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Does it move?

Euclidean and projective rigidity of hypergraphs.

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Abstract

Rigidity theory is the mathematical study of rigidity and flexibility of discrete structures. Rigidity theory, and the related field of kinematics, have a wide range of applications to fields such as material science, robotics, architecture, and computer aided design.

In rigidity theory, rigidity and flexibility are often studied as properties of an underlying combinatorial object. In this thesis, the aim is to study rigidity theoretic problems where the underlying combinatorial object is an incidence geometry. Firstly, we study rigidity problems for realisations of incidence geometries of rank 2 as points and straight lines in the plane. Finding realisations of incidence geometries as points and straight lines in the plane is an interesting problem in its own right that can be formulated as a problem of realisability of rank 3 matroids over the real numbers.

We study motions of rod configurations, which are realisations of incidence geometries as points and straight line segments in the plane, where each line segment is treated as a rigid rod. Specifically, motions of a rod configuration preserve the distance between any two points on a rod. We introduce and investigate a new notion of minimal rigidity for rod configurations. We also prove that rigidity of a rod configuration is equivalent to rigidity of a graph, under certain geometric conditions on the rod configuration. We also find realisations of v_3-configurations that are flexible as rod configurations for $v \geq 28$. We show that all regular realisations of v_3-configurations for $v \leq 15$, and triangle-free v_3-configurations for $v \leq 20$ are rigid as rod configurations.

We also consider motions of realisations of incidence geometries as points and straight lines in the plane which preserve only incidences between points and lines. We introduce the notion of projective motions, which are motions of realisations of incidence geometries as points and straight lines in the projective plane which preserve incidences. Furthermore, we introduce the basic tools for investigating rigidity with respect to projective motions. We also investigate the relationship between projective rigidity and higher-order projective rigidity.

Finally, we introduce a sparsity condition on graded posets, and introduce an algorithm which can determine whether a given graded poset satisfies the sparsity condition. We also show that sparsity conditions define a greedoid.

Sammanfattning

Stelhetsteori är det matematiska område som behandlar stelhet och rörlighet av diskreta strukturer. Stelhetsteori, och det nära relaterade området kinematik, har tillämpningar inom områden som till exempel materialvetenskap, robotik, arkitektur och datorstödd konstruktion (CAD).

Inom stelhetsteori så studeras ofta stelhet och rörlighet som egenskaper hos ett underliggande kombinatoriskt objekt. I denna avhandling är målet att studera stelhetsteoretiska problem där det underliggande kombinatoriska objektet är en incidensgeometri. Vi studerar först stelhetsproblem för realiseringar av incidensgeometrier av rang 2 som punkter och räta linjer i planet. Att hitta realiseringar av incidensgeometrier som punkter och räta linjer i planet är ett intressant problem i sig, som till exempel kan formuleras som ett problem om realiserbarhet av matroider av rang 3 över de reella talen.

Vi studerar rörelser av stavkonfigurationer, det vill säga realiseringar av incidensgeometrier som punkter och linjesegment i planet, där varje linjesegment behandlas som en stel stav. Mer specifikt så bevarar rörelser av stavkonfigurationer avståndet mellan alla par av punkter som tillhör samma linjesegment. Vi introducerar och undersöker en ny definition av minimal stelhet för stavkonfigurationer. Vi bevisar också att stelhet av en stavkonfiguration är ekvivalent med stelhet av en graf, givet vissa geometriska villkor på stavkonfigurationen. Vi hittar också v_3-konfigurationer för $v \geq 28$ som är stela som stavkonfiguratoner. Vi visar också att alla reguljära realiseringar av v_3-konfigurationer för $v \leq 15$ och triangelfria v_3-konfigurationer för $v \leq 20$ är stela som v_3-konfigurationer.

Vi behandlar också rörelser av realiseringar av incidensgeometrier som punkter och linjer som bara bevarar incidenser mellan punkter och linjer. Vi introducerar begreppet projektiva rörelser, som är rörelser av realiseringar av incidensgeometrier som punkter och linjer som bevarar incidenser. Vi introducerar också de grundläggande verktyg som behövs för att undersöka stelhet med avseende på projektiva rörelser. Vi undersöker också relationen mellan projektiv stelhet och projektiv stelhet av högre ordning.

Slutligen så introducerar vi ett gleshetsvillkor på graderade partialordnade mängder, och inför en algoritm som kan avgöra ifall en given graderad partialordnad mängd uppfyller gleshetsvillkoret. Vi visar också att gleshetsvillkoret definierar en greedoid.

List of papers

The following papers are included in this thesis:

Paper I. Exploring the infinitesimal rigidity of planar configurations of points and rods.
S. Lundqvist, K. Stokes and L-D. Öhman, *Discrete Applied Mathematics* **336**, 68-82 (2023)

Paper II. When is a planar rod configuration infinitesimally rigid?
S. Lundqvist, K. Stokes and L-D. Öhman, *Discrete Comput. Geom.* **73**, 25–48 (2025)

Paper III. Projective rigidity of point-line configurations in the plane.
L. Berman, **S. Lundqvist**, B. Schulze, B. Servatius, H. Servatius, K. Stokes and W. Whiteley, *arXiv preprint, arXiv:2407.17836*

Paper IV. Counting for rigidity under projective transformations in the plane
L. Berman, **S. Lundqvist**, B. Schulze, B. Servatius, H. Servatius, K. Stokes and W. Whiteley, *arXiv preprint, arXiv:2503.07228*

Paper V. Sparse Posets and Pebble Game Algorithms.
S. Lundqvist, T. Randrianarisoa, K. Stokes and J. Vermant, *manuscript*

Paper VI. Applying the pebble game algorithm to rod configurations.
S. Lundqvist, K. Stokes and L-D. Öhman, In: EuroCG 2023: Book of abstracts (2023), *Extended Abstract*

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Contents

1 Introduction

Consider a quadrilateral built from rigid bars connected in joints (Figure 1). As indicated in Figure 1, there is a motion of the quadrilateral that preserves the lengths of the bars. In other words, the quadrilateral is flexible. A structure consisting of rigid bars connected in joints that is not flexible is rigid. Notice that the motion of the quadrilateral changes the distance between any pair of points not connected by a bar. Hence, if a diagonal bar is added to the quadrilateral, it will no longer be possible to move the quadrilateral in such a way that the lengths of the bars are preserved. In other words, adding a diagonal makes the quadrilateral rigid. Determining whether a structure consisting of rigid bars connected in joints is flexible or rigid is a classical question in rigidity theory.

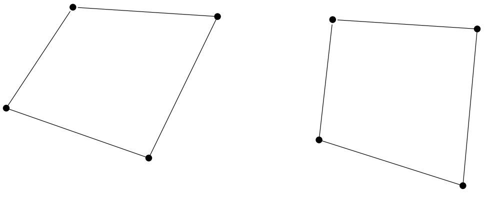

Figure 1: A motion of the quadrilateral that preserves edge-lengths.

Determining whether a structure consisting of rigid bars connected in joints is rigid can be treated as a combinatorial problem. The underlying combinatorial object of the square is a *graph*, i.e. a pair $G = (V, E)$ where V is a finite set of *vertices* and $E \subseteq V \times V$ is a set of *edges*, which consist of pairs of vertices. In this thesis, we will always consider undirected, finite, simple graphs without loops, although directed graphs, graphs with loops, infinite graphs and multigraphs all appear in rigidity theory. We will denote an edge between two vertices u and v by (u, v). The underlying graph has been given a geometric realisation by assigning a point in $\mathbb{R}^2$ to each vertex of the graph. Such a realisation is called a *framework* of the graph. More generally, one can consider frameworks in $\mathbb{R}^d$.

Maxwell showed that a graph that has a rigid framework must satisfy a set of counting conditions on the number of edges and vertices [29]. In the plane, any graph that satisfies Maxwell's necessary conditions, also has rigid frameworks [21, 34]. In Euclidean space of dimension three and higher, there are graphs that satisfy Maxwell's necessary conditions that do not have any rigid frameworks. Finding a characterisation of the graphs that have minimally rigid frameworks in $\mathbb{R}^d$ is one of the main open problems in the field.

In the rigidity problem described above, the motions preserve lengths of edges. There are, however, many other variations of the rigidity problem for frameworks of graphs. Rigidity can also be studied with respect to for example motions which preserve angles between edges, or directions of edges. A variant of the rigidity problem, with applications in computer aided design, is studying motions that preserve the lengths of some edges, and the directions of other edges [40]. Another variant is to consider frameworks in special position, for example symmetric frameworks, which is the topic of Section 4. Rigidity of frameworks of graphs can also be studied in other ambient spaces, for example in other normed spaces or on surfaces.

The aim of this thesis is to study rigidity problems where the underlying combinatorial object is an *incidence geometry*. An *incidence geometry* (of rank 2) is a triple of sets $S = (P, L, I)$, where P is a set of *points*, L is a set of *lines* and $I \subseteq P \times L$ is a set of *incidences*. We will assume throughout that there is at most one line incident to any pair of points. An incidence geometry of rank n consists of n sets $\{P_1, P_2, ..., P_n\}$ and a set of incidences $I \subset \cup_{i<j\leq n} P_i \times P_j$.

There are also other descriptions of hypergraphs of rank 2. A *hypergraph* is a pair $G = (V, E)$, where V is a finite set of *vertices*, and E is a set of *hyperedges*, which are subsets of vertices. Note that incidence geometries of rank 2 are hypergraphs, with vertex set P and hyperedges consisting of the sets of points that are incident to a common line. An incidence geometry can also be represented as a bipartite graph $G = (V, E)$, where $V = P \cup L$ and $E = I$.

Firstly, we consider motions of realisations of incidence geometries as points and straight lines [26, 25, 24]. We consider rigidity of rod configurations, which are realisations of incidence geometries as points and straight line segments where the line segments move as rigid bodies. The motions of rod configurations generalise distance-preserving motions of graphs, such as the motion of the quadrilateral in Figure 1, to structures of points and lines where lines can contain more than two points.

We also consider projective rigidity of realisations of incidence geometries as points and straight lines in the real projective plane. Projective motions preserve only incidences of points and lines, i.e. which points lie on which lines. Projective rigidity turns out to be closely related to incidence theorems in the real projective plane [3, 4].

