Numerische Mathematik (2026) 158:249–280
https://doi.org/10.1007/s00211-025-01502-6

Numerische Mathematik

On the design of locking free ghost penalty stabilization and the relation to CutFEM with discrete extension

Erik Burman[1] · Peter Hansbo[2] · Mats G. Larson[3]

Received: 4 March 2024 / Revised: 15 March 2025 / Accepted: 23 July 2025 /
Published online: 7 November 2025
© The Author(s) 2025

Abstract
In this note, we introduce a novel stabilization mechanism for Cut Finite Element Methods (CutFEM), generalizing previous ghost penalty techniques in two key aspects: (1) the choice of stabilized quantities and (2) the selection of elements involved in stabilization. This approach notably allows for flexible and precise definitions of the stabilized quantities, including various functionals associated with the discrete solution, such as finite element degrees of freedom. We demonstrate that the kernel of our proposed ghost penalty operator defines a finite element space characterized by discrete extensions, closely related to those previously presented in Burman et al. (Numer Math 152(2):331–369, 2022).

Mathematics Subject Classification 65N30

1 Introduction

Contributions. Cut finite element methods rely on embedding a computational domain within a background mesh that need not conform to the domain boundary, resulting in so-called cut elements near the boundary. By adding stabilization terms, we can control the behavior of discrete functions near the boundary, enabling proofs of stability, condition number estimates, and optimal-order a priori error bounds. Alternatively, a discrete extension operator may be used, allowing the problem to be solved within a subspace of the finite element space, effectively removing unstable degrees of freedom while preserving optimal approximation properties. Although these two approaches share the same objective-stabilizing the method-they initially seem distinct.

In this note, we demonstrate that stabilization terms added to the weak formulation can be generalized in two significant ways:

✉ Mats G. Larson
mats.larson@umu.se

1 UCL, London, United Kingdom

2 Jönköping University, Jonkoping, Sweden

3 Umeå University, Umeå, Sweden

- The stabilized quantity may be a general functional of the discrete function, such as finite element degrees of freedom. This allows a more precise stabilization of unstable modes than standard, typically element-based, methods. Additionally, this approach enables flexible tailoring or customization in the stabilization design.
- The choice of connected elements can also be generalized. While standard methods usually employ face neighbors or connected patches, we may instead stabilize by connecting boundary-intersecting elements to those located within a distance proportional to the mesh parameter.

We further show the following theoretical results:

- The generalized stabilization framework adheres to standard abstract stability requirements, thereby yielding stable methods with optimal-order convergence for second-order elliptic problems. Moreover, by comparing our results with those in [11] and the companion paper [12], it is straightforward to extend our findings to elliptic problems of higher order.
- For a robust design of the generalized stabilization form, the stabilization parameter can be allowed to approach infinity without causing locking. In this limit, the stabilization corresponds to strongly enforcing certain algebraic constraints, which align exactly with constraints implemented in specific extension operator frameworks, see [11]. This reveals the close connection between stabilization and extension methodologies.

Previous work. Extending polynomial approximation from the interior to the boundary to enhance the stability of numerical schemes was considered in [16] within the context of fictitious domain methods. A related stabilization effect, including additional control over the condition number of the system matrix, was achieved by introducing penalty terms as discussed in [7], and further explored in [9, 21–23, 26] for CutFEM methods utilizing Nitsche's approach for weak imposition of interface conditions, as well as in [10] for unfitted finite element approximations of surface PDEs. Stabilization procedures localized to macro elements were developed in [20] and applied to time dependent problems in [24]. Parallel developments have employed agglomeration techniques for boundary and bulk elements to achieve stability, effectively extending polynomial approximation spaces from the interior towards the boundary. For nonconforming FEM, such approaches have been developed in fictitious-domain formulations using discontinuous Galerkin methods as described in [14, 19], and in hybrid high-order methods for interface problems in [8]. Advanced cell merging algorithms are presented in [28] and an approach using discontinuous Trefftz spaces was developed in [17]. While agglomeration approaches are naturally suited for discontinuous approximation spaces, their application to conforming spaces presents additional challenges. Agglomeration strategies using C^0 approximation spaces with element merging and hanging nodes were explored in [18]. Alternative approaches based on extensions of shape functions were developed in [1, 3, 4, 25]. A general framework for discrete extension operators to enable higher conformity of FEM spaces was introduced in [11] and extension operators for trimmed splines were developed in [12]. Additionally, weak stabilization methods based on penalizing differences between finite element functions and their extended counterparts were considered in [2]. For additional references and discussions of various stabilization techniques, see [13].

In this work, our objective is to develop a stabilization framework with minimal assumptions. We first demonstrate that standard stabilization forms fit naturally within this framework and then exploit its flexibility to derive nodal stabilization forms targeting precisely the unstable modes. Consequently, the nodal stabilization remains robust for large stabilization parameters, avoids locking, and eliminates the over-stabilization present in standard stabilization approaches for conforming methods. For discontinuous methods, localization to non-overlapping macro elements [20] is sufficient to circumvent locking, owing to the increased flexibility of the space.

Outline. In Sect. 2, we introduce a general framework for designing ghost penalty terms and illustrate its application with examples from the literature. We also introduce nodal stabilization and discuss its implementation. Additionally, the concept of locking is presented along with the design criteria necessary to avoid it. Section 3 focuses on the analysis of these methods. Initially, we identify the required conditions on the penalty term and subsequently demonstrate that under certain localization conditions for couplings in the penalty term, locking does not occur, even for very large stabilization parameters. The main contribution of this paper is demonstrating that nodal stabilization fulfills the criteria for robust and accurate ghost penalty terms. Finally, in Sect. 4, we illustrate the theoretical findings through several numerical examples.

2 The stabilization framework

In this section, we develop a general framework for stabilization, generalizing existing approaches in two ways. First, we extend stabilization to a broader class of quantities, including specific functionals such as finite element degrees of freedom. Second, we introduce more flexible choices regarding how elements are connected within the stabilization. This approach enables precise stabilization of individual degrees of freedom and allows the penalty parameter to tend toward infinity without causing locking.

2.1 The mesh and finite element spaces

- Let $\mathcal{T}_{h,0}$ be a quasiuniform mesh, with mesh parameter $h \in (0, h_0]$ consisting of closed elements T, on a closed polygonal domain $\Omega_0 \subset \mathbb{R}^d$. Let $V_{h,0}$ be a finite element space on $\mathcal{T}_{h,0}$,

$$V_{h,0} := \{v_h \in C^0(\Omega_0) : v_h|_T \in \mathbb{P}_k(T), \quad \forall T \in \mathcal{T}_{h,0}\}$$

  where $\mathbb{P}_k(T)$ denotes the set of polynomials of degree less than or equal to k on the simplex T. Here we consider the setting of C^0 finite elements, but the discussion below is easy to extend to the case of smoother approximation spaces using the ideas from [11].
- Let $\Omega \subset \Omega_0$ be a given closed domain and let $\mathcal{T}_h = \{T \in \mathcal{T}_{h,0} : T \cap \Omega \neq \emptyset\}$ be the active mesh and let $\Omega_h = \cup_{T \in \mathcal{T}_h} T$. Let $V_h = V_{h,0}|_{\Omega_h}$ be the active finite element space. Let $V_{h,T} = V_h|_T$ be the local finite element space obtained by considering the restriction of any $v_h \in V_h$ to the element T.

- Let $\mathcal{B}_h = \{\varphi_i : i \in I\}$ be the global finite element basis in V_h indexed by the set I. For each $T \in \mathcal{T}_h$ let $\mathcal{B}_{h,T} = \{\varphi_{i,T} = \varphi_i|_T : i \in I_T\}$ be the element finite element basis in $V_{h,T}$, with $I_T \subset I$ the indices such that $T \subset \text{supp}(\varphi_i)$. Let $\mathcal{B}^*_{h,T} = \{\varphi^*_{i,T} : i \in I_T\}$ be the element degrees of freedoms such that $\varphi^*_{j,T}(\varphi_{i,T}) = \delta_{ij}$ and note that $\text{span}(\mathcal{B}_{h,T}) = V^*_{h,T}$. Since we have a conforming finite element space we have, $\varphi^*_{i,T'}(w) = \varphi^*_{i,T''}(w)$ for any two elements T' and T'' in the support of φ_i and $w \in V_h|_{\text{supp}(\varphi_i)}$.

The L^2-inner product over some domain X will be denoted by

$$(v, w)_X := \int_X uw \, \mathrm{d}X, \text{ with norm } \|v\|_X := (v, v)_X^{\frac{1}{2}}. \tag{2.1}$$

For symmetric positive semi-definite bilinear forms $s(v, w)$ the associated (semi-) norm will be denoted $\|v\|_s := s(v, v)^{\frac{1}{2}}$.

2.2 Definition of stabilization forms

This section will propose some generic design criteria for ghost penalty stabilizations. We will then show that several known stabilizations enter the proposed framework and that this abstract design leads to methods with the desired properties. For simplicity, we restrict the presentation to methods based on extensions of polynomials on an element. Another possibility is to consider a patch of elements and then first project to a global polynomial on the patch that is then extended. The below arguments can be straightforwardly extended to that case.

- Consider two elements T_1 and T_2 in $\mathcal{T}_h$ and let $v_i^e \in \mathbb{P}_k(\mathbb{R}^d)$ be the canonical extension of $v_i \in V_{T_i}$. We may then define the jump

$$[v]_{T_1,T_2} = v_1^e - v_2^e \in \mathbb{P}_k(\mathbb{R}^d) \tag{2.2}$$

  See Fig. 1 for an illustration.
- For a symmetric positive semidefinite bilinear form $b : \mathbb{P}_k(\mathbb{R}^d) \times \mathbb{P}_k(\mathbb{R}^d) \to \mathbb{R}$ and a pair of elements $T_1, T_2 \in \mathcal{T}_h$, we define the stabilization term associated with the triple (b, T_1, T_2) by

$$s_{m,b,T_1,T_2}(v, w) = \tau h^{\alpha_m} b([v]_{T_1,T_2}, [w]_{T_1,T_2}) \tag{2.3}$$

  where $\tau > 0$ and α_m are parameters. Here, α_m is determined in such a way that the form stabilises the H^m norm for $m = 0, 1$, and $\tau > 0$ is a stabilization parameter that typically is determined by the stability analysis.
- Let $\mathcal{T}_h = \mathcal{T}_h^L \cup \mathcal{T}_h^S$ be a partition of the elements, into two subsets called large and small elements, where

$$\mathcal{T}_h^L = \{T \in \mathcal{T}_h : |T \cap \Omega| \geq \gamma |T|\}, \qquad \mathcal{T}_h^S = \mathcal{T}_h \setminus \mathcal{T}_h^L \tag{2.4}$$

  with $\gamma \in (0, 1]$.

