Int. J. Fuzzy Syst.
https://doi.org/10.1007/s40815-026-02235-w

Modeling Investor Risk Perception in Qualitative Financial Portfolio Optimization Processes: Preference Driven and Interaction-Based Approaches

Ezgi Türkarslan[1,2] · **Christian Oliver Ewald**[1,3,4]

Received: 23 June 2025 / Revised: 16 September 2025 / Accepted: 17 January 2026
© The Author(s) 2026

Abstract Quantitative portfolio optimization methods are mainly based on precise numerical inputs and complete historical data. However, accessing complete historical data in real-world investment environments is not easy; often, the data are ambiguous and/or have qualitative characteristics. At this point, the use of qualitative portfolio optimization methods gains importance. This study introduces novel qualitative financial portfolio optimization approaches that integrate non-additive measures and the Shapley value to model uncertainty and interaction effects among assets for investment decision-making. By incorporating hesitant fuzzy sets, the proposed approach systematically captures subjective investor judgments, linguistic evaluations, and ambiguous data, thus allowing for personalized risk modeling. The framework distinguishes two levels of interaction: (i) Criterion-level interdependencies, modeled via non-additive measures and Choquet integrals, which enhance the structural consistency of the evaluation process by accounting for synergies and redundancies among decision criteria, and (ii) Asset-level interactions, where Shapley value-derived importance indices identify meaningful combinations of assets. In addition, the model accommodates varying investor profiles by proposing flexible preference functions that reflect different degrees of risk aversion and sensitivity to interactions. In this context, the qualitative portfolio optimization problem is solved according to various investor profiles, and the results are compared with those of established methods. Through comparative analysis, our results demonstrate that explicitly modeling both ambiguity and non-linear interdependencies leads to improved portfolio robustness and better alignment with investor preferences.

Keywords Entropy measure · Choquet integral · Fuzzy preference modeling · Shapley value-based investment allocation · Financial portfolio optimization

1 Introduction

The portfolio optimization process aims to ensure that an investor obtains maximum income by investing in multiple assets with similar or different characteristics, and at the same time, balancing the risk involved with the investment. To manage this process, some inferences and predictions must be made about the current and future status of the assets to be invested in. Markowitz developed the so-called modern portfolio theory [1] in 1952. According to this theory, which examines the effects of concepts such as diversification, risk, and correlation on portfolio returns, the risk of a portfolio is described in form of the variance of the portfolio as follows:

✉ Ezgi Türkarslan
ezgi.turkarslan@umu.se; ezgi.turkarslan@tedu.edu.tr

✉ Christian Oliver Ewald
christian.ewald@umu.se

[1] Department of Mathematics and Mathematical Statistics, Umeå University, Umeå, Sweden

[2] Department of Mathematics, TED University, Ankara, Turkey

[3] Adam Smith Business School, University of Glasgow, Glasgow, UK

[4] Inland University of Applied Sciences, Street, Lillehammer, Norway

Published online: 01 April 2026

$$V = \sum_{i,j=1}^{n} \sigma_{ij} X_i X_j \tag{1}$$

while the return of a portfolio is given by

$$E = \sum_{i=1}^{n} \mu_i X_i, \tag{2}$$

where n is the number of assets, X_i is the investment ratio of the asset i, μ_i is the expected return value of asset i, and σ_{ij} is the covariance between the returns of asset i and j with the constraints of $\sum_{i=1}^{n} X_i = 1$ and $X_i \geq 0$. Many studies have been conducted on the portfolio optimization problem and several theories have been developed based on Markowitz's modern portfolio theory, based entirely on quantitative data (see, e.g., [2–4]). After Zadeh introduced the concept of fuzzy logic [5] in 1965, integrations of fuzzy-related concepts consider the importance of modeling qualitative and quantitative uncertainties in financial data, such as fuzzy returns, fuzzy risks, and fuzzy constraints in portfolio optimization problems (see, e.g., [6,7]). However, in practice, there are investment strategies that have to rely entirely on qualitative data due to the lack of historical quantitative financial information, such as Pre-Initial Public Offerings (Pre-IPOs) investments. For this case, no market data exist prior to listing. The situation is not much different for listed Initial Public Offerings (IPOs) investments, where the limited trading history is insufficient for robust quantitative portfolio optimization. In such a case, qualitative fuzzy models are very useful to model uncertain, linguistic, and subjective factors such as investor preferences, risk perception, and market conditions, which are generally difficult to quantify explicitly in financial models. Several fuzzy investment methods have been presented in the literature (see, e.g., [8,9]). In our study, we focus on qualitative financial optimization techniques based on fuzzy set theory.