A *poset* is a pair $\mathcal{P} = (X, \leq)$, consisting of a *ground set* X and a partial order $\leq$ on X. A *chain* in a poset $\mathcal{P} = (X, \leq)$ is a subset $C \subset X$ such that C is totally ordered. A *maximal chain* in $\mathcal{P} = (X, \leq)$ is a chain C in $\mathcal{P}$ such that $C \cup \{x\}$ is not totally ordered for any $x \in X \setminus C$. A *graded poset* is a poset such that every maximal chain has the same length. A graded poset defines an incidence geometry of rank n, where n is the length of its maximal chains, by setting $P_i = \{x \in X \mid r(x) = i\}$ and $(p_i, p_j) \in I$ whenever $p_i \leq p_j$.

Maxwell's necessary counting conditions can be generalised to a class of sparsity conditions which under certain conditions define matroids (see Section 3).We define similar sparsity conditions for graded posets, and define a fast algorithm for checking whether the sparsity condition is satisfied, and show that the sparsity conditions for graded posets do not define matroids, but greedoids.

2 Rigidity of bar-joint frameworks

A *bar-joint framework*, or *framework* (G, ρ) in $\mathbb{R}^d$ consists of a graph $G = (V, E)$ and an assignment $p : V \to \mathbb{R}^d$. See Figure 2 for two examples of frameworks of the complete graph K_4.

Two frameworks of a graph $G = (V, E)$ are said to be *equivalent* if the lengths of the edges are the same in both frameworks, i.e. two frameworks (G, ρ) and (G, ρ') are equivalent if

$$||\rho(v_i) - \rho(v_j)||^2 = ||\rho'(v_i) - \rho'(v_j)||^2$$

for all edges $(v_i, v_j) \in E$. Two frameworks of G are said to be *congruent* if the distance between any two pairs or vertices is the same, i.e. two frameworks ρ and ρ' of a graph G are congruent if

$$||\rho(v_i) - \rho(v_j)||^2 = ||\rho'(v_i) - \rho'(v_j)||^2$$

for *all* pairs $(v_i, v_j) \in V \times V$. Note that the frameworks that are congruent to (G, ρ) can be obtained from (G, ρ) by an isometry of Euclidean space. Note also that congruent frameworks are equivalent, but that the converse is not necessarily true.

The two frameworks of the four-cycle in Figure 1 are equivalent, but not congruent, as the distance between the pairs of vertices that are not connected by an edge is different in the two frameworks. The two frameworks of K_4 in Figure 2 are neither equivalent nor congruent.

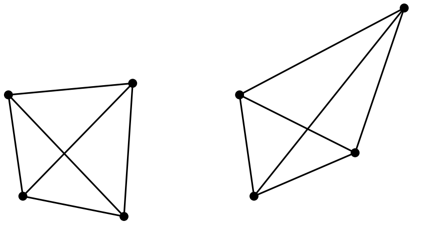

Figure 2: Two frameworks of the complete graph K_4 in $\mathbb{R}^2$.

Given a framework (G, ρ) and an edge $e = (v_i, v_j) \in E$, let $l_e = ||\rho(v_i) - \rho(v_j)||^2$. Given a framework (G, ρ), we can then define the distance map $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ by

$$f_G(\rho(v_0), ..., \rho(v_{|V|})) = (l_{e_1}, l_{e_2}, ..., l_{e_{|E|}}).$$

The *configuration space* of (G, ρ) is $f_G^{-1}(f_G(\rho(v_0), ..., \rho(v_{|V|}))) \subset \mathbb{R}^{d|V|}$. The points of the configuration space of (G, ρ) are the frameworks (G, ρ') that are equivalent to (G, ρ).

Let $K_{|V|}$ denote the complete graph on $|V|$ vertices. A framework (G, ρ) is *(locally) rigid* if there is some neighbourhood U of (G, ρ) in the configuration space such that

$$f_G^{-1}(f_G(\rho(v_0), ..., \rho(v_{|V|}))) \cap U = f_{K_{|V|}}^{-1}(f_{K_{|V|}}(\rho(v_0), ..., \rho(v_{|V|}))) \cap U.$$

In other words, a framework is rigid if all frameworks in a neighbourhood of (G, ρ) in the configuration space are congruent to (G, ρ).

A framework (G, ρ) is *flexible* if it is not rigid. Equivalently, a framework (G, ρ) is flexible if there is a continuous path

$$\gamma : [0, 1] \to f_G^{-1}(f_G(\rho(v_0), ..., \rho(v_{|V|})))$$

such that $\gamma(0) = p$, and $\gamma(t) \notin f_{K_{|V|}}^{-1}(f_{K_{|V|}}(\rho(v_0), ..., \rho(v_{|V|})))$ for some $t \in (0, 1]$ [1]. As stated in the proof of [[1], Proposition 1], the path γ can also be chosen to be analytic, rather than continuous. Since we can choose the paths to be analytic, we can also choose them to be smooth.

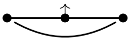

Figure 3: An infinitesimally flexible framework of K_3.

A smooth curve $\gamma(t)$ in the configuration space of a framework (G, ρ) consists of a smooth curve $\rho_t(v_i)$ for each vertex, with $\rho_t(v_i) = \rho(v_i)$. Furthermore, the lengths of the edges are preserved, so if $e = (v_i, v_j) \in E$, then

$$||\rho_t(v_i) - \rho_t(v_j)||^2 = l_e \tag{1}$$

for all $t \in [0, 1]$.

The curve $\gamma(t)$ should be understood as a motion of the vertices of the framework. A motion is *trivial* if it is an isometry of $\mathbb{R}^d$. Conversely, the framework (G, ρ) is flexible if there is a motion of the framework (G, ρ) which is not an isometry. The framework (G, ρ) is rigid if all motions of the framework are isometries.

Rather than studying the motions of a framework directly, it is common to assume that the motions of a framework are smooth, and study their first-order derivatives at time $t = 0$. At $t = 0$, the derivative of an equation of the form (1) is the linear equation:

$$(\rho(v_i) - \rho(v_j))(\rho_0'(v_i) - \rho_0'(v_j)) = 0. \tag{2}$$

Considering $p'(v_i)$ and $p'(v_j)$ as variables gives a linear system of equations. Denote the $|E| \times d|V|$-coefficient matrix of that system of equations by $R_d(G, \rho)$. The matrix $R_d(G, \rho)$ is the *d-dimensional rigidity matrix* of (G, ρ). A typical row of the d-dimensional rigidity matrix corresponding to an edge (v_i, v_j) has the form:

$$\begin{bmatrix} 0 & \dots & 0 & \overset{v_1}{\rho(v_i) - \rho(v_j)} & 0 & \dots & 0 & \rho(v_j) - \rho(v_i) & \overset{v_j}{0} & \dots & 0 & \end{bmatrix}$$

An *infinitesimal motion* of a framework (G, ρ) is an element of the kernel of $R_d(G, \rho)$. In other words, an infinitesimal motion is a vector $m \in \mathbb{R}^{d|V|}$ such that $R_d(G, \rho)m = 0$. An infinitesimal motion consists of a vector in $\mathbb{R}^d$ for each vertex $v \in V$. We will sometimes denote the vector corresponding to a vertex v by $m(v)$.

An infinitesimal motion is *trivial* if it is the linearisation of an isometry of $\mathbb{R}^d$, i.e. if the infinitesimal motion at a vertex v_i can be described as $A\rho(v_i) + b$, for a skew-symmetric matrix $A \in GL_d(\mathbb{R})$ and a vector $b \in \mathbb{R}^d$. A framework (G, ρ) is said to be *infinitesimally rigid* if all infinitesimal motions of (G, ρ) are trivial, otherwise it is said to be *infinitesimally flexible*.

Alternatively, a framework (G, ρ) in $\mathbb{R}^d$ is infinitesimally rigid if $\text{rank}(R_d(G, \rho)) = d|V| - \binom{d+1}{2}$. The dimension of the space of trivial infinitesimal motions of a framework in $\mathbb{R}^d$ is $\binom{d+1}{2}$. If $\text{rank}(R_d(G, \rho)) = d|V| - \binom{d+1}{2}$, then the kernel of $R_d(G, \rho)$ is $\binom{d+1}{2}$-dimensional, and consists of exactly the trivial infinitesimal motions. The following proposition says that infinitesimal rigidity gives a certificate for rigidity of a framework.

Proposition 2.1 ([1])**.** *If a framework (G, ρ) in $\mathbb{R}^d$ is infinitesimally rigid, then it is rigid.*

The converse of Proposition 2.1 is false in general. For example, the framework of K_3 depicted in Figure 3, where all three vertices are collinear, is rigid but infinitesimally flexible. The infinitesimal motion depicted in Figure 3 only exists because the three points are collinear. In most frameworks of K_3, the three vertices are not collinear, so most frameworks of K_3 do not have this infinitesimal motion. As the motion depicted in Figure 3 only exists for frameworks in very special position, it is natural to ask whether the converse of Proposition 2.1 holds for a "typical" frameworks of K_3.

More precisely, a framework (G, ρ) in $\mathbb{R}^d$ is *generic* if the set $\{\rho(v_0), \rho(v_1), ..., \rho(v_{|V|})\}$ is algebraically independent over $\mathbb{Q}$. Notice that the framework of K_3 in Figure 3 is not generic, since the coordinates of the vertices are linearly dependent. For generic frameworks, the converse of Proposition 2.1 holds.

Theorem 2.2 ([1], [13])**.** *A generic framework (G, ρ) in $\mathbb{R}^d$ is rigid if and only if it is infinitesimally rigid.*

Furthermore, for a given graph, all generic frameworks have the same rigidity properties.

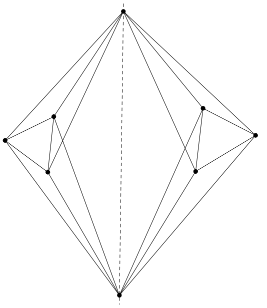

Figure 4: A framework of a flexible graph that satisfies the necessary conditions for being minimally rigid in $\mathbb{R}^3$.