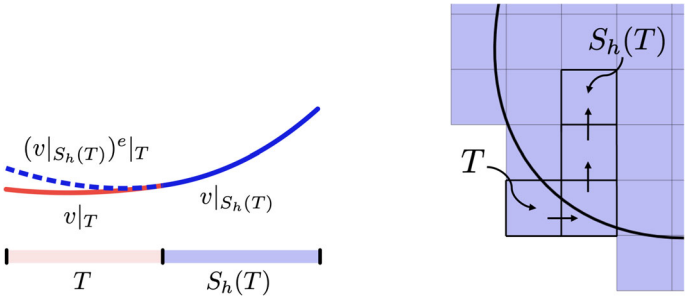


Fig. 1 Left: Illustration of the generalized jump. Right: Illustration of Assumption A2. For each small element T, the mapping $T \mapsto S_h(T)$ provides an associated large element $S_h(T)$ such that there exists a path of face-neighboring elements connecting T and $S_h(T)$. The length of this path is uniformly bounded independent of the mesh size

- Let $S_h : \mathcal{T}_h^S \to \mathcal{T}_h^L$ be a mapping, which assigns a large element $S_h(T) \in \mathcal{T}_h^L$ to each small element $T \in \mathcal{T}_h^S$. We will focus on stabilization terms of the form

$$s_{m,b,T,S_h(T)}(v, w) = \tau h^{\alpha_m} b([v]_{T,S_h(T)}, [w]_{T,S_h(T)}), \qquad T \in \mathcal{T}_h^S \tag{2.5}$$

- Given a set $\mathcal{S}$ of triples of the form $(b, T, S_h(T))$ we define

$$\boxed{s_{h,m}(v, w) = \sum_{(b,T,S_h(T)\in\mathcal{S}} s_{m,b,T,S_h(T)}(v, w)} \tag{2.6}$$

- For $\psi_T \in V_{h,T}^*$ we define the stabilization form

$$b_{m,T}(p, q) = \tau h^{\alpha_m} \psi_T(p)\psi_T(q), \qquad p, q \in \mathbb{P}_k(\mathbb{R}^d) \tag{2.7}$$

which gives

$$s_{m,b_T,T,S_h(T)}(v, w) = \tau h^{\alpha_m} \psi_T([v]_{T,S_h(T)})\psi_T([w]_{T,S_h(T)}) \tag{2.8}$$

An important special case is $\psi = \varphi_{i,T}^*$ which enables control of degree of freedom i. To define which nodes need stabilization we let

$$I = I^S \cup I^L \tag{2.9}$$

be a partition of the global index set I into the indices I^S such that the corresponding basis functions does not contain any large element in their support and the complement $I^L = I \backslash I^S$. For each $i \in I^S$ let T_i be an element such that $T_i \subset \text{supp}(\varphi_i)$, and define

$$\boxed{s_{h,m}(v, w) = \sum_{i\in I^S} \tau h^{\alpha_m} \varphi_{i,T_i}^*([v]_{T_i,S_h(T_i)})\varphi_{i,T_i}^*([w]_{T_i,S_h(T_i)})} \tag{2.10}$$

This construction enables precise stabilization of individual degrees of freedoms, and we refer to it as nodal stabilization.

- The semi-norm induced by the stabilization will be denoted

$$\|v\|^2_{s_{h,m}} = s_h(v, v) \tag{2.11}$$

where we recall that subscript m will take the values 0 or 1 depending on the stabilization is designed to give stability in the L^2-norm or the H^1-norm.

2.3 Implementation

Let us for simplicity consider piecewise linear elements on a triangulation and the nodal stabilization form (2.10). For each $i \in I^S$ we pick $T_i \subset \text{supp}(\varphi_i)$ and we let $S_h(T_i)$ be an element in $\mathcal{T}_h^L$ close to T. For instance, $S_h(T_i)$ may be chosen as the element in $\mathcal{T}_h^L$ closest to T_i. The functional $\varphi^*_{i,T_i}(v)$ is simply the nodal value in node i, denoted by x_i, i.e.,

$$\varphi_i^*(v) = v(x_i) = \hat{v}_i \tag{2.12}$$

Thus the stabilizing term for node i takes the form

$$s_{m,b,T_i,S_h(T_i)}(v, w) = \tau h^{\alpha_m}([v(x_i)]_{T_i,S_h(T_i)})([w(x_i)]_{T_i,S_h(T_i)}) \tag{2.13}$$

where

$$[v(x_i)]_{T_i,S_h(T_i)} = v(x_i) - \sum_{j \in I_{S_h(T_i)}} v(x_j)\varphi^e_{j,S_h(T_i)}(x_i) \tag{2.14}$$

and $\{\varphi_{j,S_h(T_i)}\}_{j \in I_{S_h(T_i)}}$ is the element basis on element $S_h(T_i)$. In matrix form we get

$$s_{m,b,T_i,S_h(T_i)}(v, w) = \tau h^{\alpha_m} \hat{v} \cdot \widehat{B}_i \cdot \hat{w} \tag{2.15}$$

with

$$\widehat{B}_i = \widehat{\omega}_i \otimes \widehat{\omega}_i \tag{2.16}$$

where $\widehat{\omega}_i$ is the vector

$$\widehat{\omega}_i = e_i - \sum_{j \in I_{S_h(T)}} \varphi^e_{j,S_h(T_i)}(x_i) e_j \tag{2.17}$$

with four non zero elements, $\{e_j\}_{j \in I}$ is the canonical basis in $\mathbb{R}^N$ with $N = |I|$, and $\varphi^e_{j,T}$ is the canonical extension of the elements basis functions from $S_h(T_i)$ to $\mathbb{R}^d$. For other standard finite elements spaces that satisfies the Ciarlet definition, see [5], we have the same implementation, see [11], with the modification that the number of

basis functions on the elements are different. We refer to [12] for extensions to spline spaces and further implementation details.

2.4 Examples of stabilization forms

Below we present a couple of examples of common stabilization forms to illustrate how they fit into the framework and to emphasize that the proposed stabilization forms (2.6) and (2.10) is indeed a natural extension of previous terms.

Example 1 (See [7]). We may fit the standard Ghost or Face penalty in the framework as follows. First let $\mathcal{F}_h$ be the set of all internal faces in $\mathcal{T}_h$ that belong to an element T that intersects the boundary. For piecewise linears the stabilization term takes the form

$$s_{h,m}(v, w) = \sum_{F \in \mathcal{F}_h} h^{3-2m}([\nabla_n v], [\nabla w])_F \tag{2.18}$$

where $m = 0, 1$, and

$$[\nabla_n v] = \nabla_{n_1} v_1 + \nabla_{n_2} v_1 \tag{2.19}$$

with T_1 and T_2 the elements sharing face F and $v_i = v|_{T_i}$. To set this term into our framework we have for each face the ordered triple (b_F, T_1, T_2) where

$$b_F(p, q) = (\nabla_{n_1} p, \nabla_{n_1} q)_F, \qquad p, q \in \mathbb{P}_k(\mathbb{R}^d) \tag{2.20}$$

Then taking $\alpha_m = 3 - 2m$ we get

$$\begin{aligned} s_{b,T_1,T_2}(v, w) &= \tau h^{3-2m} b_F([v]_{T_1,T_2}, [w]_{T_1,T_2}) && (2.21)\\ &= \tau h^{3-2m} (\nabla_{n_1}[v]_{T_1,T_2} \nabla_{n_1}[v]_{T_1,T_2})_F && (2.22)\\ &= \tau h^{3-2m} ([\nabla_n v], [\nabla_n v])_F && (2.23) \end{aligned}$$

where we used the identity

$$\nabla_{n_1}[v]_{T_1,T_2} = \nabla_{n_1}(v_1 - v_2) = \nabla_{n_1} v_1 + \nabla_{n_2} v_2 = [\nabla_n v] \tag{2.24}$$

which holds since $n_2 = -n_1$ on F.

Example 2 See [21, 26]. Taking

$$b_F(p, q) = (\nabla p, \nabla q)_{T_1 \cup T_2}, \qquad p, q \in \mathbb{P}_k(\mathbb{R}^d) \tag{2.25}$$

and $\alpha_1 = 0$ gives us the form

$$s_{1,b_F,T_1,T_2}(v,w) = (\nabla[v]_{T_1,T_2}, \nabla[w]_{T_1,T_2})_{T_1\cup T_2} = (\nabla(v_1^e - v_2^e), \nabla(w_1^e - w_2^e))_{T_1\cup T_2} \tag{2.26}$$

which can be used to control the H^1 seminorm. Alternatively using the L^2 product

$$b_F(p,q) = (p,q)_{T_1\cup T_2} \tag{2.27}$$

with $\alpha_m = -2m$, $m = 0, 1$, gives us the form

$$\begin{aligned} s_{m,b_F,T_1,T_2}(v,w) &= \tau h^{-2m}([v]_{T_1,T_2}, [w]_{T_1,T_2})_{T_1\cup T_2} \\ &= \tau h^{-2}(v_1 - v_2, w_1 - w_2)_{T_1\cup T_2} \end{aligned} \tag{2.28}$$

which can be used to control the L^2 norm and H^1 norm for $m = 0$ and $m = 1$, respectively.

Example 3 For each $T \in \mathcal{T}_h^S$ we consider the triple $(b_T, T, S_h(T))$ with

$$b_{1,T}(p,q) = (\nabla p, \nabla q)_T, \qquad p, q \in \mathbb{P}_k(\mathbb{R}^d) \tag{2.29}$$

leading to the stabilization form

$$s_{1,b_T,T,S_h(T)}(v,w) = \tau(\nabla[v]_{T,S_h(T)}, \nabla[w]_{T,S_h(T)})_T \tag{2.30}$$

Several variants are possible, for instance, we may take the domain of integration to be $T \cup S_h(T)$ or $S_h(T)$, we can use the L^2 inner product and $\alpha_m = -2m$, $m = 0, 1$, and we may let $S_h : \mathcal{T}_h^S \to \mathcal{T}_h^I$ where $\mathcal{T}_h^I$ is the set of elements residing in Ω, i.e. $T \subset \Omega$. The final stabilization term takes the form

$$s_{h,1}(v,w) = \sum_{T\in\mathcal{T}_h^S} s_{1,b_T,T,S_h(T)}(v,w) = \sum_{T\in\mathcal{T}_h^S} \tau(\nabla[v]_{T,S_h(T)}, \nabla[w]_{T,S_h(T)})_T \tag{2.31}$$

This stabilization shows that we may use flexible pairs of elements $(T, S_h(T))$ to construct a stabilization not only using face neighbors as in standard Ghost penalty. If we now relax the control on element T using instead a functional $\psi_T \in V_{h,T}^*$ we let

$$b_{m,T}(p,q) = h^{d-2m}\psi_T(p)\psi_T(q), \qquad p, q \in \mathbb{P}_k(\mathbb{R}^d) \tag{2.32}$$

and

$$s_{m,b_T,T,S_h(T)}(v,w) = \tau h^{d-2m}\psi_T([v]_{T,S_h(T)})\psi_T([w]_{T,S_h(T)}) \tag{2.33}$$

which gives (2.8).

Example 4 (Stabilized version of the approach from [6, 16]). Recalling that in order to establish the coercivity of Nitsche's method we need the inverse inequality

$$h\|\nabla_n v\|^2_{T\cap\partial\Omega} \lesssim \|\nabla v\|^2_T \tag{2.34}$$

for elements $T \in \mathcal{T}_h$ that intersect the boundary, and we may therefore consider

$$b_T(p,q) = (\nabla_n p, \nabla_n q)_{T\cap\partial\Omega}, \qquad p,q \in \mathbb{P}_k(\mathbb{R}^d) \tag{2.35}$$

leading to the stabilization form

$$s_{b_T,T,S_h(T)}(v,w) = \tau h(\nabla_n [v]_{T,S_h(T)}, \nabla_n [w]_{T,S_h(T)})_{T\cap\partial\Omega} \tag{2.36}$$

We then note that

$$h\|\nabla_n v\|^2_{T\cap\partial\Omega} \lesssim h\|\nabla_n (v|_{S_h(T)})^e\|^2_{T\cap\partial\Omega} + h\|\nabla_n [v]_{T,S_h(T)}\|^2_{T\cap\partial\Omega} \tag{2.37}$$

$$\lesssim \|\nabla v\|^2_{S_h(T)} + \|v\|^2_{s_{b_T,T,S_h(T)}} \tag{2.38}$$

We conclude that we have constructed a stabilization targeting the inverse inequality needed for coercivity. Although this approach leads to stable fictitious domain methods using Nitsche's method, it does not alleviate the ill-conditioning of the system matrix. Note that we omit the index m in the notation in this and the following example since we target a quantity that is not the L^2 or H^1 norm.

Example 5 As an alternative to the stabilization in Example 4, resulting in a method that also is well conditioned, we let n_T be the constant L^2 projection of the normal at $T \cap \partial\Omega$ (or the normal at any point on $T \cap \partial\Omega$) and note that

$$h\|\nabla_n v\|^2_{T\cap\partial\Omega} \lesssim h\|\nabla_{n_T} v\|^2_{T\cap\partial\Omega} + h\|n - n_T\|^2_{L^\infty(T\cap\partial\Omega)}\|\nabla v\|^2_{T\cap\partial\Omega} \tag{2.39}$$

$$\lesssim h\|\nabla_{n_T} v\|^2_{T\cap\partial\Omega} + h^3\|\nabla v\|^2_{T\cap\partial\Omega} \tag{2.40}$$

$$\lesssim \|\nabla_{n_T} v\|^2_T + \|v\|^2_T \tag{2.41}$$

This estimate suggests defining

$$b_T(p,q) = (\nabla_{n_T} p, \nabla_{n_T} q)_T + (p,q)_T \tag{2.42}$$

and

$$\begin{aligned} &s_{b_T,T,S_h(T)}(v,w) \\ &= \tau_1(\nabla_{n_T}[v]_{T,S_h(T)}, \nabla_{n_T}[w]_{T,S_h(T)})_T + \tau_2([v]_{T,S_h(T)}, [w]_{T,S_h(T)})_T \end{aligned} \tag{2.43}$$

with parameters τ_1 and τ_2. Here we get control of the $L^2(T)$ norm as well as the control of the normal derivative necessary for the inverse inequality (2.34) to hold.