Fuzzy set theory is one of the most useful qualitative mathematical frameworks, especially when quantitative/historical financial data are limited, and qualitative data provided by experts can support the investors' perception and subjective evaluations in the decision-making process. It can mathematically model subjectivity and uncertainty by allowing for partial belonging of an element to a set using the concept of a membership function. This makes it suitable for successfully analyzing qualitative data in financial decision-making. Wu et al. [10] proposed a group consensus model on the investment of newly listed IPOs using fuzzy sets type 2 intervals (IT2FS). In their study, three financial experts evaluated newly listed stocks in China using IT2FSs to recommend the optimum investment ratio of stocks for an investor. In [11], qualitative financial portfolio optimization is developed based on Hesitant Fuzzy Linguistic Term Sets (HFLTSs), and a decision matrix of the four newly listed stocks in the Growth Enterprises Market board is constructed by two financial analysts in the form of HFLTSs. In our study, we use the concept of hesitant fuzzy set (HFS) [12,13], which defines the elements belonging to the set with sets whose elements come from the closed interval [0, 1]. This fuzzy set aims to eliminate the difficulty that decision makers experience due to hesitation when faced with difficult and complex information. It is also preferred by researchers in qualitative financial portfolio optimization problems (see, e.g., [14–16]). The use of HFSs in qualitative portfolio optimization offers several advantages. First, instead of assigning a single crisp value, HFS allows the decision-maker to express hesitation through a set of possible values, thus capturing uncertainty more realistically. This feature reduces overconfidence and anchoring bias by avoiding forced precision. Moreover, HFS enables richer representation of expert opinions and investor perceptions, as multiple qualitative assessments can be incorporated simultaneously. So, this representation reduces information loss and provides more robust portfolio allocations, reflecting the complexity of investor behavior and perception under uncertainty. The common feature of the above studies [10,11,14–16] is that they all developed qualitative financial optimization methods based on the maximum-return and minimum-risk principles within the framework of fuzzy set theory, where return is evaluated through score functions of the corresponding fuzzy elements, and risk is measured by deviation-related functions or by the entropy of the corresponding fuzzy sets. Moreover, all of them use weighted mean as investment ratios and also take into consideration the investor's risk appetites. At this point, there are three important problems that occur in their approaches:

Firstly, the role of criterion weights is either completely ignored or assigned arbitrary values in the interval [0, 1] without a systematic justification. Ignoring criterion weights prevents the model from capturing investor preferences and their risk appetite, and may misrepresent the decision environment. Conversely, assigning arbitrary values makes the model prone to bias and undermines its reliability. Moreover, there are interactions among criteria. In this case, a new research question arises as follows:

RQ1. How can hesitant fuzzy aggregation operators be employed to derive optimal criterion weights and investment ratios by taking into consideration interactions in portfolio optimization, while simultaneously accounting for investor preferences and acceptable risk tolerance?

Contribution 1: Similar to the five studies above, we also adopt the maximum-return and minimum-risk principles and propose two qualitative financial optimization methods. For the return-related evaluations, a hesitant fuzzy aggregation operator based on a fuzzy measure [17] is employed to incorporate criterion weights. The resulting hesitant fuzzy

elements are converted into crisp numbers using a hesitant fuzzy score function [13]. A key aspect of our approach is that the criterion weights are not predetermined: they are automatically derived through non-linear optimization programming, which aggregates the hesitant fuzzy inputs provided either by investors or experts. In this way, the system simultaneously optimizes both the criterion weights and the investment ratios, while respecting the investor's minimum acceptable return and risk tolerance, and explicitly capturing the interaction between criteria.