Theorem 2.3 ([1])**.** *Let $G = (V, E)$ be a graph. Then either all generic frameworks (G, ρ) in $\mathbb{R}^d$ are rigid, or all generic frameworks (G, ρ) in $\mathbb{R}^d$ are flexible.*

By Theorem 2.3, rigidity of generic frameworks can be treated as a property of the graph rather than a property of the specific framework. A graph $G = (V, E)$ is *rigid* in $\mathbb{R}^d$ if all generic frameworks (G, ρ) in $\mathbb{R}^d$ are rigid, and *flexible* if all generic frameworks (G, ρ) in $\mathbb{R}^d$ are flexible. Equivalently, by Theorem 2.2, a graph is rigid in $\mathbb{R}^d$ if all its generic frameworks in $\mathbb{R}^d$ are infinitesimally rigid, and flexible otherwise. A graph is *minimally rigid* in $\mathbb{R}^d$ if it is rigid in $\mathbb{R}^d$, but removing any edge results in a flexible graph.

By Theorem 2.3, rigidity of a graph is well-defined. Genericity can, for the purposes of Theorem 2.2 and Theorem 2.3, be replaced by the weaker notion of *regularity*, where a framework is regular if it is a regular point of the distance map f_G.

Having defined rigidity of graphs, it is possible to attempt a combinatorial characterisation of the graphs that are rigid in $\mathbb{R}^d$. Maxwell proved the following necessary conditions for rigidity in $\mathbb{R}^d$.

Theorem 2.4 ([29], [30])**.** *If a graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^d$, then the following holds:*

1. $|E| = d|V| - \binom{d+1}{2}$*, and*
2. $|E'| \leq d|V(E')| - \binom{d+1}{2}$ *for any subset $E' \subseteq E$ with $|E| \geq d - 1$.*

Consider a bar-joint framework (G, ρ) in $\mathbb{R}^1$. By definition, a motion of a bar-joint framework cannot change the distance between a vertex and its neighbours. In $\mathbb{R}^1$, it also holds that if v_1 and v_2 are neighbours and v_2 and v_3 are neighbours, then there is no continuous motion which changes the distance between v_1 and v_3. In fact, given a framework in $\mathbb{R}^1$, there is no continuous motion which changes the distance between a pair of vertices v_i and v_j which are connected by a path. Hence, if the graph is connected, there is no continuous motion of (G, ρ). Conversely, if G is not connected, there clearly is a continuous motion of (G, ρ) which fixes one connected component of G and moves the other.

Hence, a graph is rigid in $\mathbb{R}^1$ if and only if it is connected, and minimally rigid in $\mathbb{R}^1$ if and only if it is a tree. Furthermore, a graph satisfies the necessary conditions of Theorem 2.4 if and only if it is a tree, so in $\mathbb{R}^1$ the conditions of Theorem 2.4 are both necessary and sufficient. The following theorem, first proven by Pollaczek-Geiringer, and later again by Laman, says that the necessary conditions are sufficient also in $\mathbb{R}^2$.

Theorem 2.5 ([21, 34])**.** *A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if:*

1. $|E| = 2|V| - 3$*, and*
2. $|E'| \leq 2|V(E')| - 3$ *for any subset $E' \subseteq E$.*

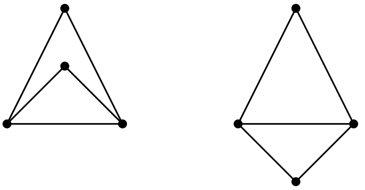

Figure 5: A rigid but not globally rigid framework.

One way of determining the rigidity of a graph in $\mathbb{R}^2$ is therefore to check whether it has a minimally rigid spanning subgraph. It may seem like checking the conditions of Theorem 2.5 would be inefficient, as the counting conditions have to be verified for all subsets of edges, but we will see in Section 3 that there are polynomial time algorithms for checking the conditions of Theorem 2.5.

Lovász and Yemini gave an alternative proof of Theorem 2.5 using graph theoretic techniques [23]. Furthermore, they proved that if a graph is 6-connected, then it is rigid in $\mathbb{R}^2$ [23]. In the same paper, they conjectured that if a graph is $d(d+1)$-connected, then it is rigid in $\mathbb{R}^d$. No progress was made on this conjecture for many years, until 2023, when Villányi showed that the conjecture is true for all d [43].

In $\mathbb{R}^d$ for $d \geq 3$, the necessary conditions of Theorem 2.4 are not sufficient. The graph in Figure 4, satisfies the necessary conditions for being minimally rigid in $\mathbb{R}^3$. However, it is flexible, since the two octahedra can be rotated independently around the dashed line. Characterising the minimally rigid graphs in $\mathbb{R}^d$ for $d \geq 3$ is one of the main open problems in the field.

Another well-studied concept of rigidity for bar-joint frameworks is the following: A framework (G, ρ) in $\mathbb{R}^d$ is *globally rigid* if any framework which is equivalent to (G, ρ) is also congruent to (G, ρ). In other words, a framework (G, ρ) is globally rigid if the configuration space of (G, ρ) consists of a unique framework, up to isometries of $\mathbb{R}^d$. Clearly, a globally rigid framework must also be rigid. However, there are rigid frameworks that are not globally rigid. As an example, the edges have the same lengths in the two frameworks in Figure 5, so they are equivalent, but they are not congruent, i.e. the same up to rotation or translation.

Connelly proved sufficient conditions for a graph to be generically globally rigid in $\mathbb{R}^d$ [9]. Gortler, Healy and Thurston later proved that these conditions are also necessary [14].

Stating the characterisation of generically globally rigid graphs requires the concept of *stresses*. Let $G = (V, E)$ be a graph. An *equilibrium stress* of a framework (G, ρ) is an assignment of a number ω_{ij} to each edge $(v_i, v_j) \in E$ such that the following equation is satisfied for all vertices $v_i \in V$:

$$\Sigma_{v_j \in V | (v_i, v_j) \in E} w_{ij}(\rho(v_i) - \rho(v_j)) = 0. \tag{3}$$

Equilibrium stresses are elements of the cokernel of the rigidity matrix. A *stress matrix* of a framework (G, ρ) is a $|V| \times |V|$- matrix Ω such that

1. $\Omega_{ij} = \Omega_{ji}$ for all $1 \leq i, j \leq |V|$,
2. $\Omega_{ij} = 0$ for all (i, j), $i \neq j$, such that $(v_i, v_j) \notin E$,
3. for all $v_i \in V$, $\Sigma_{j=1}^{|V|} \Omega(v_i, v_j) = 0$, and
4. for all vertices $v_i \in V$, $\Sigma_{v_j \in V} \Omega_{ij} \rho(v_j) = 0$.

The following theorem characterises generic global rigidity of frameworks in $\mathbb{R}^d$.

Theorem 2.6 ([14])**.** *A graph $G = (V, E)$ with at least $d + 2$ vertices is generically globally rigid in $\mathbb{R}^d$ if and only if all stress matrices Ω of generic frameworks of G in $\mathbb{R}^d$ have a kernel of dimension at least $d + 1$.*

3 Matroids and combinatorial rigidity theory

3.1 Matroids

Matroids, introduced by Whitney [47], are combinatorial structures which capture the notion of dependence. Matroids generalise linear independence in vector spaces, as well as the notion of cycles from graph theory. There are many definitions of matroids, all of which are cryptomorphic, i.e. equivalent, but not necessarily obviously so. In this section we will give some of those definitions. In terms of independent sets, a *matroid* is a pair $\mathcal{M} = (E, \mathcal{I})$, where E is a finite set (the *ground set* of the matroid), and $\mathcal{I}$ is a family of subsets of E (the *independent sets*) such that

(I1) $\emptyset \in \mathcal{I}$,

(I2) if $I' \subseteq I \subseteq E$ and $I \in \mathcal{I}$, then $I' \in \mathcal{I}$, and

(I3) if I and I' are in $\mathcal{I}$ and $|I| > |I'|$, then there is an element $e \in I \setminus I'$ such that $I' \cup \{e\} \in \mathcal{I}$.

Independent sets generalise the concept of an independent set in a vector space. Two matroids $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ are *isomorphic*, denoted $\mathcal{M}_1 \cong \mathcal{M}_2$, if there is a bijection $\psi : E_1 \to E_2$ such for any subset $A \subseteq E_1$, $\psi(A)$ is an independent set in $\mathcal{M}_2$ if and only if A is an independent set in $\mathcal{M}_1$.

A matroid is also uniquely defined by its bases. In terms of bases, a matroid is a pair $\mathcal{M} = (E, \mathcal{B})$, where E is a finite set and $\mathcal{B}$ is a family of subsets of E (the *bases*) such that

(B1) $\mathcal{B}$ is non-empty, and

(B2) if B_1, $B_2 \in \mathcal{B}$ are distinct and $b \in B_1 \setminus B_2$, then there is an element $b' \in B_2 \setminus B_1$ such that $(B_1 \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$.

The independent sets of a matroid $\mathcal{M} = (E, \mathcal{B})$ are the subsets of bases, and a basis of a matroid $\mathcal{M} = (E, \mathcal{I})$ is a maximally independent set.

In terms of its rank function, a matroid is a pair $\mathcal{M} = (E, r)$, where E is a finite set and $r : E \to \mathbb{N}$ is a function such that

(R1) for any subset $A \subseteq E$, $r(A) \leq |A|$,

(R2) for any two subsets A_1, $A_2 \subseteq E$, $r(A_1) + r(A_2) \geq r(A_1 \cup A_2) + r(A_1 \cap A_2)$, and

(R3) for any subset $A \subseteq E$ and $e \in E$, $r(A \cup \{e\}) \leq r(A) + 1$.

The independent sets of the matroid $\mathcal{M} = (E, r)$ are the subsets $I \subseteq E$ such that $r(I) = |I|$.

The rank of a subset $A \subseteq E$ is the size of the largest independent subset of A. The rank of a basis of $\mathcal{M}$ is the rank of the matroid. In analogy with bases of vector spaces, all bases of a matroid have the same size, so the rank of a matroid is well defined.

3.2 Examples from combinatorial rigidity theory

There are many examples of matroids that appear in the context of rigidity of graphs in $\mathbb{R}^d$. In this section, we are going to introduce some of them.

Example 3.1 (Linear matroids and the d-dimensional rigidity matroid)**.** Let $\mathbb{F}$ be a field, and let $V = \{v_0, ..., v_n\}$ be a set of vectors in $\mathbb{F}^d$. Let $\mathcal{I}$ be the family of independent subsets of V. Then $\mathcal{M}[V] = (V, \mathcal{I})$ is a matroid. Matroids defined in this way are called *linear matroids*. In particular, given a matrix M with row vectors $V = \{r_0, ..., r_n\}$, the linear matroid $\mathcal{M}[V]$ is the *row matroid* of M. Similarly, the *column matroid* of a matrix M with column vectors $W = \{c_1, ..., c_m\}$ is the matroid $\mathcal{M}[W]$.