The L^2 control in example 5 is needed to derive condition number estimates. More precisely the crucial stability property that allows to prove bounds on the condition number [7, 9, 10, 15] takes the form

$$\boxed{\|\nabla^m v\|^2_{\Omega_h} \lesssim \|\nabla^m v\|^2_{\Omega} + \|v\|^2_{s_{h,m}}, \qquad m = 0, 1, \quad v \in V_h} \tag{2.44}$$

where the semi-norm induced by the stabilization satisfies, $\|v\|^2_{s_{h,m}} \lesssim \|\nabla^m v\|^2_{\Omega_h}$ for all v in the finite element space (see Lemma 3.3 below). This type of bound is known to hold for the stabilizations of Examples 1, 2, 3 and 5. In this paper we will prove the bound for the penalty operator on the form (2.10).

2.5 Locking

The stabilization provides additional control of the solution in the interface zone and enhances the approximation's accuracy. However, increasing the stabilization parameter τ may lead to growing constants $C(\tau)$ in error estimates while maintaining optimal convergence order for each fixed value of the parameter τ. This type of phenomenon is referred to as locking; for fixed h and large τ the finite element space may not be sufficiently rich to satisfy the constraint imposed by the penalty operator while at the same time providing a good approximation. Usually, the choice of τ is straightforward, and locking is not a significant issue in CutFEM; however, when dealing with systems of equations and nonlinearities, a robust variant is preferable. How pronounced the locking problem is depends on the specific nature of the stabilization. For instance, the face based stabilization forms (2.18) and (2.28) leads to locking since in the limit when $\tau \to \infty$ all elements at the boundary will be coupled, and the only free function is a global polynomial. The coupling becomes less pronounced when the stabilization is based on pairs $(T, S_h(T))$ of elements. Indeed we will prove that the stabilizations of the form (2.6) can be designed to avoid locking under suitable assumptions on the mappings S_h, both for the element-based stabilization and the nodal stabilization (2.10). The key to these results is to show that any function $u \in H^1(\Omega)$ admits an interpolant $\pi_h u \in V_h$ with optimal approximation properties such that $s_h(\pi_h u, v_h) = 0$ for all $v_h \in V_h$. This form of discrete strong consistency is satisfied when the kernel of s_h is large enough. We note that the nodal stabilization is locking-free under very mild assumptions. In contrast, the (sufficient) conditions on the element-based stabilization appear to be more difficult to satisfy.

3 Analysis

In this section we establish the basic properties of the stabilization s_h defined by (2.6) and (2.10). We start with the straightforward analysis of the general stabilization form (2.6) and present a stability result and a consistency result. Then we turn to nodal stabilization, which demands slightly more complex arguments. Here we first show a stability result in the natural norms. Then we show that the stabilization parameter τ may tend to infinity without introducing locking or loss of the order of convergence.

3.1 Properties of the General Stabilization

We start by specifying certain properties of the stabilization required for the analysis.

Assumptions. The following holds uniformly for all triples $(b, T, S_h(T)) \in \mathcal{S}$:

A1. There is a constant such that

$$h^{\alpha_m} \|p\|_b^2 \lesssim \|\nabla^m p\|_T^2, \qquad p \in \mathbb{P}_k(\mathbb{R}^d) \tag{3.1}$$

A2. There is a path $\mathcal{P}_h(T, S_h(T)) = \{T_j \in \mathcal{T}_h : j = 1, \ldots, n\}$ of face neighboring elements starting in $T_1 = T$ and ending in $T_n = S_h(T)$ with bounded length $n \lesssim 1$. See Fig. 1.

Remark 3.1 It is straightforward to verify that the examples given in Sect. 2.4 satisfy the assumptions A1 and A2.

Lemma 3.1 *(Stability) If A1 and A2 hold then there is a constant such that*

$$\boxed{\sum_{(b,T,S_h(T))\in\mathcal{S}} h^{\alpha_m} \|v\|_b^2 \lesssim \|\nabla^m v\|_\Omega^2 + \|v\|_{s_{h,m}}^2} \tag{3.2}$$

Proof Adding and subtracting $(v|_{S_h(T)})^e$, using the Definition (2.5) of $s_{b,T,S_h(T)}$, and finally assumption A1, we directly get

$$h^{\alpha_m} \|v\|_b^2 \lesssim h^{\alpha_m} \|(v|_{S_h(T)})^e\|_b^2 + h^{\alpha_m} \|[v]_{T,S_h(T)}\|_b^2 \tag{3.3}$$

$$\lesssim \|\nabla^m (v|_{S_h(T)})^e\|_T^2 + \|v\|_{s_{b,T,S_h(T)}}^2 \tag{3.4}$$

$$\lesssim \|\nabla^m v\|_{S_h(T)}^2 + \|v\|_{s_{b,T,S_h(T)}}^2 \tag{3.5}$$

Summing over all stabilizing forms, and using the fact that there is a uniformly bounded number of stabilization forms associated with each element $T \in \mathcal{T}_h^S$, we obtain the desired estimate. ■

In preparation for the next lemma, we establish the following Poincaré type estimate for the jump operator.

Lemma 3.2 *Let S_h satisfy assumption A2 and $T_0 \in \mathcal{T}_h$ an element such that $T_0 \cup T \cup S_h(T) \subset B_\delta$. Then there is a constant such that*

$$\|[v]_{T,S_h(T)}\|_{T_0} \lesssim h \|\nabla v\|_{\mathcal{P}_h(T,S_h(T))} \tag{3.6}$$

Proof Using the notation $T_1 = T$, $T_2 = S_h(T)$, and $v|_{T_i} = v_i$ for $i = 1, 2$, we get by adding and subtracting a constant function $w \in P_0(\mathbb{R}^d)$,

$$\|[v]_{T_1,T_2}\|_{T_0} \lesssim \|v_1^e - w\|_{T_0} + \|w - v_2^e\|_{T_0} \tag{3.7}$$

$$\lesssim \|v_1^e - w\|_{T_1} + \|w - v_2^e\|_{T_2} \tag{3.8}$$

$$= \|v - w\|_{T_1} + \|w - v\|_{T_2} \tag{3.9}$$
$$\lesssim \|v - w\|_{\mathcal{P}_h(T_1,T_2)} \tag{3.10}$$
$$\lesssim h\|\nabla v\|_{\mathcal{P}_h(T_1,T_2)} \tag{3.11}$$

Here we used the stability

$$\|p\|_{T'} \lesssim \|p\|_T, \qquad p \in \mathbb{P}_k(\mathbb{R}^d) \tag{3.12}$$

which holds when $T' \cup T \subset B_\delta$ for some $\delta \lesssim h$. To verify the final estimate (3.7) there is by assumption A2 a path $\mathcal{P}_h(T_1, T_2) = \{T^j\}_{j=1}^n$ of face neighboring elements such that $T^1 = T_1$ and $T^n = T_2$. Then recursively applying, the Poincaré inequality

$$\|v\|_{T'}^2 \lesssim \|v\|_{T''}^2 + h^2\|\nabla v\|_{T'\cup T''}^2, \qquad v \in H^1(T' \cup T'') \tag{3.13}$$

for elements T', T'' sharing a face, we get

$$\|v - w\|_{\mathcal{P}_h(T_1,T_2)}^2 \lesssim \|v - w\|_{T_1}^2 + h^2\|\nabla v\|_{\mathcal{P}_h(T_1,T_2)}^2 \lesssim h^2\|\nabla v\|_{\mathcal{P}_h(T_1,T_2)}^2 \tag{3.14}$$

where in the final inequality we choose w to be the L^2 projection on constants on the element T_1. ■

We will see in the analysis presented below that it is natural to require that a weak consistency estimate, see (3.21), holds. This bound ensures that the stabilization is not too strong and that the convergence rate in a finite element method is not negatively affected by the stabilization.

To discuss the consistency of the method we start by defining an interpolation operator $\pi_h : L^2(\Omega) \to V_h$. To that end we introduce the local L^2-projection $P_T : L^2(T) \to \mathbb{P}_k(T)$ and recall that there is an universal extension operator $E : H^s(\Omega) \ni v \mapsto v^E \in H^s(\mathbb{R}^d)$, see [27], satisfying the stability

$$\|v^E\|_{H^s(\mathbb{R}^d)} \lesssim \|v\|_{H^s(\Omega)}, \qquad s \geq 0 \tag{3.15}$$

Next we define a Clément interpolation operator of form

$$\pi_{h,Cl} : L^2(\Omega_h) \ni v \mapsto \sum_{i\in I} \varphi_i^*(P_{T_i} v|_{T_i})\varphi_i \in V_h \tag{3.16}$$

where $T_i \in \mathcal{T}_h$ is an element in the support of φ_i. Then we define the interpolation operator by composing the Clément operator and the extension operator,

$$\pi_h : L^2(\Omega) \ni v \mapsto \pi_{h,Cl} E v \in V_h \tag{3.17}$$

By standard arguments we have the stability

$$\|\pi_h v\|_T \lesssim \|v\|_{\mathcal{T}_h(T)} \tag{3.18}$$

where $\mathcal{T}_h(T)$ is the set of all elements that share a node with T and the the error estimate

$$\|v - \pi_h v\|_{H^m(\Omega_h)} \lesssim h^{k+1-m}\|v\|_{H^{k+1}(\Omega)}, \qquad m = 0, 1 \tag{3.19}$$

Lemma 3.3 (Weak consistency general interpolants) *If A1 and A2 hold then there are constants such that*

$$\boxed{\|v\|_{s_{h,m}} \lesssim \tau^{1/2}\|\nabla^m v\|_{\Omega_h}, \qquad v \in V_h} \tag{3.20}$$

and

$$\boxed{\|\pi_h v\|_{s_{h,m}} \lesssim \tau^{1/2} h^{k+1-m}\|v\|_{H^{k+1}(\Omega)}, \qquad v \in H^{k+1}(\Omega)} \tag{3.21}$$

Proof First, we prove (3.20). We have for $v \in V_h$,

$$\begin{aligned}
\|v\|_{s_h}^2 &= \sum_{(b,T,S_h(T))\in\mathcal{S}} \tau h^{\alpha_m}\|[v]_{T,S_h(T)}\|_b^2 && (3.22)\\
&\lesssim \sum_{(b,T,S_h(T))\in\mathcal{S}} \tau\|\nabla^m[v]_{T,S_h(T)}\|_T^2 && (3.23)\\
&\lesssim \sum_{(b,T,S_h(T))\in\mathcal{S}} \tau h^{-2m}\|[v]_{T,S_h(T)}\|_T^2 && (3.24)\\
&\lesssim \sum_{(b,T,S_h(T))\in\mathcal{S}} \tau h^{2(1-m)}\|\nabla v\|_{\mathcal{P}_h(T,S_h(T))}^2 && (3.25)\\
&\lesssim \tau\|\nabla^m v\|_{\mathcal{T}_h}^2 && (3.26)
\end{aligned}$$

where we used some inverse estimates and Lemma 3.2. For the weak consistency estimate let $v \in H^{k+1}(\Omega)$ and consider the form associated with the triple $(b, T, S_h(T))$. We have for each $w \in \mathbb{P}_k(\mathbb{R}^d)$,

$$\begin{aligned}
\|\pi_h v\|_{s_{b,T,S_h(T)}} &= \|\pi_h(v-w)\|_{s_{b,T,S_h(T)}} \lesssim h^{\alpha_m/2}\|[\pi_h(v-w)]_{T,S_h(T)}\|_b && (3.27)\\
&\lesssim \|\pi_h(v-w)\|_{H^m(T\cup S_h(T))} \lesssim \|v-w\|_{H^m(\mathcal{T}_h(\mathcal{P}_h(T,S_h(T))))}\\
&\lesssim h^{k+1-m}\|v\|_{H^{k+1}(B_{\delta'})} && (3.28)
\end{aligned}$$

where in the last inequality we choose w according to the Bramble-Hilbert lemma, see [5], and $B_{\delta'}$ is an open ball containing $\mathcal{T}_h(\mathcal{P}_h(T, S_h(T)))$, the set of all elements that share a node with an element in the chain $\mathcal{P}_h(T, S_h(T))$ that connects T and $S_h(T)$, with $\delta' \sim h$. Summing the contributions we obtain

$$\|\pi_h v\|_{s_{h,m}}^2 = \sum_{(b,T,S_h(T))\in\mathcal{S}} \|\pi_h v\|_{s_{b,T,S_h(T)}}^2 \tag{3.29}$$

$$\lesssim \sum_{(b,T,S_h(T))\in\mathcal{S}} \tau h^{2(k+1)} \|v\|^2_{H^{k+1-m}(B_{\delta'})} \lesssim \tau h^{2(k+1-m)} \|v\|^2_{H^{k+1}(\Omega)} \quad (3.30)$$

where we used the fact that the set of balls $B_{\delta'}$ containing the chains $\mathcal{P}_h(T, S_h(T))$ have finite overlap, i.e, the number of balls an arbitrary point belongs to is bounded. ■