At this point, there is a disadvantage to using the hesitant fuzzy score functions due to their tendency to produce ties. However, Alcantud et al. [18] have demonstrated that the occurrence of ties is mathematically unavoidable for score functions defined over typical finite hesitant fuzzy sets. This is a consequence of foundational results such as Cantor's theorem, which states that there is no injective mapping from the power set of a nonempty set X to X itself, and they show that tie-free scoring is theoretically achievable within a restricted class of closed non-degenerate intervals. These findings reinforce the understanding that ties are not flaws inherent to particular scoring methods but rather structural features of total evaluations over HFSs. Therefore, the existence of ties does not inherently undermine the validity or reliability of the results, especially when the decision-making process is guided by a structurally separate aggregation logic, as in our case. However, our proposed methods do not rely on score-based rankings or algebraic operators over HFSs as the core decision-making tool; instead, our decision logic is thoroughly grounded in fuzzy measure-driven interaction modeling.

For the risk-related evaluations, we introduce a novel parametric hesitant fuzzy entropy measure. Because qualitative information includes uncertainties in the decision environment, it contributes to the overall risk. In fuzzy decision environments, entropy is considered the amount of unclear/inconsistent information. Hence, the aim is to minimize entropy to reduce epistemic uncertainty and enhance decisional clarity. The significance of this measure lies in its ability to calculate the entropy of each hesitant fuzzy element in the decision matrix separately and then aggregate them using the Choquet integral. In this framework, fuzzy measures are determined automatically and dynamically through non-linear programming. The entropy parameter generates a new decision matrix and defines an updated acceptable risk tolerance range for the investor. Section 3 provides an illustrative example demonstrating that as the entropy parameter increases, the entropy of a hesitant fuzzy set decreases. Furthermore, Sect. 7 examines its relationship with the minimum-return and maximum-risk tolerance ranges.

One of the contributions of this study is to demonstrate the use of fuzzy measures in qualitative portfolio optimization, as their application in financial portfolio models, particularly in qualitative settings, remains relatively rare (for quantitative ones see, e.g., [19–22]). Nevertheless, the concept of fuzzy measures has been integrated into portfolio selection models, particularly within the domain of portfolio selection for investment projects [23,24]. One reason for this discrepancy lies in the fundamental differences between the two decision contexts: project portfolio problems involve discrete and interdependent alternatives, where interactions such as resource conflicts or strategic complementarities are explicit and central to the decision process. In contrast, financial portfolio models have traditionally assumed independence among evaluation criteria (e.g., risk and return), and have been dominated by additive and probabilistic structures rooted in classical mean variance theory. This study contributes to this emerging line of research by employing a fuzzy measure-based approach to explicitly model the interactions among evaluation criteria and asset dependencies within a financial portfolio selection framework. Our approach fundamentally differs from project portfolio selection models in terms of objectives, data structure, and application domain.

Secondly, in the above existing studies, investment ratios are typically obtained by optimizing a weighted mean of asset scores under certain constraints. However, this approach is essentially linear and ignores the interaction structure among assets, which may lead to biased allocations. Moreover, the use of weighted means cannot guarantee 'fair' capital allocation, as well as ignoring all possible contributions of financial instruments to the portfolio. In conclusion, the following research question appears:

RQ2. How can investment ratios be determined in a fair and interaction-aware manner beyond the limitations of weighted mean aggregation?

Contribution 2: Contrary to the above-mentioned studies, we propose to determine the investment ratios using the Shapley value [25]. The concept of Shapley value is a game-theoretical concept used in cooperative game theory. It reflects the average contribution of a single element in all coalitions. The Shapley value provides a 'fair' allocation mechanism by quantifying each element's marginal contribution to the overall value across all possible coalitions. The non-linear nature of Shapley value allocation provides three key advantages: (i) it captures interaction effects among criteria that are ignored by linear weighted sums, (ii) it ensures fair allocation by distributing capital according to marginal contributions, and (iii) it increases robustness by delivering more stable allocations under varying investor preferences. This non-linearity allows the proposed model to represent investor behavior in a more realistic and explainable way.

In our approach, each asset is treated as a 'player' in a cooperative game, and its investment share is calculated as its

marginal contribution averaged over all possible coalitions. This guarantees a fair and interaction-aware allocation of the total investment, without relying on weighted mean aggregation. Moreover, the results obtained show that the proposed two methods satisfy the diversity of the portfolio when investment rates are calculated by the Shapley value instead of the weighted mean (see, e.g., Tables 7, 9 in Sect. 6).