The *d-dimensional rigidity matroid* of a graph $G = (V, E)$ is the row matroid of $R_d(G, \rho)$, for a generic framework (G, ρ). Since the rows of $R_d(G, \rho)$ are indexed by the edge-set of G, the rigidity matroid can be viewed as a matroid with ground set E.

A graph with $d+1$ or fewer vertices is rigid if and only if it is complete. A graph with at least $d+2$ vertices is rigid if and only if its d-dimensional rigidity matroid has rank $d|V| - \binom{d+1}{2}$. Equivalently, a graph with at least $d+2$ vertices is rigid if and only if it contains a spanning subgraph which is a basis of the d-dimensional rigidity matroid. Characterising the bases of the d-dimensional rigidity matroid therefore becomes a very important problem in combinatorial rigidity theory.

Example 3.2 (Count matroids)**.** Let $G = (V, E)$ be a graph. We say that a subset $F \subseteq E$ is (k, l)-sparse if $|F'| \leq k|V(F')| - l$ for all non-empty subsets $F' \subseteq F$. We say that a subset $F \subseteq E$ is (k, l)-tight if it is (k, l)-sparse and $|F| = k|V(F)| - l$. Note that if $l > 2k$, then a single edge is not (k, l)-sparse, so if $l > 2k$ the only (k, l)-sparse set is the empty set.

Let $\mathcal{I}$ be the family of (k, l)-sparse subsets. Then $\mathcal{M} = (E, \mathcal{I})$ is a matroid, the (k, l)-*sparsity matroid*. The (k, l)-tight subsets are the bases of the (k, l)-sparsity matroid.

Note that the edge-set of a graph is $(1, 1)$-tight if and only if it is a tree, meaning that a graph is a basis of the 1-dimensional rigidity matroid if and only if it is $(1, 1)$-tight.

Furthermore, the Geiringer-Laman theorem (Theorem 2.5) says that a set of edges is a basis of the 2-dimensional rigidity matroid if and only if it is $(2, 3)$-tight.

Note that for $d \geq 3$, $\binom{d+1}{2} > 2d$, so the only $(d, \binom{d+1}{2})$-sparse set of edges is empty. The graphs with $(d, \binom{d+1}{2})$-tight sets of edges are therefore not those that are bases of the d-dimensional rigidity matroid for $d \geq 3$.

Graver introduced the concept of an abstract d-dimensional rigidity matroid, and conjectured that the 3-dimensional rigidity matroid is the unique maximal abstract 3-dimensional rigidity matroid [15]. Recently, independence in the unique maximal abstract 3-dimensional rigidity matroid was characterised by Clinch, Jackson, and Tanigawa [7, 8].

Example 3.3 (Graphic matroids)**.** Let $G = (V, E)$ be a graph. Then the graphic matroid of G is the matroid $\mathcal{M}(G) = (E, \mathcal{I})$, where the independent sets are the forests in G. The bases of $\mathcal{M}(G)$ are the spanning forests of G. If G is connected, the bases are spanning trees.

Nash-Williams proved the following theorem, which states that a graph is independent in the (k, k)-sparsity matroid if and only if it is the edge-disjoint union of independent sets in the graphic matroid.

Theorem 3.4 ([32])**.** *A graph is (k, k)-sparse if and only if it is the union of k edge-disjoint spanning forests, and (k, k)-tight if and only if it is the union if k spanning forests.*

Lovász and Yemini connected the Nash-Williams Theorem (Theorem 3.4 to rigidity in $\mathbb{R}^2$.

Theorem 3.5 ([23])**.** *A graph $G = (V, E)$ is rigid in $\mathbb{R}^2$ if and only if doubling any edge of E results in a graph which is the edge-disjoint union of two spanning trees.*

3.3 Algorithms and computational aspects

Finding independent sets of count matroids on graphs is computationally easy. The *pebble game algorithm* (Algorithm 1) is an algorithm for finding a maximum size spanning $(2, 3)$-sparse subgraph, which was introduced by Jacobs and Hendricksen [18]. The input of the pebble game algorithm is a graph $G = (V, E)$ together with an ordering of the edges. The output of the algorithm is a spanning subgraph of G that is independent of the $(2, 3)$-sparsity matroid, which is of maximum size in the sense that there are no larger subgraphs of G that are independent in the $(2, 3)$-sparsity matroid. From the end state of the algorithm one can also decide whether the output is $(2, 3)$-tight. As noted in [18], the edges can be considered in any order, but changing the order of the edges can affect which maximally independent subgraph is produced by the algorithm.

Lee and Streinu generalised the pebble game algorithm to recognise (k, l)-sparse multigraphs [22]. Streinu and Theran later generalised the pebble game algorithm to recognise (k, l)-sparse hypergraphs [41]. The pebble game algorithm was generalised to general count matroids by Frank [11].

Berg and Jordán introduced an algorithm for determining independence in count matroids, which was inspired by the pebble game algorithm due to Jacobs and Hendricksen, but uses orientations as its main algorithmic tool [2]. Gabow and Westerman [12] introduced another algorithm for finding $(2, 3)$-sparse spanning subgraphs [12]. All of the algorithms mentioned, including all versions of the pebble game, are of polynomial time complexity.

Algorithm 1 The pebble game algorithm.

Input: A graph $G = (V, E)$, and an ordering of the edges.
Output: $(2, 3)$-sparsity and/or $(2, 3)$-tightness of G

```
For each vertex, set peb(v) = 2 for each v ∈ V (Initialise number of pebbles).
D ← (V, ∅) (Initialise D).
for e = (v_i, v_j) ∈ E do
    while e has not been processed do
        if peb(v_i) + peb(v_j) > 4 then
            Set peb(v_i) = peb(v_i) − 1.
            Add the directed edge from v_i to v_j to D.
            The edge e has been processed.
        else
            Search for a directed path (v′...u) in D, where v′ ∈ {v_i, v_j}, {v′, u} ≠ {v, w}, with peb(u) > 0
            if (v′...u) with peb(u) > 0 has been found then
                Set peb(u) = peb(u) = 1.
                Redirect the edges in the directed path (v′...u).
                Set peb(v′) = peb(v′) + 1.
            else
                return : G is not (2, 3)-sparse.
            end if
        end if
    end while
end for
if ∑_{v∈V} peb(v) = 3 then
    return G is (2, 3)-tight.
else
    return G is (2, 3)-sparse, but not (2, 3)-tight.
end if
```

4 Symmetries and group actions in rigidity theory

Many results in rigidity theory are concerned with generic frameworks. However, non-generic frameworks can also be of interest, theoretically and in applications. In this section, we will discuss frameworks which are symmetric, i.e. invariant under the action of a group, and therefore non-generic.

Let $G = (V, E)$ be a graph. An *automorphism* of G is a map $\gamma : V \to V$ such that $(v_i, v_j) \in E$ if and only if $(\gamma(v_i), \gamma(v_j)) \in E$. The automorphisms of a graph G form a group under composition, called the *automorphism group* of G. Let $G = (V, E)$ be a graph, and let $\mathrm{Aut}(G)$ denote its automorphism group. Suppose that Γ is a subgroup of $\mathrm{Aut}(G)$, and that there is a representation $\phi : \Gamma \to O(d, \mathbb{R})$. A d-dimensional framework (G, ρ) is Γ-*symmetric* if $\phi(\gamma)p(v) = p(\gamma v)$ for all $\gamma \in \Gamma$ and all vertices $v \in V$.

The literature about symmetric frameworks often treats one of the following two questions:

1. Is the framework infinitesimally rigid?
2. Is there an infinitesimal motion of the framework that preserves the symmetry?

Some non-generic frameworks are more (infinitesimally) flexible than generic frameworks of the same graph. The first problem, the *incidental symmetry problem*, is fundamentally the problem of determining which symmetric frameworks remain rigid despite being non-generic. Fowler and Guest introduced tools for solving the incidental symmetry problem [10]. Characterisations of rigidity of Γ-symmetric frameworks are known for several groups Γ [36, 37, 38].

In the second problem, the *forced-symmetric rigidity problem*, we also require that the (infinitesimal) motion is symmetric. Consider the framework in Figure 6. It is symmetric with respect to the reflection in the dashed line. It also has an infinitesimal motion which preserves that reflectional symmetry. Note that

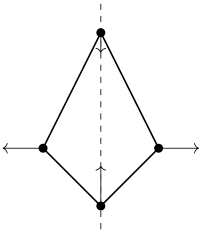

Figure 6: A framework with a reflectional symmetry which is preserved by a motion

the infinitesimal motion has the same symmetry as the framework, which is why it preserves the symmetry of the framework.

Suppose that (G, ρ) is a Γ-symmetric framework. For simplicity, we will assume that the action of Γ on V is free. Then an infinitesimal motion m of (G, ρ) is Γ-symmetric if $\phi(\gamma)m(v) = m(\gamma v)$ for all $\gamma \in \Gamma$ and $v \in V$. If m is a Γ-symmetric motion of a framework, and we know $m(v)$ for one representative v of an orbit of vertices under the action of Γ, then by symmetry we also know $m(v')$ for all vertices v' in the same orbit. When considering Γ-symmetric motions, we can restrict to one representative of each orbit of vertices and one representative of each orbit of edges.

Let v_i and v_j be representatives of orbits of vertices under the action of a group Γ. We need not assume that v_i and v_j are distinct. Let $\gamma \in \Gamma$, and suppose that there is an edge $e = (v_i, \gamma v_j)$ in a Γ-symmetric framework (G, ρ).