To see that the method is locking free provided the coupling induced by the stabilization s_h is sufficiently local we introduce the following partition of the elements,

$$\widetilde{\mathcal{T}}_h^L = \mathcal{T}_h^L \setminus S_h(\mathcal{T}_h^S) = \{T \in \mathcal{T}_h^L : \nexists T' \in \mathcal{T}_h^S \text{ such that } T = S_h(T')\} \quad (3.31)$$

and

$$\mathcal{T}_h^M = \{T^M : T^M = S_h^{-1}(T) \cup T \text{ for some } T \in \mathcal{T}_h^L \setminus \widetilde{\mathcal{T}}_h^L\} \quad (3.32)$$

Observe that $\mathcal{T}_h^M$ is the set of macro elements defined by the operator S_h. We further reduce this set by merging any two (or more) $T_1^M, T_2^M \in \mathcal{T}_h^M$ for which $T_1^M \cap T_2^M \neq \emptyset$ into $\widetilde{T}^M = T_1^M \cup T_2^M$. The resulting set of merged macro elements is denoted $\widetilde{\mathcal{T}}_h^M$. This means that we merge all macro elements that share a node into larger macroelements that are isolated from each other in the sense that there is a layer of elements that are not macro elements between them. It follows by the definition that the union of all elements in $\widetilde{\mathcal{T}}_h^L$ and $\widetilde{\mathcal{T}}_h^M$ is Ω_h.

To each merged macro element $\widetilde{T}^M \in \widetilde{\mathcal{T}}_h^M$ we associate a ball $B_{\delta(\widetilde{T}^M),\widetilde{T}^M}$ of diameter $\delta(\widetilde{T}^M)$, such that $\widetilde{T}^M \subset B_{\delta(\widetilde{T}^M),\widetilde{T}^M}$.

We next decompose the set of finite element basis functions into $\mathcal{B}_h^M$, the basis functions defining polynomials on the elements in the set $\widetilde{\mathcal{T}}_h^M$ and $\mathcal{B}_h^L = \mathcal{B}_h \setminus \mathcal{B}_h^M$, with associated index sets I_M and I_L. Observe that since the elements in $\widetilde{\mathcal{T}}_h^M$ have disjoint boundaries each basis function is either in $\mathcal{B}_h^L$ and then its support has zero intersection with the elements in $\widetilde{\mathcal{T}}_h^M$, or attributed to a single element $\widetilde{T}^M$ in $\mathcal{B}_h^M$.

We are now ready to define the modified interpolant

$$\pi_h^M v := \sum_{i\in I_L} \varphi_i^*(\pi_h v)\varphi_i + \sum_{i\in I_M} \varphi_i^*(P_{B_{\delta,\widetilde{T}^M(i)}} v^E)\varphi_i \quad (3.33)$$

where $B_{\delta,\widetilde{T}^M(i)}$ denotes a ball associated to the unique $\widetilde{T}^M \in \widetilde{\mathcal{T}}^M$ of the basis function φ_i such that $\widetilde{T}^M \subset B_{\delta,\widetilde{T}^M(i)}$. Observe that it follows by the definition that $\pi_h^M v|_{\widetilde{T}^M} = P_{B_{\delta(\widetilde{T}^M),\widetilde{T}^M}} v$ for all $\widetilde{T}^M \in \widetilde{\mathcal{T}}_h^M$. As a consequence $[\pi_h^M v]_{T,S_h(T)} = 0$ for all $T \in \mathcal{T}_h^S$. We then have the following result.

Lemma 3.4 (Strong consistency) *The interpolant defined by* (3.33) *satisfies*

$$\boxed{\|\pi_h^M v\|_{s_{h,m}} = 0} \quad (3.34)$$

If there is a constant such that $\delta(\widetilde{T}^M) \lesssim h$ *for all* $\widetilde{T}^M \in \widetilde{\mathcal{T}}_h^M$ *and the set of balls* $\{B_{\delta(\widetilde{T}^M),\widetilde{T}^M} : \widetilde{T}^M \in \widetilde{\mathcal{T}}_h^M\}$ *have uniformly finite overlap. Then there is a constant such that*

$$\boxed{\|v - \pi_h^M v\|_{H^m(\Omega_h)} \lesssim h^{k+1-m}\|v\|_{H^{k+1}(\Omega)}, \qquad m = 0, 1} \tag{3.35}$$

Proof The first claim is true by construction, since for all $\widetilde{T}^M \in \widetilde{\mathcal{T}}_h^M$ $\pi_h^M v|_{\widetilde{\mathcal{T}}_h^M} = P_{B_{\delta(\widetilde{T}^M),\widetilde{T}^M}} v$ and therefore $[\pi_H^M v]_{T,S_h(T)} = 0$. To prove the error estimate (3.35) we note that

$$\|v - \pi_h^M v\|_{H^m(\Omega_h)} \leq \|v - \pi_h v\|_{H^m(\Omega_h)} + \|\pi_h v - \pi_h^M v\|_{H^m(\Omega_h)} \tag{3.36}$$

The first term of the right hand side is bounded by the standard estimate (3.19). For the second part we observe that

$$\pi_h v - \pi_h^M v = \sum_{i \in I_M} \varphi_i^* (B_{\delta(\widetilde{T}^M),\widetilde{T}^M(i)} v - \pi_h v)\varphi_i. \tag{3.37}$$

It then follows using an inverse inequality that

$$\begin{aligned} &\|\pi_h v - \pi_h^M v\|_{H^m(\Omega_h)} \\ &\quad \lesssim (h^{-m} + 1)\|\pi_h v - \pi_h^M v\|_{\Omega_h} \lesssim (h^{-m} + 1)\|\pi_h v - \pi_h^M v\|_{\tilde{\Omega}_M} \end{aligned} \tag{3.38}$$

where $\tilde{\Omega}_M = \cup_{\widetilde{T}^M \in \widetilde{\mathcal{T}}_h^M} \widetilde{T}^M$. Applying once again the triangle inequality we see that

$$\|\pi_h v - \pi_h^M v\|_{\tilde{\Omega}_M} \leq \|\pi_h v - v\|_{\Omega_h} + \|v - \pi_h^M v\|_{\tilde{\Omega}_M}. \tag{3.39}$$

It only remains to bound the second term of the right hand side. It follows by the definition that

$$\|v - \pi_h^M v\|_{\tilde{\Omega}_M}^2 = \sum_{\tilde{T}_M \in \widetilde{\mathcal{T}}_M} \|v - \pi_h^M v\|_{\tilde{T}_M}^2 \leq \sum_{\tilde{T}_M \in \widetilde{\mathcal{T}}_M} \|v - \pi_h^M v\|_{B_{\delta(\widetilde{T}^M),\widetilde{T}^M}}^2 \tag{3.40}$$

$$\lesssim h^{2(k+1)} \sum_{\tilde{T}_M \in \widetilde{\mathcal{T}}_M} |v^E|_{H^{k+1}(B_{\delta(\widetilde{T}^M),\widetilde{T}^M})}^2 \lesssim h^{2(k+1)}|v|_{H^{k+1}(\Omega)}^2 \tag{3.41}$$

where we used the finite overlap of the $B_{\delta(\widetilde{T}^M),\widetilde{T}^M}$ and the stability of the extension in the last inequality. We conclude by collecting the above bounds. ■

We note that the assumptions on $\widetilde{\mathcal{T}}_h^M$ are quite strong. It is not obvious how to design an agglomeration map S_h that ensures the uniform bound $\operatorname{diam}(\tilde{T}^M) = O(h)$ for all $\tilde{T}^M \in \widetilde{\mathcal{T}}_h^M$ as well as finite overlap of the associated set $\{B_{\delta(\widetilde{T}^M),\widetilde{T}^M} : \widetilde{T}^M \in \widetilde{\mathcal{T}}_h^M\}$ of balls. We therefore now focus on what we call nodal-based stabilization, which does not need such strong assumptions to be locking free.

3.2 Properties of nodal stabilization

Here we specialize the analysis to the nodal stabilization defined by (2.10) and we show that the critical stability bounds (2.44) indeed hold and that we can construct an interpolation operator that satisfies strong consistency; more precisely, the interpolant of an L^2 function is in the kernel of the stabilization.

Starting with the stability estimate we note that applying Lemma 3.1 we get

$$\sum_{i\in I^S} h^{d-2m}(\varphi^*_{T_i}(v))^2 \lesssim h^{-2m}\|v\|^2_\Omega + \|v\|^2_{S_h} \tag{3.42}$$

Thus the simple general analysis does not work directly in the case $m = 1$, since we have the term $h^{-2m}\|v\|^2_\Omega$ instead of $\|\nabla v\|^2_\Omega$. To handle this case, we need the following natural yet technical assumption extending A2.

A3. For elements T_1, T_2 in $\mathcal{T}_h^S$ contained in $\mathrm{supp}(\varphi_i)$ there is a path $\mathcal{P}_h(S_h(T_1), S_h(T_2))$ of uniformly bounded length consisting only of elements in $\mathcal{T}_h^L$. For $T_1 \in \mathcal{T}_h^L$ and $T_2 \in \mathcal{T}_h^S$ contained in $\mathrm{supp}(\varphi_i)$ there is a path $\mathcal{P}_h(T_1, S_h(T_2))$ of uniformly bounded length consisting only of elements in $\mathcal{T}_h^L$.

Lemma 3.5 (Stability) *Let s_h be the nodal stabilization form defined by* (2.10). *Then there is a constant such that*

$$\boxed{\|\nabla^m v\|^2_{\Omega_h} \lesssim \|\nabla^m v\|^2_\Omega + \|v\|^2_{s_{h,m}}, \qquad m = 0, 1, \quad v \in V_h} \tag{3.43}$$

(2.44) *holds.*

Proof We first recall that we have the element wise equivalence

$$\|v\|^2_T \sim h^d \|\hat{v}\|^2_{\mathbb{R}^{N_T}} \tag{3.44}$$

see [15], where $\hat{v}_i = \varphi^*_{i,T}(v)$. We then have the following estimate

$$\begin{aligned}
\|\nabla^m v\|^2_{\Omega_h} &= \|\nabla^m v\|^2_{\mathcal{T}_h^L} + \|\nabla^m v\|^2_{\mathcal{T}_h^S} && (3.45)\\
&\lesssim \|\nabla^m v\|^2_{\mathcal{T}_h^L} + \sum_{T\in\mathcal{T}_h^S}\left(\|\nabla^m (v|_{S_h(T)})^e\|^2_T + \|\nabla^m [v]_{T,S_h(T)}\|^2_T\right) && (3.46)\\
&\lesssim \|\nabla^m v\|^2_{\mathcal{T}_h^L} + \underbrace{\sum_{T\in\mathcal{T}_h^S}\|\nabla^m v\|^2_{S_h(T)}}_{\le \|\nabla^m v\|^2_{\mathcal{T}_h^L}} + \sum_{T\in\mathcal{T}_h^S} h^{-2m}\|[v]_{T,S_h(T)}\|^2_T && (3.47)\\
&\lesssim \|\nabla^m v\|^2_{\mathcal{T}_h^L} + \underbrace{\sum_{T\in\mathcal{T}_h^S}\sum_{i\in I_T} h^{d-2m}|\varphi^*_{i,T}[v]_{T,S_h(T)}|^2}_{\bigstar} && (3.48)
\end{aligned}$$

$$\lesssim \|\nabla^m v\|^2_{\mathcal{T}_h^L} + \|v\|^2_{S_{h,m}} \tag{3.49}$$

where we for each element $T \in \mathcal{T}_h^S$ added and subtracted $(v|_{S_h(T)})^e$, used the stability $\|\nabla^m (v|_{S_h(T)})^e\|_T \lesssim \|\nabla^m v\|_{S_h(T)}$ of polynomial extension, the equivalence (3.44), and finally the estimate

$$\bigstar \lesssim \|\nabla^m v\|^2_{\mathcal{T}_h^L} + \|v\|^2_{S_{h,m}} \tag{3.50}$$

which we verify next. To that end we will consider the global degrees of freedom in I^S and I^L separately.

