. 50 % and 25 %). The second task was slightly more difficult, but students were able to draw pie charts by making estimates (e.g. 20 %). The third task was designed so that students would see the need to translate % into degrees (e.g. 17 %). Conjectures for the second lesson were that all students would promptly construct the charts of 50 % and 25 %. Thereafter, they would estimate the chart of 20 %. When finding the chart of 17 % difficult, they would engage in reasoning how to, based on the knowledge that a full pie-chart is both 100 % and 360°, calculate how many degrees are 17 %. Conjectures on how teacher-student interactions would support students were to not verify correct or incorrect answers but rather ask the students how they could be sure whether their answers were correct or not. Difficulties anticipated were to translate percent into degrees. Here the teacher could encourage them to use the knowledge that a full circle is 360°, and if necessary, ask what percentage a full circle represents.

The structure of the lessons and data collection

The teacher had a brief introduction informing the students they would work on tasks in pairs. During the first lesson, the students were told they had met the mathematical content before and for the second that they would have a chance to learn how to construct pie charts. The students in both lessons were informed that they could ask the teacher for help whenever they needed, and they presented the solutions on separate papers. The teacher and a researcher carried a voice-recording device each. The teacher had the routine to in every interaction mention the students' names in purpose to facilitate pairing of recordings from both lessons. The researcher followed the teacher and after a teacher-student interaction the researcher stayed a while to catch reasoning between students after interacting with the teacher. Thereafter the researcher caught up with the teacher. The students were told that they could not expect any help from the researcher. However, the researcher could ask questions aiming at clarifying parts in students' reasoning. Hence, data consisted of recordings of teacher-student interactions, students reasoning after the interactions, and students' written solutions.

Analysis method

The voice-recordings were transcribed verbatim. Teacher-student interactions were then identified where the students either recalled prior knowledge, related this to new knowledge or gave explanations thereof. Then, a first step was to characterize the interactions based on the contributions from students and the teacher, respectively. Patterns in the interactions were searched for and categories were formed. Thereafter, based on conjectures of how the design would work, it was described how the teacher aided students to retrieve prior knowledge (lesson 1) and supported students to utilize prior knowledge when learning how to construct pie charts (lesson 2).

Results

During lesson 1, the retrieval lesson, 23 teacher-student interactions were recorded. In five of them the students had solved a task (or several tasks) correctly and explained their solutions for the teacher. In the other 18, the students needed teacher support. Initially, the students rather easy retrieved factual knowledge such as labels of different angles and procedures like measuring acute angles. Some arguments, typically on request from the teacher, were expressed. Most students succeeded in measuring acute angles while some discussions between students and the teacher could be observed when measuring obtuse angles. However, difficulties appeared when measuring angles larger than 180°.

Student C: This one is not 50°. This is not the outside. Is it this one you must measure?

Teacher: You do not know which one to measure. How did you do in the earlier ones? Why don't you know now?

Student D: This one is in the wrong direction. It is on the outside. Outside the [small] angle.

Teacher: Okay, how do you do then?

Student D: Well, the small angle (students work in silence).

Teacher: Ah, you are measuring the small angle. Why do you subtract from 360°?

Student D: Because that is the whole [circle].

The teacher, typically, did not verify whether the students were correct or not. The questions support the students to use knowledge on how to measure angles smaller than 180° in reasoning how to with the aid of a 180°-protractor calculate an angle larger than 180°. What was not obvious but reasonable is that the students realized that if they know the smaller angle, they could use the knowledge that a whole circle is 360° to calculate the larger angle. This is an example where the teacher could support students to retrieve knowledge both as fact and through reasoning without providing facts or procedures.

The second lesson on learning how to construct pie charts included 18 teacher-student interactions. Five of them could be categorized as the teacher supported students to retrieve content from the retrieval lesson and to use that knowledge to solve the current tasks. In five interactions the students managed to retrieve knowledge from the retrieval lesson but needed teacher support to solve the current tasks. In eight interactions the students had successfully solved tasks and explained them for the teacher. Initially the lesson followed the anticipated path, the students promptly constructed pie charts with a 50 % pie and with a 25 % pie. Some differences in the approach of the 20 % and 80 % chart were observed. Most students recognized that 20 % is 1/5 and estimated how to create a corresponding pie in the chart. Some students successfully realized that 1/5 of 360° is 72° and in the third task, constructing a chart with a 17 % pie, they replicated the approach. The following is an example where the teacher supported students to retrieve and use knowledge from the previous lesson. Student O and F had without problems constructed pies of 50 % and 25 %. Approaching the task to construct a pie of 20 %, they initially retrieved knowledge that 20 % is 1/5, which they used to estimate a pie. However, when the teacher asked them how they could be sure that the pie was 20 %, they could not answer. The teacher asked them if they could measure how many degrees the pie constitutes.

Student O: I don't know how to do that.

Teacher: Do you remember what we did yesterday? You constructed and measured angles.

Student F: [student measures the angle] 45°

Teacher: Okay, is that correct? 45° is 20 %? Can you check that?

Students: [silence]

Teacher: Is that correct, that 45° is 1/5? What is 1/5?

Student O: 1/5 of 360.

Teacher: Try that [leaves the students]

This is an example of an interaction when the teacher supported the students to retrieve content from the earlier lesson. Later the teacher returns to students O and F.

Teacher: Okay, tell me what you have done.

Student F: This is 17 % and I need to know what that is in degrees. So, I divided 360 by 100 to know how many degrees is 1 %. Then I multiplied 3.6 with 17 and that is 61,2 which is approximately 61. Then I constructed an angle of 61°.