Then any Γ-symmetric motion of the framework satisfies

$$\langle \rho(v_i) - \phi(\gamma)\rho(v_j), m(v_i) - \phi(\gamma)m(v_j)\rangle = 0. \tag{4}$$

By bilinearity of the inner product, and since $\phi(\gamma)$ is an element of the orthogonal group, equation (4) can be rewritten as

$$\langle \rho(v_i) - \phi(\gamma)\rho(v_j), m(v_i)\rangle + \langle \rho(v_j) - (\phi(\gamma))^{-1}\rho(v_i), m(v_j)\rangle = 0. \tag{5}$$

The corresponding equation for another edge $\beta e = (\beta v_i, \beta\gamma v_j)$ in the same orbit as e is

$$\langle \phi(\beta)\rho(v_i) - \phi(\beta)\phi(\gamma)\rho(v_j), \phi(\beta)m(v_i) - \phi(\beta)\phi(\gamma)m(v_j)\rangle = 0. \tag{6}$$

Again, since $\phi(\beta)$ is an element of the orthogonal group, equation (5) and equation (6) are equivalent. It follows that we only need to consider one equation of the form 5 for each orbit of edges under the action of Γ. The *orbit rigidity matrix* of the Γ-symmetric framework (G, ρ) is the coefficient matrix of the system of equations of the form (5). Its kernel is in one-to-one correspondence with the Γ-symmetric infinitesimal motions of (G, ρ) [39].

The orbit rigidity matrix can be defined similarly when the action of Γ on V is not free, but some care needs to be taken to ensure that vertices which are fixed by the action of Γ remain fixed by the symmetry throughout the motion [39].

There are characterisations of symmetry-forced rigidity for graph frameworks in the plane with reflectional, rotational symmetry and D_{2n}-symmetry, when n is odd [19]. When n is even, the natural necessary conditions are not sufficient [19].

The analogue of the forced-symmetry problem has also been extensively studied for periodic frameworks in the plane, which are infinite frameworks in the plane that are invariant with respect to the action of a group. See for example [5, 27, 28].

A framework on the torus can be modelled as an infinite framework in $\mathbb{R}^2$ which is invariant under translations. A motion of a framework on the torus can then be modelled as a motion of the infinite framework which preserves the translational symmetry. In that way, rigid graphs on the torus could be characterised by considering the forced symmetry problem for infinite frameworks [35].

5 Incidence geometries in rigidity theory

5.1 Drawing incidence geometries

In this chapter, we are going to discuss rigidity problems for realisations of incidence geometries. In particular, we are interested in realisations of incidence geometries as points and straight lines in the projective plane. Such a realisation consists of an assignment of a point $\mathbf{p}$ in the projective plane to each element $p \in P$, and a line $\mathbf{l}$ in the projective plane to each element $l \in L$, such that the point $\mathbf{p}$ lies on the line $\mathbf{l}$ if p and l are incident. There are realisations of any incidence geometry such that all elements of P are mapped to the same point, and realisations such that all elements of L are mapped to the same line. We say that a realisation of an incidence geometry is *proper* if no points coincide and no lines coincide. Not all incidence geometries have proper realisations as points and straight lines.

Example 5.1 (The Fano plane)**.** The Fano plane is an incidence geometry with 7 points, and lines $\{p_0, p_1, p_2\}$, $\{p_0, p_3, p_4\}$, $\{p_0, p_5, p_6\}$, $\{p_1, p_3, p_6\}$, $\{p_1, p_4, p_5\}$, $\{p_2, p_3, p_5\}$,$\{p_2, p_4, p_6\}$.

The Sylvester-Gallai theorem states that for any finite set of points in the Euclidean plane, there is either a line passing through exactly two of the points, or a line passing through all of the points [31].

Notice that for each pair of points, there is a line of the incidence geometry containing that pair of points, and a third point of the incidence geometry (see Figure 7). Hence, there is no line containing exactly two points in a realisation of the Fano plane as points and straight lines. By the Sylvester-Gallai theorem, there must therefore be a line passing through all points in any realisation of the Fano plane as points and straight lines in the Euclidean plane. Hence the lines must coincide in all realisations of the Fano plane where all points are assigned distinct coordinates.

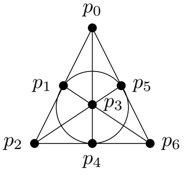

Figure 7: The Fano plane.

As incidence geometries, unlike graphs, do not necessarily have realisations as points and straight lines, the problem of finding geometric realisations becomes important when considering rigidity problems for incidence geometries. One way to find realisations of incidence geometries is finding $\mathbb{R}$-realisations of rank 3 matroids.

Given an incidence geometry $S = (P, L, I)$, there is a corresponding matroid $\mathcal{M}(S)$ of rank 3 with P as its ground set, and with bases given by the triples of points that are not incident to a common line of the incidence geometry.

A matroid $\mathcal{M} = (E, \mathcal{I})$ is $\mathbb{F}$-*realisable* (or $\mathbb{F}$-*representable*) if there is a set of vectors $V = \{v_0, ..., v_n\}$ in $\mathbb{F}^d$ such that $\mathcal{M} \cong \mathcal{M}[V]$. We say that $\mathcal{M}[V]$ is a *realisation* of $\mathcal{M}$ if $\mathcal{M} \cong \mathcal{M}[V]$. In this context, we are interested in $\mathbb{R}$-realisations of $\mathcal{M}(S)$. An $\mathbb{R}$-realisation of $\mathcal{M}(S)$ is an assignment of a vector $\mathbf{p}_i \in \mathbb{R}^3$ to each element $\mathbf{p}_i \in P$, such that $\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$ are linearly independent if and only if p_i,p_j and p_k are not incident to a common line in the incidence geometry. Now, each vector $\mathbf{p}_i \in \mathbb{R}^3$ corresponds to a point in the real projective plane, and $\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$ are linearly independent in $\mathbb{R}^3$ if and only if the points lie on a line in the real projective plane. Hence, a realisation of $\mathcal{M}(S)$ corresponds to a drawing of S as points and straight lines in the real projective plane. However, not all drawings of an incidence geometry $S = (P, L, I)$ necessarily correspond to a realisation of $\mathcal{M}(S)$.

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid of rank m, and let $\mathbb{F}$ be a field. We can then consider the space of all realisations of $\mathcal{M}$ over $\mathbb{F}$. There are two models of this space; the *matroid realisation space*, utilising the Grassmannian, and the more recently introduced *slack realisation space*.

The *Grassmannian* $\mathrm{Gr}(m, n; \mathbb{F})$ is a parametrisation of the m-dimensional subspaces of $\mathbb{F}^n$. In particular, we are interested in the Plücker embedding of $\mathrm{Gr}(m, n; \mathbb{F})$ into $PG(\mathbb{F}, \binom{n}{m} - 1)$. Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{F}^n$, and let V be an m-dimensional subspace of $\mathbb{F}^n$ with basis $\{v_1, ..., v_m\}$. Let

$$A(V) = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

be the $m \times n$-matrix with the vectors $\{v_1, ..., v_m\}$ as its row vectors. Given a sequence $1 \leq i_1 < i_2 < ... < i_m$, let $A_{i_1 \dots i_m}(V)$ be the $m \times m$ minor given by the columns of $A(V)$ indexed by $\{i_1, ..., i_m\}$. Let $\{A_1(V), ..., A_k(V)\}$ be the set of all such $m \times m$-minors. The Plücker embedding ι then takes an m-dimensional subspace of $\mathbb{F}^n$ to the vector in $PG(\mathbb{F}, \binom{n}{m} - 1)$ with coordinate vector $[A_1(V) : ... : A_{\binom{n}{m}}(V)]$.

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid of rank m. Without loss of generality, we can assume that $E = \{1, ..., n\}$ for some positive integer n. The *realisation space*, or *matroid stratum* of $\mathcal{M}$ is the set

$$\mathcal{R}(\mathcal{M}) = \{V \in Gr(m, n) \mid A_{i_1 \dots i_m} \neq 0 \text{ if and only if } \{i_1, i_2, ..., i_m\} \in \mathcal{B}\}.$$

The realisation space consists of all subspaces $V \subseteq \mathbb{F}^n$ such that the column matroid of $A(V)$ is a realisation of $\mathcal{M}$.

The slack realisation space of a matroid, introduced by Brandt and Wiebe, is a different algebraic model for the realisation space of a matroid [6]. Suppose that $\mathcal{M}$ is the column matroid $\mathcal{M}[V]$ of a $k \times n$-matrix M with column vectors $\{v_1, ..., v_n\}$, and that $\mathcal{M}$ has rank m.

A *flat* of a matroid $\mathcal{M} = (E, r)$ is a set $F \subseteq E$ such that $r(F \cup \{x\}) > r(F)$ for all $x \in E \setminus F$. A *hyperplane* of a matroid of rank m is a flat of rank $m - 1$. Let $\mathcal{H} = \{H_1, ..., H_k\}$ be the set of hyperplanes of the linear matroid $\mathcal{M}[V]$. Note that each column vector $v_i \in V$ can be interpreted as a point in $\mathbb{F}^m$. When the column vectors are interpreted as points, the hyperplanes of the matroid can be interpreted as hyperplanes in $\mathbb{F}^m$. As such, each $H_i \in \mathcal{H}$ can be defined by a vector $h_i \in \mathbb{F}^m$ such that $h_i^T v_j = 0$ if and only if $v_j \in H_i$. Let W be the matrix with column vectors $\{h_1, ..., h_k\}$. The *slack matrix* $S_{\mathcal{M}[V]}$ of the matroid $\mathcal{M}[V]$ is the matrix $S_{\mathcal{M}[V]} = V^T W$.

Proposition 5.2 ([6]). *Let $\mathcal{M}[V]$ be a linear matroid. Then the row matroid of the slack matrix is an $\mathbb{F}$-realisation of $\mathcal{M}[V]$.*

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid of rank m, which is not necessarily linear. Let the *symbolic slack matrix* $\mathcal{S}_{\mathcal{M}}(x)$ be the matrix with rows indexed by the elements of E, and columns indexed by the hyperplanes of $\mathcal{M}$ and

$$\mathcal{S}_{\mathcal{M}}(x)_{ij} = \begin{cases} x_{ij} & \text{if } i \notin H_j \\ 0 & \text{if } i \in H_j \end{cases}$$

The *slack ideal* $I_{\mathcal{M}}$ is the ideal generated by the $m+1$-minors of $\mathcal{S}_{\mathcal{M}}(x)_{ij}$, saturated by $\langle \Pi_{i=1}^m \Pi_{j:i \notin H_j} x_{ij} \rangle$, the product of all variables x_{ij}. Let t be the number of variables x_{ij}, and let $\mathcal{V}(I_{\mathcal{M}})$ be the variety in $\mathbb{F}^t$ defined by the slack ideal. Given a point $s \in \mathbb{F}^m$ in $\mathcal{V}(I_{\mathcal{M}})$, there is a corresponding matrix $\mathcal{S}_{\mathcal{M}}(s)$. These matrices are exactly those that realise $\mathcal{M}$ [6].