Case 1: $i \in I^S$. Consider a global degree of freedom $i \in I^S$ with $T_i \subset \text{supp}(\varphi_i)$ the element assigned in the stabilization. For each $T \in \mathcal{T}_h^S$ such that $T \subset \text{supp}(\varphi_i)$, we wish to express $\varphi^*_{i,T}[v]_{T,S_h(T)}$ in terms of the stabilized functional $\varphi^*_{i,T_i}[v]_{T_i,S_h(T)}$. Adding and subtracting $(v|_{S_h(T_i)})^e$ we have the identity

$$[v]_{T,S_h(T)} = [v]_{T,S_h(T_i)} + [v]_{S_h(T_i),S_h(T)} \tag{3.51}$$

Here we observe that

$$\varphi^*_{i,T}[v]_{T,S_h(T_i)} = \varphi^*_{i,T_i}[v]_{T_i,S_h(T_i)} \tag{3.52}$$

since the finite element space is conforming. Using these observations we have the estimate

$$h^{d-2m}|\varphi^*_{i,T}[v]_{T,S_h(T)}|^2 \lesssim h^{d-2m}|\varphi^*_{i,T}[v]_{T,S_h(T_i)}|^2 + h^{d-2m}|\varphi^*_{i,T}[v]_{S_h(T_i),S_h(T)}|^2 \tag{3.53}$$

$$\lesssim h^{d-2m}|\varphi^*_{i,T_i}[v]_{T_i,S_h(T_i)}|^2 + h^{-2m}\|[v]_{S_h(T_i),S_h(T)}\|^2_T \tag{3.54}$$

$$\lesssim h^{d-2m}|\varphi^*_{i,T_i}[v]_{T_i,S_h(T_i)}|^2 + h^{2-2m}\|\nabla v\|^2_{\mathcal{P}_h(S_h(T_i),S_h(T))} \tag{3.55}$$

$$\lesssim h^{d-2m}|\varphi^*_{i,T_i}[v]_{T_i,S_h(T_i)}|^2 + \|\nabla^m v\|^2_{\mathcal{P}_h(S_h(T_i),S_h(T))} \tag{3.56}$$

where we used identity (3.52) to replace T by T_i, the boundedness of $\varphi^*_{i,T}$, Lemma 3.2, and finally an inverse estimate in the case $m = 0$.

Case 2: $i \in I^L$. For $i \in I^L$ we instead let T_i be a large element in the support of φ_i, i.e. $T_i \in \mathcal{T}_h^L$ and $T \subset \text{supp}(\varphi_i)$, which gives, using the same arguments as in Case 1,

$$h^{d-2m}|\varphi^*_{i,T}[v]_{T,S_h(T)}|^2 \lesssim h^{d-2m}|\varphi^*_{i,T}[v]_{T,T_i}|^2 + h^{d-2m}|\varphi^*_{i,T}[v]_{T_i,S_h(T)}|^2 \tag{3.57}$$

$$\lesssim h^{-2m}\|[v]_{T_i,S_h(T)}\|^2_{T_i} \tag{3.58}$$

$$\lesssim h^{2-2m}\|\nabla v\|^2_{\mathcal{P}_h(T_i,S_h(T))} \tag{3.59}$$

$$\lesssim \|\nabla^m v\|^2_{\mathcal{P}_h(T_i,S_h(T))} \tag{3.60}$$

Conclusion. Using the estimates in Case 1 and 2 we obtain (3.50) as follows

$$\bigstar = \sum_{T\in\mathcal{T}_h^S}\sum_{i\in I_T} h^{d-2m}|\varphi_{i,T}^*[v]_{T,S_h(T)}|^2 \tag{3.61}$$

$$= \sum_{T\in\mathcal{T}_h^S}\sum_{i\in I_T\cap I^S} h^{d-2m}|\varphi_{i,T}^*[v]_{T,S_h(T)}|^2 + \sum_{T\in\mathcal{T}_h^S}\sum_{i\in I_T\cap I^L} h^{d-2m}|\varphi_{i,T}^*[v]_{T,S_h(T)}|^2 \tag{3.62}$$

$$\lesssim \sum_{T\in\mathcal{T}_h^S}\sum_{i\in I_T\cap I^S} h^{d-2m}|\varphi_{i,T_i}^*[v]_{T_i,S_h(T_i)}|^2 + \|\nabla^m v\|^2_{\mathcal{P}_h(S_h(T_i),S_h(T))} \tag{3.63}$$

$$+ \sum_{T\in\mathcal{T}_h^S}\sum_{i\in I_T\cap I^L} \|\nabla^m v\|^2_{\mathcal{P}_h(T_i,S_h(T))} \tag{3.64}$$

$$\lesssim \|v\|^2_{S_{h,m}} + \|\nabla^m v\|^2_{\mathcal{T}_h^L} \tag{3.65}$$

which together with estimate (3.45) conclude the proof. ■

Next we study the limit when the stabilization parameter becomes large we will see that for the the nodal stabilization (2.10) we retain optimal order approximation properties. More precisely, we show that there is an interpolation operator $\Pi_h : L^2(\Omega) \to V_h$ such that

$$\boxed{\|\Pi_h u\|_{s_h} = 0} \tag{3.66}$$

which we may view as a strong version of (3.21), and the optimal order interpolation estimate (3.69) holds.

To that end we define an interpolation operator $\widetilde{\Pi}_h : V_h \to V_h$, with the special property that only $v|_{\mathcal{T}_h^L}$ is used to determine the nodal values,

$$V_h \ni v \mapsto \widetilde{\Pi}_h v = \sum_{i\in I^S} \varphi_{i,T_i}^*((v|_{S_h(T_i)})^e)\varphi_i + \sum_{i\in I^L} \varphi_{i,T_i}^*(v)\varphi_i \in V_h \tag{3.67}$$

and then we define the interpolation operator

$$\boxed{\Pi_h : L^2(\Omega) \ni v \mapsto \widetilde{\Pi}_h \pi_h v \in V_h} \tag{3.68}$$

By construction the strong consistency (3.66) holds.

Lemma 3.6 (Approximation) *There is a constant such that*

$$\boxed{\|v - \Pi_h v\|_{H^m(T)} \lesssim h^{k+1-m}\|v^E\|_{H^{k+1}(B_{\delta,T})}} \tag{3.69}$$

where $B_{\delta,T}$ is a ball containing $\mathcal{T}_h(T) \cup \mathcal{T}_h(S_h(T))$ with radius $\delta \sim h$ and we recall that $\mathcal{T}_h(T')$ is the set of elements that share a node with T'.

Proof Let $I_T = I_T^S \cup I_T^L$ be the indices to the basis functions φ_i with $T \subset \text{supp}(\varphi_i)$. We first note that we have the L^2 stability

$$\|\Pi_h v\|_T \lesssim \|v^E\|_{B_{\delta,T}} \tag{3.70}$$

since

$$\begin{aligned}
\|\Pi_h v\|_T^2 &\lesssim \sum_{i\in I_T^S} h^d |\varphi_i^*(\pi_h v|_{S_h(T_i)})^e)|^2 + \sum_{i\in I_T^L} h^d |\varphi_i^*(\pi_h v)|^2 && (3.71)\\
&\lesssim \sum_{i\in I_T^S} \|(\pi_h v|_{S_h(T_i)})^e\|_{T_i}^2 + \sum_{i\in I_T^L} \|\pi_h v\|_{T_i}^2 && (3.72)\\
&\lesssim \sum_{i\in I_T^S} \|\pi_h v\|_{S_h(T_i)}^2 + \sum_{i\in I_T^L} \|\pi_h v\|_{T_i}^2 && (3.73)\\
&\lesssim \sum_{i\in I_T^S} \|v^E\|_{\mathcal{T}_h(S_h(T_i))}^2 + \sum_{i\in I_T^L} \|v^E\|_{\mathcal{T}_h(T_i)}^2 && (3.74)\\
&\lesssim \|v^E\|_{B_{\delta,T}}^2 && (3.75)
\end{aligned}$$

where T_i is a stabilized element for $i \in I_T^S$ and a large element for $i \in I_T^L$, and $B_{\delta,T}$ is a ball such that $\mathcal{T}_h(S_h(T_i)) \cup \mathcal{T}_h(T_i) \subset B_{\delta,T}$. We note that there is such a ball with radius $\delta \lesssim h$, due to shape regularity and assumption A2, for each T and that the set of all such balls has uniformly bounded intersection. Furthermore, we used the stability of polynomial extension and the stability of the element L^2 and the stability $\|v\|_T = \|\pi_{h,Cl} v^E\|_{T_i} \lesssim \|v\|_{\mathcal{T}_h(T_i)}$.

Next to prove (3.69) we first note that for $w \in \mathbb{P}_k(B_{\delta,T})$ we have

$$(\pi_h w)|_T = w|_T \tag{3.76}$$

which gives

$$\begin{aligned}
\|v - \pi_h v\|_{H^m(T)} &\le \|v - w\|_{H^m(T)} + \|w - \pi_h v\|_{H^m(T)} && (3.77)\\
&\le \|v - w\|_{H^m(T)} + h^{-m}\|\pi_h(w - v)\|_{H^m(T)} && (3.78)\\
&\lesssim \|v - w\|_{H^m(T)} + h^{-m}\|w - v\|_{B_{\delta,T}} && (3.79)
\end{aligned}$$

Finally, using the Bramble–Hilbert Lemma, see [5], we may choose $w \in \mathbb{P}_k(B_{\delta,T})$ such that

$$\|v - \pi_h v\|_{H^m(T)} \lesssim h^{k+1-m}\|v\|_{H^{k+1}(B_{\delta,T})} \tag{3.80}$$

as was to be shown. ■

3.3 Relation to discrete extension operators

We note that

$$\widetilde{\Pi}_h : V_h \to V_h^E \subset V_h \tag{3.81}$$

where $V_h^E = \mathrm{Im}(\widetilde{\Pi}_h) = \widetilde{\Pi}_h V$ is a proper subspace of V_h in fact

$$\ker(\widetilde{\Pi}_h) = \mathrm{span}\{\varphi_i \in \mathcal{B}_h : i \in I^S\} \tag{3.82}$$

with dimension $\dim(\ker(\widetilde{\Pi}_h)) = |I^S|$ and as a consequence $\dim(V_h^E) = |I^L|$. The operator $\widetilde{\Pi}_h$ restricted to V_h is indeed identical to extension operators developed in [4] and [11].

3.4 Remarks

Remark 3.2 In [2], a stabilizing form penalizing a suitable norm of the difference between the finite element solution and the discrete extension is added. This approach requires explicit access to the discrete extension operator. In contrast, our approach does not require implementing the discrete extension operator explicitly. Instead, our framework allows the construction of a weak stabilizing form whose kernel coincides with the discrete extended finite element space.

Remark 3.3 The localized macro-element stabilization introduced in [20] can be viewed as a weak implementation of an approach where, for discontinuous elements, the kernel of a patch-based stabilization form is used to define a stable approximation space on a patch or macro-element, as discussed in [17], Section 5.2. In the limit of a large stabilization parameter, the discrete solution becomes constrained to lie in this kernel. For polynomial approximations, the kernel consists of global polynomials defined over the entire patch.