In this sequence, the students have solved the task, reasonably aided by the support they earlier had from the teacher. Quite a few students did not interact with the teacher until they had solved all the tasks. The following is an example where the students had solved the task before interacting with the teacher, who typically asked them to explain what they had done and how they were thinking.

Student A: We divided 360 by 5.

Teacher: Why did you divide it by 5?

Student A: Because 20 % equals 1/5.

Teacher: In the other task [17 %] it seems like you had another strategy?

Student A: Yes, we divided by 100 and then multiplied with 17.

Teacher: And?

Student A: Then we had 61,2 which is approximately 61, and then we constructed the angle.

These students use content from the retrieval lesson with confidence. The teacher does not need to support the students' learning, but her questions allow the students to verify that they have the knowledge according to the lesson's learning goal. However, there were examples when students apparently had used content from the retrieval lesson but still had problems with the task asking them to construct a chart with a 17 % pie. The following is an example where the teacher supported students to solve the task with aid of the knowledge from the earlier lesson that the students have retrieved independently. Student H and I had successfully calculated the degrees for 20 % and constructed the angle. Now they reason that 17 % must be slightly smaller.

Teacher: Okay, but how can you do to get the exact size?

Student I: I think that the whole circle is 100 %. We can divide 360 by 100.

Student H: And then we can multiply with 17.

Teacher: So, you divide 360 by 100. What did you get?

Student I: 1 %, 3.6°.

Teacher: And then?

Student I: 3.6 times 17.

From this point the students promptly constructed a 61° angle in the chart. This is an example in which the students obviously used content from the retrieval lesson and needed minor support from the teacher to translate 17 % into degrees. Notably, the teacher support does not include direct instructions on how to calculate the angle, but a question that helped students to focus on the problem; How to transfer 17 % to a fraction. In summary, the teacher's strategy to start any interaction with asking the students to explain their thinking helped her to adjust further support. Students who had solved the tasks independently could be challenged to explain and justify their solutions. Students who in their reasoning revealed that they had retrieved necessary prior knowledge could be guided to focus on the challenging parts of new content, and students who did not use content from the retrieval lesson could be notified of that.

Discussion

This study asks how a teacher can support students to retrieve prior knowledge and how a teacher can support students to use the retrieved knowledge when learning new mathematical content. The result indicates that quite a few students need to reason in retrieving mathematical content not taught recently. In the first lesson most students in this sample were able to do that. A reason can be the teacher's approach to ask students to explain their thinking and give them time to do so. Furthermore, splitting the activities of activating prior knowledge and learning new content may help students to concentrate on retrieving earlier learned mathematics rather than try to understand new content and retrieving necessary knowledge simultaneously. From the teacher's perspective, it may also be easier to support students' knowledge retrieval in a separate lesson than when students need support to use prior knowledge when learning new mathematics. In our earlier studies a recurrent issue has been to support students' retrieval of prior knowledge when solving CMR-tasks aiming at learning new content (Olsson & Granberg, 2022; Olsson & D'Arcy, 2022).

In the second lesson, some students did not spontaneously use content from the retrieval lesson. However, after the teacher encouraged them to retrieve and use such content, there were examples when students promptly solved all tasks including the most challenging one with a 17 % pie. It is reasonable that even though the students needed support for the retrieval, knowledge from the first lesson could be retrieved more easily compared to if there had not been such a lesson. For the teacher, the knowledge that the students retrieved appropriate prior knowledge in the previous lesson may help formulating supporting questions. Both the first and the second lesson were designed based on the principle for CMR, which means the teacher encourages the students to initiate, develop, justify, and verify their reasoning. Experiments building on similar theories have been found increasing the ability to transfer knowledge into new tasks (Jonsson et al., 2020, Olsson & Granberg, 2019). In summary, in this intervention, the design seems to facilitate some of the known difficulties associated with learning new mathematical content through CMR. Anyhow, further research is needed, particularly the significance of the retrieval activities must be investigated.

The principle for CMR-teaching has not primarily been tested in this study. It has guided the design of the intervention and much speaks for that the conditions allowed students to initiate, develop, justify, and verify reasoning, which is a result replicated from other studies (e.g. Olsson & Granberg, 2022). However, the proposition for RP which guides the design of separating activation of prior knowledge from using prior knowledge when learning new content, may be evaluated based on the results. All students to some extent needed to reason when retrieving prior knowledge. Objections to teaching where students learn from problem solving often state that there is too much load on working memory

both to retrieve prior knowledge and implement a solution strategy (Kirschner et al., 2006; Sweller, 2020). We consider that allowing students to activate prior knowledge in a separate lesson close to learning new content will facilitate the retrieval of necessary prior knowledge when solving problems aiming at learning new content. The results showed quite a few students who fluently or with minor support from the teacher retrieved knowledge from the first lesson when constructing pie charts. However, there were some students who in the second lesson needed extended teacher support to retrieve necessary prior knowledge. These students may benefit from an additional separate retrieval activity. Studies of RP have shown that repeated retrieval activities will further strengthen memory and learning, provided the retrieval is active on the student's behalf (Dunlosky et al., 2013). We suggest that the design of the intervention should be developed to, in the second lesson, include a short activity to retrieve content from the first lesson before introducing the tasks aiming at learning new content. Thus, we propose the proposition for RP will be developed as: *If students are supposed to use prior knowledge when learning new mathematical content, students should, in a separate lesson before learning new mathematics, have opportunities to retrieve adequate prior knowledge, and in the following lessons, start with an activity engaging them in retrieving the content from the previous lesson*.

Acknowledgement

The research is funded by Skolforskningsinstitutet.

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