Any incidence geometry $S = (P, L, I)$ can be realised as points and straight lines in such a way that two or more points coincide, or such that two or more lines coincide. It may also be that, for example, three points are collinear despite not being incident to a common line in L. Realisations where points or lines coincide, and realisations where three points are collinear despite not being incident to a common line do not correspond to $\mathbb{R}$-realisations of $\mathcal{M}(S)$.

For example, consider the realisation of the incidence geometry $S = (P, L, I)$ in Figure 8. In the drawing in Figure 8, the points p_6, p_7 and p_8 lie on a line in the plane, although they are not incident to a common line in the incidence geometry. Hence, the drawing in Figure 8 does not correspond to a realisation of $\mathcal{M}(S)$.

Pappus theorem says that if the points p_0, p_1 and p_2 lie on a line in $PG(\mathbb{R}, 2)$, and the lines p_3, p_4 and p_5 lie on a line in $PG(\mathbb{R}, 2)$, then the points p_6, p_7 and p_8 also lie on a line in $PG(\mathbb{R}, 2)$. Since in any $\mathbb{R}$-realisation of $\mathcal{M}(S)$, three points lie on a line in $PG(\mathbb{R}, 2)$ if and only if they are incident to a common

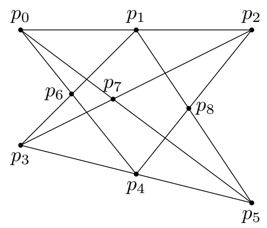

Figure 8: A subgeometry of the Pappus configuration.

line in the incidence geometry, it follows that S has no $\mathbb{R}$-realisations. In fact, Pappus theorem applies over any field $\mathbb{F}$, so S is not realisable over any field.

A v_k-configuration is an incidence geometry with v points and v lines such that k points lie on each line and k lines go through each point. Realisability of v_k-configurations as points and straight lines is comparatively well-studied. There are many families of v_k-configurations for which there are constructions of geometric realisations [16, 33].

5.2 Concepts of rigidity for incidence geometries

The remainder of this section will be an overview of concepts of rigidity for incidence geometries.

The central problem in scene analysis, introduced in Section 5.2.1 is finding the space of hyperplanes and points in $\mathbb{R}^d$ which satisfy a given incidence relation, such that the first $d-1$ coordinates of each points are fixed. Parallel redrawings, the concept introduced in Section 5.2.2, is the equivalent of direction-preserving rigidity for bar-joint framework for incidence geometries. Section 5.2.3 introduces the theory of body-joint frameworks and rod configurations.

5.2.1 Scenes

Let $S = (P, L, I)$ be an incidence geometry. A $(d-1)$-*picture* (S, x) of the incidence geometry S is an assignment of a point $x(p) \in \mathbb{R}^{d-1}$ to each element $p \in P$.

Given a $(d-1)$-picture (S, x), a *d-scene* lifting (S, x) is an assignment of a coordinate $y(p) \in \mathbb{R}$ to each element $p \in P$, and a vector $n(l) \in \mathbb{R}^{d-1}$ and an affine shift $h(l) \in \mathbb{R}$ to each element $l \in L$ such that

$$\langle (n(l), 1, h(l)), (x(p), y(l), 1)\rangle = 0 \tag{7}$$

for all incidences (p, l).

Given a picture (S, x), finding the scenes that lift (S, x) amounts to solving an equation of the form

$$x(p)_1 n(l)_1 + x(p)_2 n(l)_2 + ... + x(p)_{d-1} n(l)_{d-1} + y(p) + h(l) \tag{8}$$

for each incidence (p, l), where $n(l)_1$,$n(l)_2$,...,$n(l)_{d-1}$, $h(l)$ and $y(p)$ are variables. Let the d-dimensional *lifting matrix* $A_d(S, x)$ be the $|I| \times (|P| + d|L|)$-coefficient matrix of the system of equations of the form (8).

Equation (8) says that the point $(x(p), y(p))$ lies on the hyperplane with normal $(n(l)_1, ..., n(l)_{d-1}, 1)$ and height $h(p)$. Note that the assumption that the last coordinate of the normal is 1 is an assumption that the hyperplane is not parallel to the d:th coordinate axis.

A d-scene is *trivial* if all hyperplanes of the scene are the same, i.e. if $(n(l_j), 1, h(l_j)) = (n(l_k), 1, h(l_k))$ for all pairs of elements l_j and l_k of L. A scene is *sharp* if all hyperplanes of the scene are distinct, i.e. if $(n(l_j), 1, h(l_j)) \neq (n(l_k), 1, h(l_k))$ for all pairs of elements l_j and l_k of L. There is a d-dimensional space of trivial d-scenes lifting all $(d-1)$-pictures (S, x).

A picture is *generic* if the set of coordinates assigned to the points, i.e. the set $\{x(p) \mid p \in P\}$, is algebraically independent over $\mathbb{Q}$. A natural first question is which incidence geometries have generic $(d-1)$-dimensional pictures that lift to sharp scenes in $\mathbb{R}^d$. That question was answered by Whiteley [46].

Let $S = (P, L, I)$ be an incidence geometry. The *d-plane matroid* at L is the matroid with ground set I, where a set $I' \subseteq I$ is independent if

$$|I''| \leq |P(I'')| + d|L(I'')| - d$$

for all subsets $I'' \subseteq I'$.

Theorem 5.3 ([46])**.** *Let $S = (P, L, I)$ be an incidence geometry. The rows of $A_d(S, x)$ are independent for a generic picture (S, x) in $d-1$-dimensional space if and only if I is independent in the d-plane matroid at L.*

Now, suppose that $S = (P, L, I)$ is an incidence geometry such that I is a basis of the d-plane matroid. Then $A_d(S, x)$ has a d-dimensional kernel for any generic picture $\mathbf{p}$ of S. It follows that (S, x) has only the d-dimensional space of trivial liftings, and does not lift to a sharp scene.

Corollary 5.4 ([46])**.** *Let $S = (P, L, I)$ be an incidence geometry. Then a generic picture (S, x) in $\mathbb{R}^{d-1}$ lifts to a sharp scene if and only if*

$$|I'| \leq |P(I')| + d|L(I')| - (d+1)$$

for all subsets $I' \subseteq I$.

5.2.2 Parallel redrawings

Again, let $S = (P, L, I)$ be an incidence geometry. Consider a hyperplane arrangement (S, n), where n is an assignment $n : L \to \mathbb{R}^{d-1}$, and $(n(l), 1)$ is the normal of a hyperplane in $\mathbb{R}^d$. A parallel redrawing of (S, n) is an assignment $h(l) \in \mathbb{R}$ to each element $l \in L$, and assignments $x(p) \in \mathbb{R}^{d-1}$ and $y(p) \in \mathbb{R}$ such that Equation (7) holds for all incidences $(p, l) \in I$.

Finding a parallel redrawing of a hyperplane arrangement (S, p) amounts to solving an equation

$$x(p)_1 n(l)_1 + x(p)_2 n(l)_2 + ... + x(p)_{d-1} n(l)_{d-1} + y(p) + h(l) \tag{9}$$

for each incidence $(p, l) \in I$, where $x(p)_1, ..., x(p)_{d-1}$, $y(p)$ and $h(l)$ are variables. Let the *concurrence geometry matrix*, denoted $A^*_d(S, n)$, be the $(|I| \times |L| + |P|)$-coefficient matrix of the system of equations of the form (9).

A parallel redrawing is *trivial* if all points are assigned the same coordinates, i.e. if $(x(p_i), 1, y(p_i)) = (x(p_k), 1, y(p_k))$ for all pairs of points p_i and p_k in P. A parallel redrawing is *proper* if all points are assigned distinct coordinates, i.e. if $(x(p_i), 1, y(p_i)) \neq (x(p_k), 1, y(p_k))$ for all pairs of points p_i and p_k in P. There is a d-dimensional space of trivial parallel redrawings of all hyperplane arrangements (S, n).

A hyperplane arrangement (S, n) is *generic* if the set of entries of the normals, i.e. the set $\{n(l) \mid l \in L\}$, is algebraically independent over $\mathbb{Q}$. Again, the natural first question is which incidence geometries have proper parallel redrawings for generic hyperplane arrangements (S, n).

Let $S = (P, L, I)$ be an incidence geometry. The *d-plane matroid* at P is the matroid with ground set I, where a set $I' \subseteq I$ is independent if

$$|I''| \leq |L(I'')| + d|P(I'')| - d$$

for all subsets $I'' \subseteq I'$.

Theorem 5.5 ([46])**.** *Let $S = (P, L, I)$ be an incidence geometry. The rows of $A^*_d(S, n)$ are independent for a generic hyperplane arrangement (S, n) in $d-1$-dimensional space if and only if I is independent in the d-plane matroid at P.*

Again, if an incidence geometry S is a basis of the d-plane matroid at P, then generic hyperplane arrangements (S, n) have only trivial parallel redrawings.

Corollary 5.6 ([46])**.** *Let $S = (P, L, I)$ be an incidence geometry. Then a generic hyperplane arrangement (S, n) in $\mathbb{R}^{d-1}$ has proper parallel redrawings if and only if*

$$|I'| \leq |L(I')| + d|P(I')| - (d+1)$$

for all subsets $I' \subseteq I$.

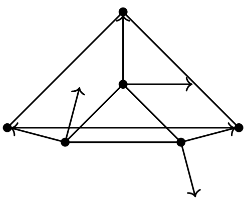

Figure 9: There is a correspondence between the infinitesimal motions of a graph frameworks and its parallel redrawings.

There is a duality of the projective plane that sends the points of the projective plane to the lines of the projective plane. This duality takes liftings of pictures to parallel redrawings of hyperplane arrangements, and vice versa.

Given a framework (G, ρ) in $\mathbb{R}^2$, the parallel redrawings of (G, ρ) are in one-to-one correspondence with its infinitesimal motions. Specifically, if m is an element of the kernel of $R_2(G, \rho)$, then the orthogonal vector $m^\perp$ is an element of the kernel of $A_2^*(G, n)$, where n is the set of normals of the edges in (G, ρ). See Figure 9 for an illustration of what the one-to-one correspondence looks like.