3.5 Application to CutFEM

For completeness, we include an application of the stabilization forms to CutFEM. We present two error estimates, one based on the weak consistency and one on the strong consistency provided by the operator Π_h. To that end let us consider the elliptic model problem

$$-\Delta u = f \quad \text{in}\,\Omega, \qquad u = 0 \quad \text{on}\,\partial\Omega \tag{3.83}$$

and the cut finite element method: find $u_h \in V_h$ such that

$$A_h(u_h, v) = l_h(v) \tag{3.84}$$

where the forms are defined by

$$A_h(v, w) = a_h(v, w) + s_h(v, w) \tag{3.85}$$

$$a_h(v, w) = (\nabla v, \nabla w)_\Omega - (\nabla_n v, w)_{\partial\Omega} - (v, \nabla_n w)_{\partial\Omega} + \beta h^{-1}(v, w)_{\partial\Omega} \tag{3.86}$$

$$l_h(v) = (f, v)_\Omega \tag{3.87}$$

where $\beta \in [\beta_0, \infty)$ for some $\beta_0 > 0$. Using partial integration we note that the exact solution to (3.83) satisfies the unstabilized equation

$$a_h(u, v) = l_h(v) \qquad \forall v \in V_h \tag{3.88}$$

In the following we shall account for the effect of the stabilization parameter τ, which is hidden inside the stabilization form, and to make that dependence clear we from here on adopt the notation $s_{h,\tau}$. We also assume that there is a parameter $\tau_0 > 0$ such that $\tau \in [\tau_0, \infty)$. Define the Nitsche norms

$$|||v|||_h^2 = \|\nabla v\|_\Omega^2 + \beta^{-1} h \|\nabla_n v\|_{\partial\Omega}^2 + \beta h^{-1} \|v\|_{\partial\Omega}^2 \tag{3.89}$$

$$|||v|||_{h,\bigstar}^2 = |||v|||_h^2 + \|v\|_{s_{h,\tau}}^2 \tag{3.90}$$

Assuming that the stabilization form $s_{h,\tau}$ is defined in such a way that

$$h\|\nabla_n v\|_{\partial\Omega}^2 \lesssim \|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau_*}}^2 \tag{3.91}$$

for some fixed positive parameter τ_*. Then the form a_h is coercive on V_h for sufficiently large $\beta \in [\beta_0, \infty)$,

$$|||v|||_{h,\bigstar}^2 \lesssim A_h(v, v) \qquad v \in V_h \tag{3.92}$$

and $\tau \geq \tau_0 \gtrsim \beta^{-1}$. For the convenience of the reader we have included the proof of this result in Appendix B. We also have the continuity

$$A_h(v, w) \lesssim |||v|||_{h,\bigstar} |||w|||_{h,\bigstar}, \qquad v, w \in V_h + H^{3/2+\epsilon}(\Omega) \tag{3.93}$$

which follows directly from the Cauchy-Schwarz inequality.

Remark 3.4 The inverse inequality (3.91) holds for instance for the different stabilization forms in Examples 1–5 with $m = 1$.

Proposition 3.1 *If the stabilization form $s_{h,\tau}$ is such that the stability estimate (2.44) and the consistency estimate (3.21) hold with $m = 1$, and the stabilization form implies satisfaction of the inverse inequality (3.91). Then there is a constant such that for $\tau \geq \tau_0$,*

$$|||u - u_h|||_h \lesssim (1 + \tau^{1/2}) h^k \|u\|_{H^{k+1}(\Omega)} \tag{3.94}$$

If in addition there exists an interpolant π_h satisfying (3.34) *and* (3.35)*, then*

$$|||u - u_h|||_h \lesssim h^k \|u\|_{H^{k+1}(\Omega)} \tag{3.95}$$

Proof Splitting the error by adding and subtracting an interpolant we get

$$|||u - u_h|||_h \leq |||u - \pi_h u|||_h + |||\pi_h u - u_h|||_h \lesssim h^k \|u\|_{H^{k+1}(\Omega)} + |||\pi_h u - u_h|||_h \tag{3.96}$$

For the second term we use the coercivity as follows

$$|||\pi_h u - u_h|||_h \leq |||\pi_h u - u_h|||_{h,\bigstar} \lesssim \sup_{v \in V_h} \frac{A_h(\pi_h u - u_h, v)}{|||v|||_{h,\bigstar}} \tag{3.97}$$

where we have the identity

$$\begin{aligned} A_h(\pi_h u - u_h, v) &= A_h(\pi_h u, v) - l_h(v) && (3.98)\\ &= A_h(\pi_h u, v) - a_h(u, v) && (3.99)\\ &= a_h(\pi_h u - u, v) + s_{h,\tau}(\pi_h u, v) && (3.100) \end{aligned}$$

Estimating the right hand side gives

$$\begin{aligned} |A_h(\pi_h u - u_h, v)| &\lesssim |||\pi_h u - u|||_h |||v|||_h + \|\pi_h u\|_{s_{h,\tau}} \|v\|_{s_{h,\tau}} && (3.101)\\ &\lesssim (|||\pi_h u - u|||_h + \|\pi_h u\|_{s_{h,\tau}}) |||v|||_{h,\bigstar} && (3.102) \end{aligned}$$

for $v \in V_h$. Combining the bounds (3.96), (3.97), and (3.101), we obtain

$$\begin{aligned} |||u - u_h|||_h &\lesssim |||\pi_h u - u_h|||_h + \|\pi_h u\|_{s_{h,\tau}} && (3.103)\\ &\lesssim h^k \|u\|_{H^{k+1}(\Omega)} + \tau^{1/2} h^k \|u\|_{H^{k+1}(\Omega)} && (3.104) \end{aligned}$$

where we used the interpolation estimate (3.17) and the weak consistency estimate (3.21). The second claim (3.95) follows by noting that $\|\pi_h u\|_{s_{h,\tau}}$ in (3.103) is zero and hence $s_{h,\tau}$ does not contribute to the upper bound. ■

Remark 3.5 Note that the constants in the resulting estimate will depend on the choices of the stabilization parameter τ and the Nitsche penalty parameter β. It follows from the proof of the coercivity estimate (3.92) that we may indeed take $\tau \sim \beta^{-1}$. Thus τ is a parameter of moderate size or even quite small in practice. Nevertheless, formulations that are less sensitive to parameters are, in general preferable.

Remark 3.6 The interpolant Π_h associated to the stabilization form $s_{h,m}$ defined by (2.10) with $m = 1$ and $\alpha_1 = d - 2$ satisfies (3.34) and (3.35) as was shown in (3.66) and (3.69).

Proposition 3.2 *If the stabilization form $s_{h,\tau}$ is defined by (2.10) with $\alpha = d-2$. Then*

$$\lim_{\tau\to\infty} u_h = \widetilde{\Pi}_h u_h \tag{3.105}$$

Proof First we observe that it follows from the definition (3.67) of $\widetilde{\Pi}_h$ that $\varphi_i^*(\widetilde{\Pi}_h v) = \varphi_i^*(v)$ for all $i \in I^L$. Thus the nodal values corresponding to indices $i \in I^L$ do not change and it remains to study the nodal values corresponding to $i \in I^S$. Setting $v = u_h$ in (3.84) and using coercivity we get

$$l_h(u_h) = A_h(u_h, u_h) \gtrsim |||u_h|||^2_{h,\bigstar} = |||u_h|||^2_h + \|u_h\|^2_{s_{h,\tau}} \tag{3.106}$$

Furthermore, we have

$$|l_h(v)| \lesssim \|f\|_\Omega \|v\|_\Omega \lesssim \|f\|_\Omega |||v|||_h \tag{3.107}$$

where we used the Poincaré estimate $\|v\|_\Omega \lesssim |||v|||_h$ for $v \in V_h$. Combining the estimates we get

$$|||u_h|||^2_h + \|u_h\|^2_{s_{h,\tau-\tau_0}} \le C\|f\|_\Omega |||v|||_h \le \frac{1}{2}C^2\|f\|^2_\Omega + \frac{1}{2}|||v|||^2_h \tag{3.108}$$

which directly implies

$$\frac{1}{2}|||u_h|||^2 + \|u_h\|^2_{s_{h,\tau}} \le \frac{1}{2}C^2\|f\|^2_\Omega \tag{3.109}$$

Next using the special choice (2.10) of the stabilization form we have

$$\|v\|^2_{s_{h,\tau}} = \sum_{i\in I^S} \tau h^{d-2} |\varphi^*_{i,T_i}([v]_{T_i,S_h(T_i)})|^2 \tag{3.110}$$

for $v \in V_h$, and from definition (3.67) of $\widetilde{\Pi}_h$ we obtain the identity

$$\varphi^*_{i,T_i}([v]_{T_i,S_h(T_i)}) = \varphi^*_{i,T_i}(v - (v|_{S_h(T_i)})^e) = \varphi^*_{i,T_i}(v - \widetilde{\Pi}_h v) \tag{3.111}$$

which gives

$$\|v\|^2_{s_{h,\tau}} = \sum_{i\in I^S} \tau h^{d-2} |\varphi^*_{i,T_i}(v - \widetilde{\Pi}_h v))|^2 \tag{3.112}$$

We conclude that

$$\sum_{i\in I^S} h^{d-2} |\varphi^*_{i,T_i}(u_h - \widetilde{\Pi}_h u_h)|^2 \lesssim \frac{1}{\tau}\|f\|^2_\Omega \tag{3.113}$$

and the desired result follows. ■

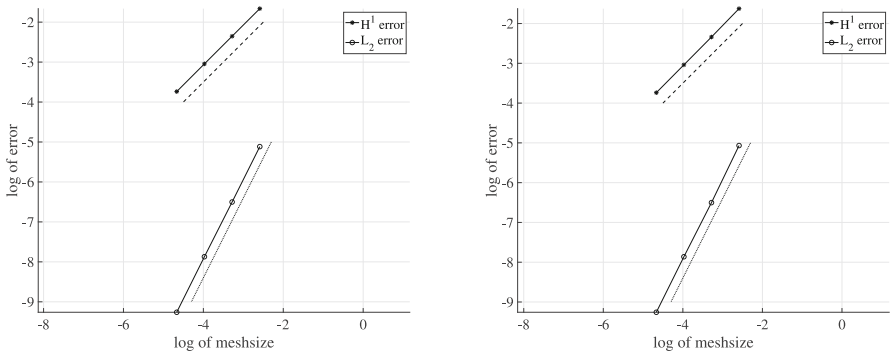


Fig. 2 Convergence for point penalty with $\tau = 10^{-1}$ (left) and $\tau = 10$ (right)

4 Numerical examples

In the numerical examples below, we compare the forms taken from Example 1, (face penalty, (2.18)), from Example 3 (L^2 penalty on the gradient of the difference of the solution and local extension from the interior, (2.30)) and from the nodal stabilization (2.10). We show that face penalty is most sensitive to locking for large τ and that the nodal stabilization is completely robust.

We consider the Poisson's equation on a circle of radius $1/2$ with center at the origin. On this circle we use the constructed solution

$$u = \cos(\pi r), \quad r = \sqrt{x^2 + y^2} \tag{4.1}$$

corresponding to the right–hand side

$$f = (\pi(\sin(\pi r) + \pi r \cos(\pi r)))/r \tag{4.2}$$

We set the penalty parameter in the different cases to $\tau \in \{10^{-1}, 10, 10^3\}$ and display convergence and condition numbers for the different approaches for increasing τ.

In Figs. 2 and 3 we give convergence plots for the point oriented method. We note that the convergence pattern is unaffected by the increased penalty (Figs. 4, 5); in Figs. 6, 7, 8 and 9 we give the corresponding results for the gradient penalty and in Figs. 10 and 11 for the face penalty. The dashed and dotted lines, indicating first and second-order convergence, respectively, are fixed in all diagrams (Fig. 12). We note the slight locking for the gradient penalty and severe locking for face penalty on coarse meshes. This is also visible in the elevation plots on coarse and fine meshes with $\tau = 10^3$, for point oriented in Fig. 5, for gradient penalty in Fig. 9, where small instabilities near the boundary are visible on the coarse mesh, and for the face penalty in Fig. 13, which exhibits severe locking on coarse meshes and has a visible effect also on the fine mesh.