Thirdly, the above studies (except for [11]) adopt score-trisection and deviation–trisection methods to categorize investors' risk appetites into three discrete groups (risk averse, risk neutral, risk seeking). Such a classification is inherently discrete; infinite intermediate attitudes disappear, and, for instance, a 'moderately risk-averse' investor cannot be represented. Not every investor must fit into three categories. Therefore, the following research question arises:

RQ3. How can investor risk preferences, including intermediate attitudes such as moderate risk aversion or partial risk seeking, be more accurately reflected in the portfolio optimization process?

Contribution 3: Unlike existing studies that discretize investor risk appetite into only three categories, this study introduces a parametric representation of the [min, max] score and risk range considering criteria weights with interactions and asset interdependencies. By introducing a parametric formulation of risk tolerance, the model is able to represent the entire spectrum of investor preferences from risk seeking to risk averse without collapsing them into crisp categories. This continuous representation not only captures intermediate attitudes such as moderate risk aversion but also enhances the flexibility, realism, and robustness of the portfolio optimization process.

Consequently, the novelty of this study lies in providing a unified qualitative portfolio optimization framework that integrates three methodological innovations. First, instead of relying on predetermined criterion weights, the model automatically derives them through non-linear optimization that aggregates hesitant fuzzy inputs, thereby jointly optimizing both criterion weights and investment ratios while explicitly capturing criterion and asset interactions. Second, a novel parametric hesitant fuzzy entropy measure is introduced to represent risk, which quantifies and minimizes epistemic uncertainty by calculating the entropy at the element level and aggregating via the Choquet integral. Third, investment ratios are allocated using Shapley values, ensuring a fair and interaction-aware distribution that cannot be achieved by linear weighted mean aggregation. Investor heterogeneity and intermediate risk attitudes are represented realistically through a parametric formulation. Taken together, these contributions enable a more realistic, explainable, and robust modeling of investor behavior in qualitative and uncertain decision environments.

Our work builds upon the approaches of [15,16] by extending their hesitant fuzzy set-based methods with automatic weight optimization, a novel parametric entropy measure, and Shapley value-based allocation.

Our paper is organized as follows. In Sect. 2, some concepts related to hesitant fuzzy theory, fuzzy measures, and integral theories are reviewed, and some references to existing portfolio optimization models within the hesitant fuzzy environment are provided. In Sect. 3, a new parametric entropy measure based on the Choquet integral is proposed to measure a portfolio's risk, and the parameter's effect on the proposed entropy measure is investigated. Section 4 develops the new fuzzy methods for risk-neutral and risk-sensitive investors for portfolio optimization processes under the hesitant fuzzy set setting. In Sect. 5, a portfolio optimization problem is illustrated to demonstrate the effectiveness of the proposed fuzzy models. A comprehensive comparison and sensitivity analysis are presented in Sects. 6 and 7, respectively. In Sect. 8, we summarize our conclusion.

2 Preliminaries

In this section, we first recall some of the foundations of hesitant fuzzy set theory. Then, we recollect some concepts of fuzzy measures and integral theories, and finally, we review the portfolio optimization models of Zhou and Xu [15] and Deng et al. [16], which provide the basis of our extensions.

2.1 Hesitant Fuzzy Theory in Qualitative Financial Portfolio Optimization Problem

Considering that the difficulty in determining the belonging of an element to a set arises from the fact that this belonging has many possible values, the concept of a hesitant fuzzy set (HFS) is defined to eliminate this complexity. This approach aims to model the uncertainty resulting from hesitancy, which can arise when assigning imprecise information. Defining uncertainty as a set of possible values rather than an exact value accounts for the possibility that this set considers subjectivity. For this reason, scientists have frequently used HFSs to mathematically model the subjectivity of opinions of investors, experts, or decision makers and qualitative data in the portfolio optimization problem.

Definition 1 ([12,13]) Let X be a fixed set, a HFS A on X is given as

$$A = \{\langle x, h_A(x)\rangle : x \in X\} \tag{3}$$

where $h_A(x)$ is a subset of $[0, 1]$ indicating the possible membership degrees of the element x to the set A. For convenience, $h = h_A(x)$ is called a hesitant fuzzy element (HFE), and H is the set of all HFEs.