5.2.3 Body-joint frameworks and rod configurations

In this section, we will consider realisations of incidence geometries as rigid bodies connected in joints. Let $S = (P, L, I)$ be an incidence geometry. A *body-joint framework* realising S in $\mathbb{R}^d$ is an assignment $\rho : P \to \mathbb{R}^d$, and a replacement of each element $l_i \in L$ by a minimally rigid bar-joint framework including the joint $\rho(p_j)$ if and only if $(p_j, l_i) \in I$. Note that a body-joint framework may contain points which do not realise elements of P. A body-joint frameworks is rigid if it is rigid as a bar-joint framework, and similarly infinitesimally rigid if it is infinitesimally rigid as a bar-joint framework. A body-joint framework is independent if it is independent in the d-dimensional rigidity matroid.

A body-joint framework realising an incidence geometry such that each element of P is incident to exactly two elements of L is a *body-hinge* framework. Tay and Whiteley independently characterised which incidence geometries have realisations as body-hinge frameworks in $\mathbb{R}^d$ [42, 45].

They then formulated the molecular conjecture, which has since been proven. A *panel-hinge framework* realising an incidence geometry $S = (P, L, I)$ is a body-hinge framework such that the hinges realising the points incident to a common line are collinear. The molecular conjecture, states that infinitesimally rigid body-hinge frameworks in $\mathbb{R}^d$ remain rigid when the bodies are flattened to panels.

Theorem 5.7 (The molecular theorem)**.** *An incidence geometry has a realisation as an infinitesimally rigid body-hinge framework in $\mathbb{R}^d$ if and only it has a realisation as an infinitesimally rigid panel-hinge framework in $\mathbb{R}^d$.*

In 1989, Whiteley proved a version of the molecular conjecture in the plane, which applies to body-joint frameworks (i.e. there is no requirement that the points of the incidence geometry are incident to exactly two points), but only to independent body-joint frameworks [46]. In 2008, Jackson and Jordán proved the molecular conjecture in the plane [17]. Finally, Katoh and Tanigawa proved the molecular conjecture in arbitrary dimension [20]. In 2011, Jackson and Jordán and Katoh and Tanigawa proved the molecular conjecture for body-hinge frameworks, i.e. with the requirement that each point is incident to exactly two points.

A *rod configuration* in the plane realising an incidence geometry S is a proper realisation ρ of the incidence geometry as points and straight lines in $\mathbb{R}^2$, where each line moves as a rigid body. That each line moves as a rigid body means that the distances between all points on a line are preserved by motions of the rod configuration. Formally, a continuous motion of a rod configuration consists of a curve $\tau_p(t) \in \mathbb{R}^2$ for each element $p \in P$ such that $\tau_p(0) = \rho(p)$ for all $p \in P$, and $||\tau_p(t) - \tau_q(t)|| = ||\tau_p(0) - \tau(0)||$ for all $t \in [0, 1]$. A rod configuration is *rigid* if all possible continuous motions of the rod configuration are rigid body motions, and *flexible* otherwise. Figure 10 shows a flexible rod configuration.

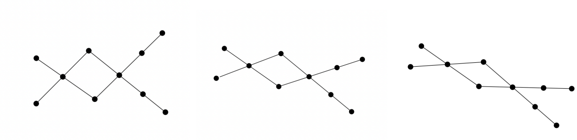

Figure 10: A flexible rod configuration

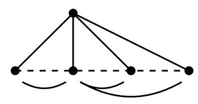

Figure 11: The framework representing a line incident to four points.

An infinitesimal motion is an assignment of a vector $v \in \mathbb{R}^2$ to each element of P such that for each rod, the vectors assigned to the points on the rod are the linearisation of a trivial motion.

The version of the molecular conjecture that Whiteley proved was the following:

Theorem 5.8 ([46])**.** *For an incidence geometry $S = (P, L, I)$, the following are equivalent:*

1. *S has a realisation as an independent body-joint framework in $\mathbb{R}^2$,*
2. *$2|I'| \leq 3|L(I')| + 2|P(I')| - 3$ for any subset $I' \subseteq I$, and*
3. *S has a realisation as an independent body-joint framework in $\mathbb{R}^2$ such that all joints incident to a common line are collinear.*

Whiteley proved Theorem 5.8 by considering the body-joint framework realising an incidence geometry $S = (P, L, I)$ where the points incident to each element $l \in L$ are collinear, and l is represented by a framework on the set of points incident to l, and a vertex v not representing an element of P. The framework representing a line $l \in L$ is given by a tree on the points incident to l, and edges from v to all points incident to l (see Figure 11).

Remark 5.9. Given a graph $G = (V, E)$, its *cone graph* G^c is the graph given by adding a new vertex (a *cone vertex*) which is adjacent to all vertices in V. Then G is rigid in $\mathbb{R}^d$ if and only if G^c is rigid in $\mathbb{R}^{d+1}$. In fact, even more is true. Suppose that (G, ρ) is a d-dimensional framework of G. Then (G, ρ) can be interpreted as a framework in $\mathbb{R}^{d+1}$ which is contained in a subspace of dimension d. The coned framework of (G, ρ) is the $(d+1)$-dimensional framework (G^c, ρ^c) given by extending the framework (G, ρ) by adding the cone vertex at the origin. Then (G^c, ρ^c) is a non-generic framework in $\mathbb{R}^d$ which is infinitesimally rigid in $\mathbb{R}^d$ if and only if the framework (G, ρ) is infinitesimally rigid in $\mathbb{R}^d$ [44].

The frameworks representing lines in Whiteley's proof of Theorem 5.8 are cones of frameworks of trees in $\mathbb{R}^1$. As trees are minimally rigid in $\mathbb{R}^1$, coned frameworks of trees are minimally rigid in $\mathbb{R}^2$.

To prove Theorem 5.8, Whiteley proves that if an incidence geometry satisfies condition 2 of Theorem 5.8, then replacing each line by the cone of a tree gives a framework such that the rows of its concurrence geometry matrix are independent. Theorem 5.8 then follows from the one-to-one correspondence between parallel redrawings and infinitesimal motions of a bar-joint framework in $\mathbb{R}^2$.

Theorem 5.8 can be interpreted as a statement about rod configurations, by observing that replacing each rod by the cone of a tree, gives a graph framework with the same infinitesimal motions as the rod configuration. See Figure 12 for an example of a rod configuration and a bar-joint framework that has the same infinitesimal motions.

To prove Theorem 11, Whiteley shows that generic frameworks of the graph given by replacing each line with the cone of a tree are independent if and only if frameworks where the vertices representing points incident to the same line are collinear are independent. See Figure 13 for an example.

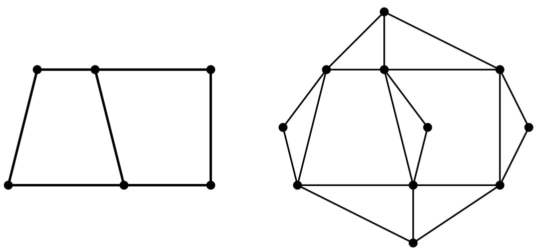

Figure 12: A rod configuration and a non-generic bar-joint framework given by replacing each rod with a tree.

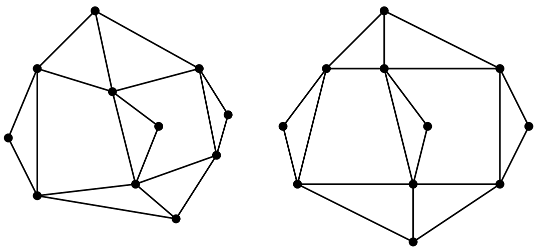

Figure 13: The framework on the left remains rigid when the point incident to a common line are collinear.

6 Summary of papers

Paper I - Exploring the infinitesimal rigidity of planar configurations of points and rods.

S. Lundqvist, K. Stokes and L-D. Öhman, *Discrete Applied Mathematics* **336**, 68-82 (2023)

In this paper, we introduce the following notion of minimal rigidity for rod configurations: A rod configuration realising an incidence geometry is minimally rigid if removing any rod results in a flexible rod configuration.

Our notion of minimal rigidity is different to the notion of minimal rigidity that Whiteley used in Theorem 5.8. The notion of minimal rigidity used in Theorem 5.8, is that a rod configuration is minimally rigid if the graph given by replacing each rod by the cone of a tree in minimally rigid as a bar-joint framework.

Any rod configuration which is minimally rigid with respect to Whiteley's notion of minimality is also minimally rigid with respect to our new notion of minimality, as removing a rod corresponds to removing multiple edges in the bar-joint framework where each rod is replaced by the cone of a tree. However, there are many rod configurations which are minimally rigid with respect to our notion of minimality that are rigid, but not minimally rigid, with respect to Whiteley's notion of minimality. In fact, we give an example of a family of rod configurations which are minimally rigid with respect to our notion of minimality, but that can be arbitrarily far from satisfying the counting conditions of Theorem 5.8. We also show that if a rod configuration is close to satisfying the counting conditions of Theorem 5.8, then it is minimally rigid with our notion of minimality.

Finally, a v_k-configuration is an incidence geometry with v points and v lines such that each line is incident to k points and each point is incident to k lines. We construct examples of families of v_3-configurations which are flexible as rod configurations, and prove that there are v_3-configurations which are flexible as rod configurations for all $v \geq 28$.

Paper II - When is a Planar Rod Configuration Infinitesimally Rigid?

S. Lundqvist, K. Stokes and L-D. Öhman, *Discrete Comput. Geom.* **73(1)**, 25–48 (2025)

Whiteley's version of the molecular conjecture (Theorem 5.8) applies to incidence geometries which have realisations as independent body-joint frameworks. In this paper, we generalise Whiteley's version of the molecular conjecture to incidence geometries which have realisations as dependent body-joint frameworks and satisfy a geometric condition.

If an incidence geometry has a realisation as an independent body-joint framework, then it also has a realisation as a proper rod configuration, where the rods can be chosen to have generic line slopes. The biggest obstacle to generalising Theorem 5.8 was that incidence geometries which have realisations as body-joint frameworks that are not independent may not have realisations as proper rod configurations. Even if they do, there are no guarantees that the proper rod configurations realising the incidence geometry are not in very special position.

To solve this problem, we define the notion of *regularity*. The definition of regularity says that the rows of the concurrence geometry matrix of the framework given by replacing each rod with the cone over a tree are as independent as they can be in a proper realisation. As we consider rod configurations in the plane, parallel redrawings are in one-to-one correspondence with infinitesimal motions, so regularity is a condition which can be used when investigating the space of infinitesimal motions.