Finally, in Figs. 4, 8, and 12, we give the plots of the condition number for the different methods, for $\tau = 10^{-1}$ and for $\tau = 10^3$. The rate given by the dashed line

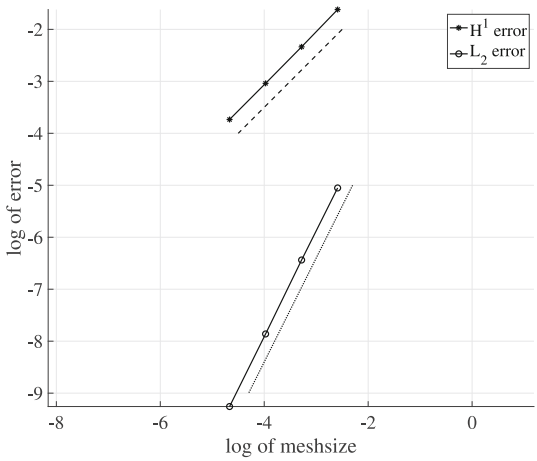


Fig. 3 Convergence for point penalty with $\tau = 10^3$

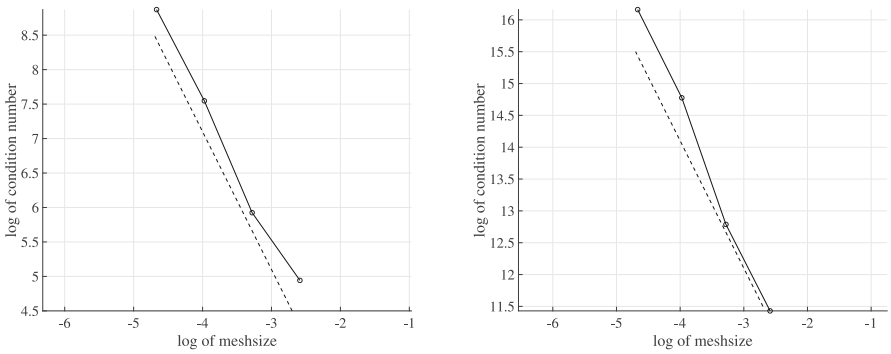


Fig. 4 Conditioning for point penalty with $\tau = 10^{-1}$ (left) and $\tau = 10^3$ (right)

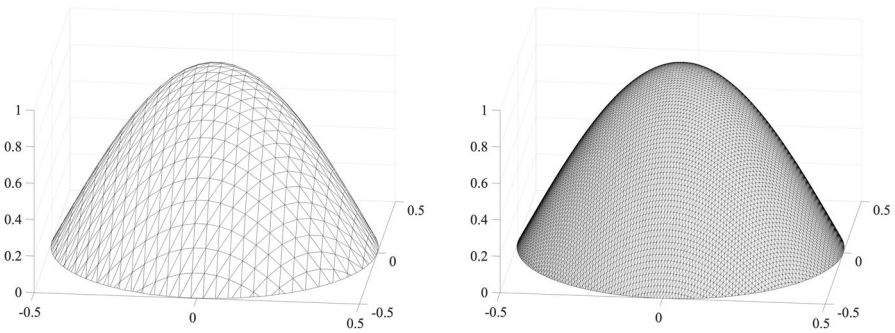


Fig. 5 Elevation of the computed solution on a coarse and on a fine mesh for point penalty with $\tau = 10^3$

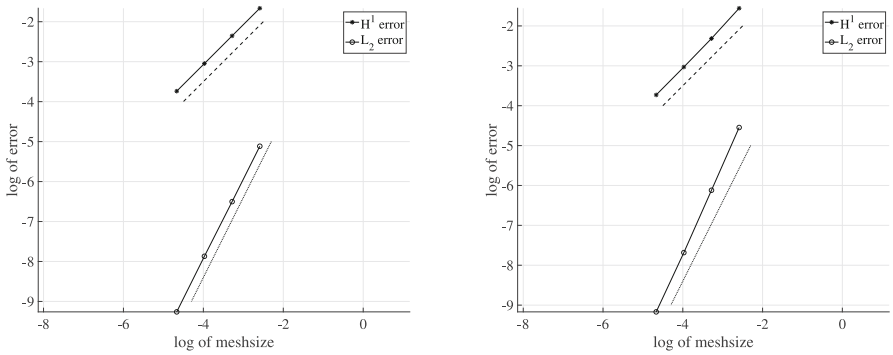
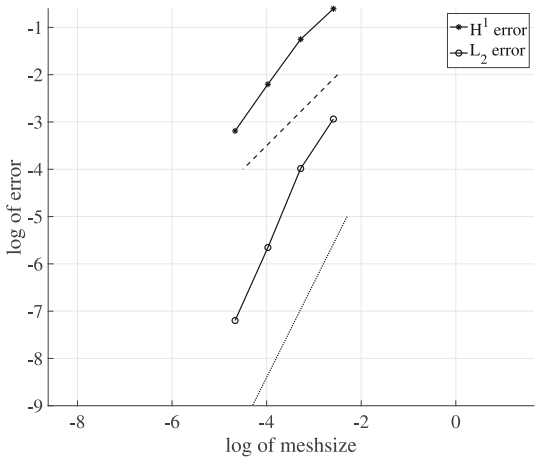


Fig. 6 Convergence for $L2$ gradient penalty with $\tau = 10^{-1}$ (left) and $\tau = 10$ (right)

Fig. 7 Convergence for L^2 gradient penalty with $\tau = 10^3$



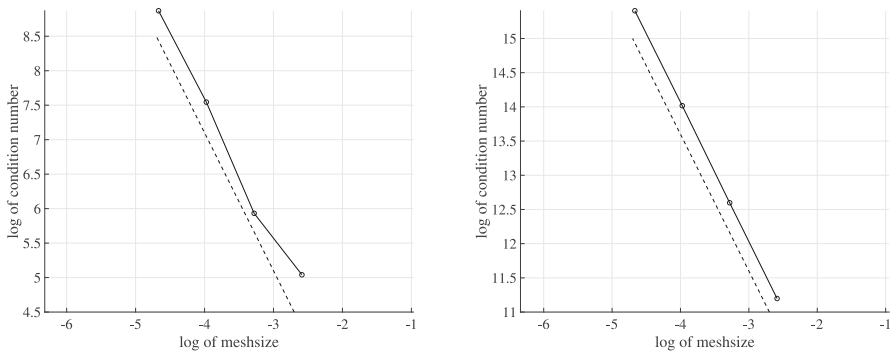


Fig. 8 Conditioning for L^2 gradient penalty with $\tau = 10^{-1}$ (left) and $\tau = 10^3$ (right)

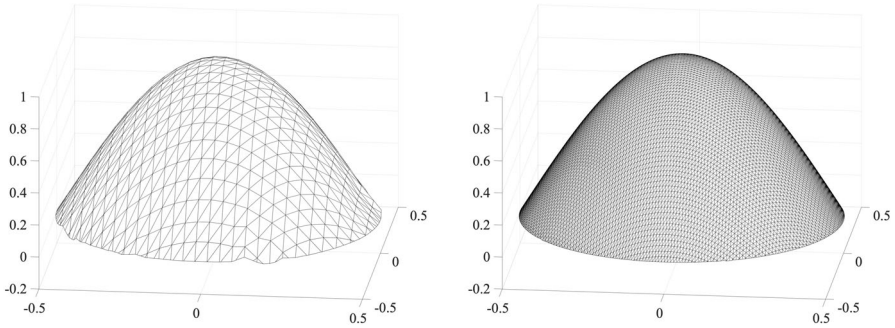


Fig. 9 Elevation of the computed solution on a coarse and on a fine mesh for L^2 gradient penalty with $\tau = 10^3$

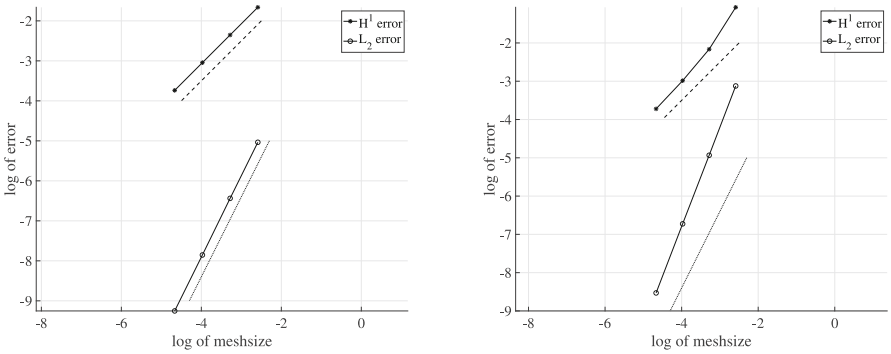
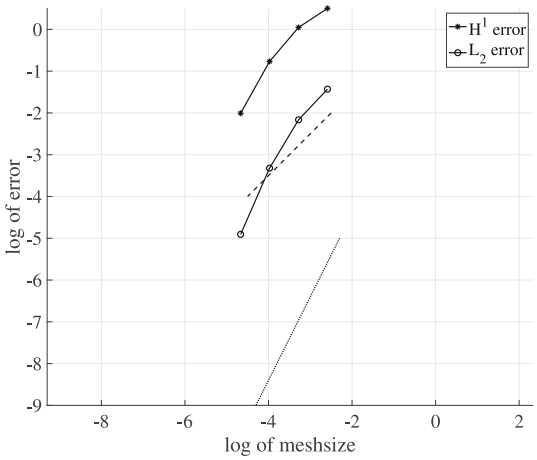


Fig. 10 Convergence for face penalty with $\tau = 10^{-1}$ (left) and $\tau = 10$ (right)

Fig. 11 Convergence for face penalty with $\tau = 10^3$



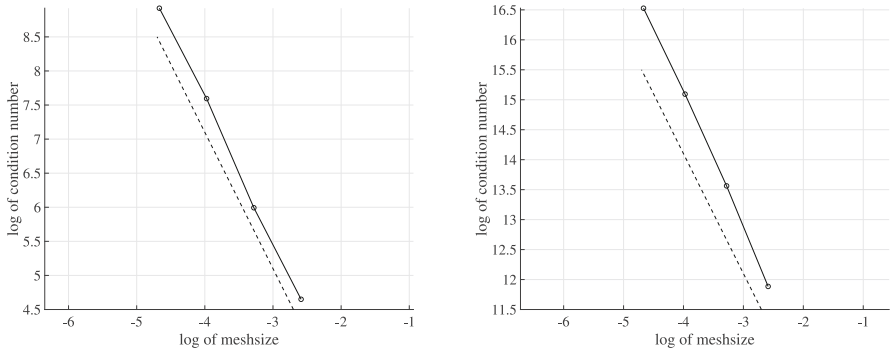


Fig. 12 Conditioning for face penalty with $\tau = 10^{-1}$ (left) and $\tau = 10^3$ (right)

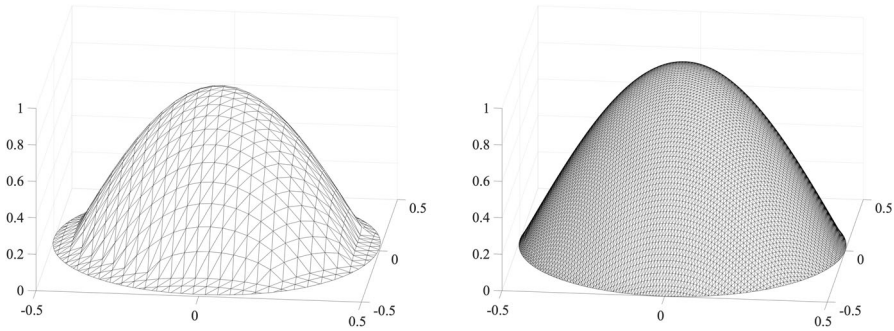


Fig. 13 Elevation of the computed solution on a coarse and on a fine mesh for face penalty with $\tau = 10^3$

is $O(h^{-2})$, and we note that all methods give approximately the same conditioning of the discrete system, with increasing τ adversely affecting the condition number. In all graphics, the natural logarithm of the plotted quantity is reported on the ordinate.

Appendix

A L^2 Stability for nodal stabilization

Here we include the simplified proof of the stability estimate (3.43) in the case $m = 0$,

$$\|v\|^2_{\Omega_h} \lesssim \|\nabla^m v\|^2_{\Omega} + \|v\|^2_{s_{h,0}}, \quad v \in V_h \tag{A.1}$$

for the nodal stabilization $s_{h,0}$ defined by (2.10) with $m = 0$.

Proof We have

$$\|v\|^2_{\Omega_h} = \|v\|^2_{\mathcal{T}_h^S} + \|v\|^2_{\mathcal{T}_h^L} \tag{A.2}$$

$$\lesssim h^d \sum_{T \in \mathcal{T}_h^S} \|\hat{v_T}\|^2_{\mathbb{R}^{N_T}} + \|v\|^2_{\mathcal{T}_h^L} \tag{A.3}$$

$$\lesssim h^d \|\hat{v^S}\|^2_{\mathbb{R}^{N_S}} + \|v\|^2_{\mathcal{T}_h^L} \tag{A.4}$$

Thus it follows that, if the stabilization form satisfies

$$h^d \|\hat{v^S}\|^2_{\mathbb{R}^{N_S}} \lesssim \|v\|^2_{\mathcal{T}_h^L} + \|v\|^2_{S_{h,0}} \tag{A.5}$$

the desired stability estimate (2.44) holds. To verify (A.5) we start from the definition (2.10) of the stabilization form. For each $i \in I_S$, we have the identity

$$\hat{v_i} = \varphi^*_{i,T_i}(v) = \varphi^*_{i,T_i}((v|_{S_h(T_i)})^e) + \varphi^*_{i,T_i}([v]_{T_i,S_h(T_i)}) \tag{A.6}$$

Using the triangle inequality we get

$$h^d|\hat{v_i}|^2 = h^d|\varphi^*_{i,T_i}(v)|^2 \tag{A.7}$$

$$\lesssim h^d|\varphi^*_{i,T_i}((v|_{S_h(T_i)})^e)|^2 + h^d|\varphi^*_{i,T_i}([v]_{T_i,S_h(T_i)})|^2 \tag{A.8}$$

$$\lesssim \|(v|_{S_h(T_i)})^e\|^2_T + h^d|\varphi^*_{i,T_i}([v]_{T_i,S_h(T_i)})|^2 \tag{A.9}$$

$$\lesssim \|v\|^2_{S_h(T_i)} + h^d|\varphi^*_{i,T_i}([v]_{T_i,S_h(T_i)})|^2 \tag{A.10}$$

where we used an inverse inequality and the stability of the canonical extension. Summing over all $i \in I^S$ we get

$$h^d\|\hat{v^S}\|^2_{\mathbb{R}^{N_S}} = \sum_{i \in I^S} h^d|\hat{v_i}|^2 \tag{A.11}$$

$$\lesssim \sum_{i \in I^S} \|v\|^2_{S_h(T_i)} + h^d|\varphi^*_{i,T_i}([v]_{T_i,S_h(T_i)})|^2 \tag{A.12}$$

$$\lesssim \|v\|^2_{\mathcal{T}_h^L} + \sum_{i \in I^S} h^d|\varphi^*_{i,T_i}([v]_{T_i,S_h(T_i)})|^2 \tag{A.13}$$

$$\lesssim \|v\|^2_{\Omega} + \|v\|^2_{S_{h,0}} \tag{A.14}$$

as was to be shown. ■

B Coercivity

Lemma 4.1 *The form A_h, defined in (3.85) is coercive on V_h for sufficiently large $\beta \in [\beta_0, \infty)$,*

$$|||v|||^2_{h,\bigstar} \lesssim A_h(v, v) \qquad v \in V_h \tag{B.1}$$

and $\tau \in [\tau_0, \infty)$ where $\tau_0 \gtrsim \beta^{-1}$.

Proof of (3.92) We have

$$A_h(v,v) = \|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau}}^2 - 2(\nabla_n v, v)_{\partial\Omega} + \beta h^{-1}\|v\|_{\partial\Omega}^2 \tag{B.2}$$

$$\geq \|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau}}^2 - 2|(\nabla_n v, v)_{\partial\Omega}| + \beta h^{-1}\|v\|_{\partial\Omega}^2 \tag{B.3}$$

Recalling (3.91), we have

$$h\|\nabla_n v\|_{\partial\Omega}^2 \lesssim \|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau_*}}^2 \tag{B.4}$$

for some fixed positive parameter τ_*, and we may estimate the negative term as follows

$$2|(\nabla_n v, v)_{\partial\Omega}| \leq 2\beta^{-1}h\|\nabla_n v\|_{\partial\Omega}^2 + 2^{-1}\beta h^{-1}\|v\|_{\partial\Omega}^2 \tag{B.5}$$

$$\leq 2\beta^{-1}C(\|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau_*}}^2) + 2^{-1}\beta h^{-1}\|v\|_{\partial\Omega}^2 \tag{B.6}$$

where C is the hidden constant in (3.91). Combining the estimates we get

$$A_h(v,v) = 2^{-1}(\|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau}}^2) + (2^{-1} - 2\beta^{-1}C)\|\nabla v\|_\Omega^2 \tag{B.7}$$

$$+ (2^{-1}\|v\|_{s_{h,\tau_0}}^2 - 2\beta^{-1}C\|v\|_{s_{h,\tau_*}}^2) + 2^{-1}\beta h^{-1}\|v\|_{\partial\Omega}^2 \tag{B.8}$$

$$\geq 2^{-1}(\|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau}}^2 + \beta h^{-1}\|v\|_{\partial\Omega}^2) \tag{B.9}$$

if β is chosen such that $2^{-1} - 2\beta^{-1}C \geq 0$ and τ_0 satisfies $\tau_0 \geq 4\beta^{-1}C$. Here we used the simple identity $\alpha_1\|v\|_{s_{h,\tau_1}}^2 + \alpha_2\|v\|_{s_{h,\tau_2}}^2 = \|v\|_{s_{h,\alpha_1\tau_1+\alpha_2\tau_2}}^2$ to conclude that

$$2^{-1}\|v\|_{s_{h,\tau_0}}^2 - 2\beta^{-1}C\|v\|_{s_{h,\tau_*}}^2 = \|v\|_{s_{h,2^{-1}\tau_0-2\beta^{-1}\tau_*}}^2 \tag{B.10}$$

Finally, we use the fact that $\tau \geq \tau_0 \geq c\beta^{-1}\tau_*$ to conclude that

$$\|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau}}^2 \geq \|\nabla v\|_\Omega^2 + \|v\|_{s_{h,c\beta^{-1}\tau_*}}^2 \geq \|\nabla v\|_\Omega^2 + c\beta^{-1}\|v\|_{s_{h,\tau_*}}^2 \tag{B.11}$$

$$\geq \min(1, c\beta^{-1})(\|\nabla v\|_\Omega^2 + \|v\|_{s_{h,\tau_*}}^2) \gtrsim \beta^{-1}h\|\nabla_n v\|_{\partial\Omega}^2 \tag{B.12}$$

where we used (3.91) and the estimate $\min(1, c\beta^{-1}) \gtrsim \beta^{-1}$ since $\beta \geq \beta_0 > 0$. Together the estimates (B.7) and (B.11) prove the desired result. ■

Acknowledgements This research was supported in part by the Swedish Research Council Grants Nos. 2021-04925, 2022-03908, and the Swedish Research Programme Essence. EB acknowledges funding from EP/T033126/1 and EP/V050400/1.

Funding Open access funding provided by Umea University.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If

material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Badia, S., Martin, A.F., Verdugo, F.: Mixed aggregated finite element methods for the unfitted discretization of the Stokes problem. SIAM J. Sci. Comput. **40**(6), B1541–B1576 (2018)
2. Badia, S., Neiva, E., Verdugo, F.: Linking ghost penalty and aggregated unfitted methods. Comput. Methods Appl. Mech. Eng. **388**, 114232 (2022)
3. Badia, S., Neiva, E., Verdugo, F.: Robust high-order unfitted finite elements by interpolation-based discrete extension. Comput. Math. Appl. **127**, 105–126 (2022)
4. Badia, S., Verdugo, F., Martín, A.F.: The aggregated unfitted finite element method for elliptic problems. Comput. Methods Appl. Mech. Eng. **336**, 533–553 (2018)
5. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, volume 15 of Texts in Applied Mathematics, 2nd edn. Springer, New York (2002)
6. Buffa, A., Puppi, R., Vázquez, R.: A minimal stabilization procedure for isogeometric methods on trimmed geometries. SIAM J. Numer. Anal. **58**(5), 2711–2735 (2020)
7. Burman, E.: Ghost penalty. C. R. Math. Acad. Sci. Paris **348**(21–22), 1217–1220 (2010)
8. Burman, E., Ern, A.: An unfitted hybrid high-order method for elliptic interface problems. SIAM J. Numer. Anal. **56**(3), 1525–1546 (2018)
9. Burman, E., Hansbo, P.: Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method. Appl. Numer. Math. **62**(4), 328–341 (2012)
10. Burman, E., Hansbo, P., Larson, M.G.: A stabilized cut finite element method for partial differential equations on surfaces: the Laplace-Beltrami operator. Comput. Methods Appl. Mech. Eng. **285**, 188–207 (2015)
11. Burman, E., Hansbo, P., Larson, M.G.: Cutfem based on extended finite element spaces. Numer. Math. **152**(2), 331–369 (2022)
12. Burman, E., Hansbo, P., Larson, M.G., Larsson, K.: Extension operators for trimmed spline spaces. Comput. Methods Appl. Mech. Eng. **403**, 115707 (2023)
13. de Prenter, F., Verhoosel, C.V., van Brummelen, E.H., Larson, M.G., Badia, S.: Stability and conditioning of immersed finite element methods: analysis and remedies. Arch. Comput. Methods Eng. **30**(6), 3617–3656 (2023)
14. Engwer, C., Heimann, F.: Dune-udg: a cut-cell framework for unfitted discontinuous Galerkin methods. In: Dedner, A., Flemisch, B., Klöfkorn, R. (Eds.) Advances in DUNE, pp. 89–100. Springer, Berlin (2012)
15. Ern, A., Guermond, J.-L.: Evaluation of the condition number in linear systems arising in finite element approximations. M2AN Math. Model. Numer. Anal. **40**(1), 29–48 (2006)
16. Haslinger, J., Renard, Y.: A new fictitious domain approach inspired by the extended finite element method. SIAM J. Numer. Anal. **47**(2), 1474–1499 (2009)
17. Heimann, F., Lehrenfeld, C., Stocker, P., von Wahl, H.: Unfitted Trefftz discontinuous Galerkin methods for elliptic boundary value problems. ESAIM Math. Model. Numer. Anal. **57**(5), 2803–2833 (2023)
18. Huang, P., Wu, H., Xiao, Y.: An unfitted interface penalty finite element method for elliptic interface problems. Comput. Methods Appl. Mech. Eng. **323**, 439–460 (2017)
19. Johansson, A., Larson, M.G.: A high order discontinuous Galerkin Nitsche method for elliptic problems with fictitious boundary. Numer. Math. **123**(4), 607–628 (2013)
20. Larson, M.G., Zahedi, S.: Conservative cut finite element methods using macroelements. Comput. Methods Appl. Mech. Eng. **414**, 116141 (2023)
21. Lehrenfeld, C., Olshanskii, M.: An Eulerian finite element method for pdes in time-dependent domains. ESAIM Math. Model. Numer. Anal. **53**(2), 585–614 (2019)
22. Massing, A., Larson, M.G., Logg, A., Rognes, M.E.: A stabilized Nitsche fictitious domain method for the Stokes problem. J. Sci. Comput. **61**(3), 604–628 (2014)
23. Massing, A., Larson, M.G., Logg, A., Rognes, M.E.: A stabilized Nitsche overlapping mesh method for the Stokes problem. Numer. Math. **128**(1), 73–101 (2014)

24. Myrbäck, S., Zahedi, S.: A high-order conservative cut finite element method for problems in time-dependent domains. Comput. Methods Appl. Mech. Eng. **431**, 117245 (2024)
25. Neiva, E., Badia, S.: Robust and scalable h-adaptive aggregated unfitted finite elements for interface elliptic problems. Comput. Methods Appl. Mech. Eng. **380**, 113769 (2021)
26. Preuss, J.: Higher order unfitted isoparametric space-time fem on moving domains. Master's thesis, University of Göttingen (2018)
27. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series, vol. 30. Princeton University Press, Princeton (1970)
28. Toprak, T., Rieckmann, M., Kummer, F.: Cell agglomeration strategy for cut cells in extended discontinuous Galerkin methods. Int. J. Numer. Methods Eng. **126**(9), e70013 (2025)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.