For regular rod configurations, we can then show that generic body-joint frameworks realising an incidence geometry are infinitesimally rigid if and only if regular frameworks given by replacing each line by a cone are infinitesimally rigid, thereby generalising Theorem 5.8.

One implication of our result is that testing whether regular rod configurations realising an incidence geometry S is infinitesimally rigid is equivalent to testing whether generic body-joint frameworks realising S are rigid in $\mathbb{R}^2$. Testing whether generic body-joint frameworks realising S are rigid in $\mathbb{R}^2$ can be done in polynomial time with, for example, the pebble game algorithm.

Paper III - Projective rigidity of point-line configurations in the plane

L. Berman, **S. Lundqvist**, B. Schulze, B. Servatius, H. Servatius, K. Stokes and W. Whiteley, *arXiv preprint, arXiv:2407.17836*

In this paper, we introduce the necessary tools to study incidence-preserving (or *projective*) motions of incidence geometries in the projective plane. Studying projective motions essentially amounts to studying the space of realisations of an incidence geometry as points and straight lines in the projective plane.

We define a real projective variety which consists of all realisations of an incidence geometry as points and straight lines. We call this variety the *realisation space* of the incidence geometry. The realisation space of an incidence geometry $S = (P, L, I)$ is philosophically similar to the matroid realisation space of the matroid $\mathcal{M}(S)$, but our realisation space includes degenerate realisations of the incidence geometry, such as those where all point are coincident, which are not included in $\mathcal{M}(S)$.

We define continuous projective motions as continuous curves in the realisation space of the incidence geometry. We also define a projective rigidity matrix, which has as its kernel the *projective infinitesimal motions*. We define projective equilibrium stresses of a configuration to be elements of the cokernel of the projective rigidity matrix. The dimensions of the projective rigidity matrix give necessary counting conditions for independence in the projective rigidity matrix. We give examples to show that the necessary counting conditions are not sufficient.

Furthermore, we consider symmetry-forced projective rigidity. We define a projective orbit rigidity matrix, which has as its kernel the symmetry-preserving projective motions. We can set up the orbit rigidity matrix of a realisation with reflectional, rotational or dihedral symmetry.

Furthermore, any non-degenerate quadratic form on $\mathbb{R}^3$ defines a polarity of the projective plane, which interchanges the points and lines of the projective plane. A realisation of an incidence geometry is *autopolar* with respect to a polarity if the polarity sends the configuration to itself. We can also set up the orbit rigidity matrix with respect to a polarity defined, which has as its kernel the motions that preserve autopolarity. We give an example of an autopolar configuration which has a motion that preserves the autopolarity.

Furthermore, we give an example of a symmetric configuration (pictured on the front cover) which satisfies the necessary conditions for minimal projective rigidity, but has a two-dimensional space of non-trivial projective motions. Interestingly, the non-trivial motions preserve a reflectional symmetry of the configuration, which suggests that all realisations of the configuration might be projectively equivalent to a configuration with reflectional symmetry.

Paper IV - Counting for rigidity under projective transformations in the plane

L. Berman, **S. Lundqvist**, B. Schulze, B. Servatius, H. Servatius, K. Stokes and W. Whiteley, *arXiv preprint, arXiv:2503.07228*

In this paper, we further investigate projective rigidity. We give more examples of configurations which satisfy the necessary conditions for independence, but have projective rigidity matrices with dependent rows. Interestingly, the examples of incidence geometries which satisfy the necessary conditions for independence but have projective rigidity matrices with dependent rows all seem to come from incidence theorems. However, one incidence theorem, Pascal's theorem, satisfies the necessary conditions for independence, and in fact, the rows of its projective rigidity matrix are independent.

Furthermore, we investigate the relationship between different notions of projective rigidity. We show that projective infinitesimal rigidity implies projective rigidity. We give an example of a projectively rigid but projectively infinitesimally flexible configuration. We also show that second-order projective rigidity implies rigidity, but that the converse is not true.

Finally, we give geometric interpretations of projective infinitesimal motions and projective stresses. In particular, the geometric interpretation of projective equilibrium stresses leads to balancing conditions which need to be satisfied in any projective configuration which has a projective stress.

Paper V - Sparse Posets and Pebble Game Algorithms

S. Lundqvist, T. Randrianarisoa, K. Stokes and J. Vermant, *manuscript*

In this paper, we define a sparsity condition on graded posets, which is similar to count matroids defined on graphs. Given a graded poset $\mathcal{P}$, we can define the associated graph $G = (V, E)$ with vertex set $V = V_1 \cup ... \cup V_n$, where each set of vertices V_k consists of the elements of $\mathcal{P}$ of rank k, and edge set

$$E = \{(x, y) \mid r(x) = k, r(y) = k + 1 \text{ and } x < y\}.$$

Let $K = (k_1, ..., k_n)$ and $L = (l_{1,2}, ..., l_{n-1,n})$ be integers. Given a graded poset $\mathcal{P}$, the associated graph $G = (V, E)$ is (K, L)-sparse if and only if

$$|E'| \leq \Sigma_{i=1}^{n} k_i V_i \cap |V(E')| - l_{i,i+1}$$

for all subsets $E' \subseteq E$. We define a pebble game algorithm, which, under certain conditions on K and L, can identify graded posets with (K, L)-sparse associated graphs. The (k, l)-sparse graph are the independent sets on a matroid. We show that the posets with associated (K, L)-sparse graphs are not the independent sets of a matroid, but that the sparsity conditions do define a *greedoid*, which is a generalisation of a matroid.

Paper VI - Applying the pebble game algorithm to rod configurations

S. Lundqvist, K. Stokes and L-D. Öhman, In: EuroCG 2023: Book of abstracts (2023), *Extended Abstract*

By the main result of Paper II, a regular rod configuration is rigid if and only if the graph given by replacing each rod by the cone of a tree is rigid in $\mathbb{R}^2$. Given a rod configuration, the pebble game algorithm can be used to check whether the graph given by replacing each rod by the cone of a tree is rigid in $\mathbb{R}^2$.

We perform a computational experiment to check whether regular rod configurations realising small v_3-configurations are rigid, by applying the pebble game algorithm to the graph given by replacing each rod in the v_3-configuration by the cone of a tree. We show that all regular rod configurations realising v_3-configurations for $v \leq 15$ are rigid. We also show that all rod configurations realising triangle-free v_3-configurations for $v \leq 20$ are rigid. We conjecture that the smallest regular v_3-configuration which is flexible as a rod configuration is a 28_3-configuration that we constructed in Paper I.

Author contributions

I contributed in the following ways to the papers included in this thesis:

Paper I. The main ideas of the paper were developed in discussions among the authors. I contributed several of the proofs and examples. I also made significant contributions to the planning, writing, editing and proofreading of the paper.

Paper II. I contributed the statements and proofs of the main theorems of the paper. I was also responsible for writing the paper, and contributed to the planning, editing and proofreading of the paper.

Paper III. The ideas of this paper were developed in discussions among the authors. I contributed many of the examples and computations, as well as some of the proofs. I also contributed to the planning, writing, editing, and proofreading of the paper.

Paper IV. I contributed to the general discussions where the ideas of the paper were developed. I also contributed to the planning, writing, editing and proofreading of the paper.

Paper V. The main ideas of the paper were developed in discussions among the authors. I contributed to developing the statements and proofs of the main results of the paper. I also contributed to the planning, writing, editing, and proofreading of the paper.

Paper VI. I was responsible for the computational experiment and contributed to the writing, editing, and proofreading of the extended abstract.

7 Conclusions and future work

The objective of this thesis was to investigate rigidity problems for realisations of hypergraphs. Rigidity of rod configurations was investigated in Paper I and Paper II. In Paper I, we introduced a new notion of minimal rigidity for rod configurations, and showed that our notion can be significantly different to the existing notion of minimal rigidity for rod configurations. There is no characterisation of the rod configurations that have minimally rigid realisations as rod configurations. The main outstanding question related to Paper I would be to find such a characterisation.

In Paper II, we further investigate rigidity of rod configurations. We show that under appropriate assumptions about realisability of the hypergraph as points and straight lines in the plane, rigidity of the rod configuration is equivalent to the rigidity of an associated graph. In both Paper I and Paper II, all results apply in the plane. The main result of Paper II could potentially be extended to panel-hinge frameworks in higher dimensional Euclidean spaces. The techniques that we applied in Paper II relied heavily on Theorem 2.5, the characterisation of minimally rigid graphs in $\mathbb{R}^2$. Since there is currently no characterisation of the minimally rigid graphs in $\mathbb{R}^3$, extending the result to higher dimensions would require different techniques.

There are many unanswered questions about projective rigidity, which is the topic of Paper III and Paper IV. Perhaps the most significant open question is finding a characterisation of projective rigidity. The connection to incidence theorems makes characterising projective rigidity a challenging problem, as incidence theorems do not seem to be amenable to explicit descriptions. Realisation spaces of matroids are comparatively well-studied, and investigating the relationship between projective motions of configurations realising an incidence geometry S and properties of the matroid realisation space of $\mathcal{M}(S)$ is a possible avenue for future research.

There are also projective planes in which some incidence theorems which hold in the real projective plane do not hold. For example, there are projective planes in which Desargues theorem and Pappus theorem are not true. An interesting avenue of future research would therefore be to investigate projective rigidity in other projective planes. As different incidence theorems hold in different projective planes, projective rigidity is expected to behave differently in other projective planes. Understanding projective rigidity in these settings could help clarify the relationship between incidence theorems and projective rigidity, as well as the relationship between incidence theorems and the algebraic structure of the underlying projective plane.

Incidences can also be defined between more general objects than points and lines. The set-up of Paper III and Paper IV generalises easily to incidence-preserving motions of points and hyperplanes in real projective spaces of higher dimension. Incidences can also be defined between for example between points and conics, or between spheres, so it would be possible to study projective motions of configurations of points and conics, or sphere packings.

There are also several connections between projective rigidity and other interesting mathematical problems. As an example, scene analysis is the special case of projective motions where $d-1$ coordinates of each point are fixed throughout the motion. The research on projective rigidity also fits into a larger theory of hyperplane arrangements. The relationship between projective rigidity and the larger theory of hyperplane arrangements is largely unexplored. Exploring connections between projective rigidity and other problems is potentially a promising direction for future research.

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