HFS theory is particularly useful in qualitative evaluation when quantitative or historical data are scarce or unavailable. In such cases, qualitative information (e.g., expert judgment, investor perceptions) provides the only feasible way to capture decision-relevant knowledge. By representing hesitation and subjective evaluations explicitly, HFS makes it possible to incorporate these qualitative aspects into a structured computation system, thereby facilitating a user-adaptive and perception-driven decision process. In this respect, it is a very useful tool for pre-IPO and IPO investments. Now, let us examine the criteria and stocks used in the qualitative financial case study of the five studies we compared in the introduction to the literature.

As shown in Table 1, each study defines its alternatives (stocks or industries) based on the availability of qualitative information in newly listed IPO contexts. In our study, the chosen alternatives are consistent with this approach, serving as representative cases to examine how HFSs can capture investor perceptions under uncertainty. Since our study focuses on HFSs, let us examine how HFSs can be used in such real situations:

Example 1 Assume that an investor (or a financial analyst group) wants to evaluate some of the current and possible future financial instruments under some evaluation criteria such as the profitability capability ($\mathbf{x_1}$), the external reputation ($\mathbf{x_2}$), and the technical feasibility ($\mathbf{x_3}$). These criteria have a hybrid nature, i.e., they can be evaluated quantitatively and qualitatively under appropriate conditions. The company's status (e.g., privately held, lack of data, new IPO) determines which side these criteria lean more toward.

(1) **HFS-Based Illustrative Example on Pre-IPO Evaluation: OpenAI**
HFS theory makes a strong case for modeling pre-IPO investment decisions because the financial transparency of pre-IPO companies is limited or nonexistent. Institutional investors, venture capitalists, and potential IPO participants must make decisions based on insufficient, inconsistent, or wholly qualitative information. In such cases, developing a simulated investment strategy becomes advantageous. Within this paradigm, HFS theory is a helpful tool for representing the uncertainty and hesitancy inherent in early-stage investment decisions. HFS modeling, which draws on qualitative sources such as media analysis, expert opinions, and strategic relationships, allows for a structured and behaviorally realistic assessment of a company's potential investment worthiness in the face of ambiguity.
Pre-IPO firms typically lack transparent financial statements, making traditional quantitative portfolio optimization inapplicable. Investors, therefore, rely on qualitative and often contradictory information such as market reports, expert opinions, and media coverage. To illustrate, OpenAI's case shows both rapid revenue growth (e.g., CNBC, March 2025) and expectations of delayed profitability until 2029 (Reuters, March 2025), producing hesitant evaluations of profitability potential such as $h_{\text{OpenAI}}(x_1) = \{0.60, 0.75, 0.90\}$. Similarly, reputation is shaped by contrasting signals, strong market valuation (NYT, March 2025) versus ethical and cybersecurity concerns (Business Insider, 2024), leading to assessments like $h_{\text{OpenAI}}(x_2) = \{0.90, 0.40, 0.50, 0.60\}$. Technological strength is also debated, with optimism about GPT-4.5 (OpenAI, 2025) and skepticism about AGI progress (The Atlantic, 2025), reflected in values such as $h_{\text{OpenAI}}(x_3) = \{0.90, 0.95, 0.80, 0.75\}$. These hesitant fuzzy evaluations demonstrate how HFS can realistically capture investor hesitation and ambiguity in pre-IPO contexts based on publicly reported qualitative information.

(2) **HFS-Based Evaluation of a Newly Listed Stock: CoreWeave**
CoreWeave Inc. provides cloud infrastructure designed for artificial intelligence and high-performance computing applications. The company priced its initial public offering (IPO) at $40 per share on March 27, 2025, and began trading on the Nasdaq Global Select Market under the ticker symbol 'CRWV' on March 28, 2025 [26]. As a newly listed firm, CoreWeave has audited financial statements from its prospectus (SEC Form S-1/424B4), but its trading history is short, and volatility remains high. This makes it a suitable example for HFS-based evaluation, which integrates quantitative indicators with qualitative investor perceptions under uncertainty. $\mathbf{x_1} \rightarrow$ Profitability Potential: CoreWeave's SEC filings report a rapid revenue increase from $15.8M in 2022 to $1.92B in 2024, with operating income turning positive at $324.4M in 2024, but with persistent net losses of $863.4M. Such contradictory signals motivate hesitant evaluations. For instance, profitability can be represented as $h(x_1) = \{0.60, 0.75, 0.90\}$, reflecting pessimistic, moderate, and optimistic investor views. A fuzzification step is also possible: applying a piecewise linear membership function of the EBITDA value of The CoreWeave to the 2024 EBITDA ($480.0M) yields a fuzzy score of 0.684, which can be incorporated into the hesitant set alongside subjective values:

$$\begin{aligned}
&\mu_{\text{EBITDA}}(x) \\
&= \mu_{\text{EBITDA}}(x) \\
&= \begin{cases}
0.20 & \text{if } x < 0, \\
0.20 + \dfrac{0.30}{250{,}000} \cdot x & \text{if } 0 \le x < 250{,}000, \\
0.50 + \dfrac{0.20}{250{,}000} \cdot (x - 250{,}000) & \text{if } 250{,}000 \le x < 500{,}000, \\
0.85 & \text{if } x \ge 500{,}000,
\end{cases}
\end{aligned}$$

Table 1 Summary of qualitative portfolio optimization studies using fuzzy information

References and type of fuzzy information	Applications		
	Stock market	Stocks/industries	Criteria
[10] Interval Type-2 Fuzzy Information	The Growth Enterprises	SMIC, code 688191	Profitability
	Market Board of the Shenzhen Stock Exchange in China	Foxit, code 688095	Growth ability
		Roborock, code 688169	Solvency
		CanSinoBIO, code 688185	Operational capacity
[11] Hesitant Fuzzy Linguistic Term Information	The Growth Enterprises	SFT, code 300454	Technical feasibility
	Market Board of the Shenzhen Stock Exchange in China	SGC, code 300739	Future earnings
		TCT, code: 300504	Core team
		CAT, code: 300750	
[14] Hesitant Fuzzy Information	Scenario-based (Hypothetical Stock Market)		Running cost
		4 Newly listed stocks	Profitability
			Ability of innovation
[15,16] and This study, Hesitant Fuzzy Information	The New Tertiary Board (The National Equities and Quotations) in China	An off-shore gas recovery	Profitability capability
		A genetically modified food	External reputation
		An unmanned aerial vehicle manufacturing enterprise	Technical feasibility
		A pilot training company	

where a fuzzy semantic scaling approach is applied to convert these values into fuzzy scores, the thresholds are derived from benchmark reports and the financial literature, which indicate that EBITDA over $ 500K typically suggests solid performance, while values below zero represent operational loss by the expert. The value of 0.684 can be obtained for 2024. In this case, the profitability capability of CoreWeave is based on optimistic investor perception, with objective expert evaluation as follows:

$$h_{\text{CoreWeave}}(x_1) = \{0.684\}\,.$$

The financial data of CoreWeave Inc. were obtained from Yahoo Finance, which compiles official filings and market information from Nasdaq and the SEC.

$\mathbf{x_2}$ $\rightarrow$ Reputation: Qualitative sources highlight both strong market interest and strategic contracts, but also concerns about sustained losses and IPO volatility (The Wall Street Journal, 2 April 2025). This divergence can be encoded as $h(x_2) = \{0.40,\ 0.60,\ 0.90\}$.

$\mathbf{x_3}$ $\rightarrow$ Technological Strength: Investor sentiment on CoreWeave's technological position is shaped by optimism about AI/HPC growth and skepticism about long-term competitiveness. This can be represented by $h(x_3) = \{0.75,\ 0.80,\ 0.95\}$.

Together, these two cases OpenAI (pre-IPO) and CoreWeave (newly listed) illustrate how hesitant fuzzy modeling adapts to varying data availability and investor ambiguity across different stages of market maturity.

In conventional portfolio management approaches, the concept of the score function is used to measure the expected returns of the portfolios, while the concept of the deviation function is utilized to compute the portfolio's risk.

Definition 2 ([13]) Let $h = \bigcup_{\gamma\in h}\gamma$ be a HFE, γ be the possible membership degrees of h in [0, 1]. The score function of h is given by

$$s(h) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma, \tag{4}$$

where $\#h$ is the number of the elements in h. The deviation function of h [27] is given by

$$d(h) = \frac{1}{\#h}\sum_{\gamma\in h}|\gamma - s(h)| = \frac{1}{\#h}\sum_{\gamma\in h}\sqrt{(\gamma - s(h))^2}. \tag{5}$$

2.2 Non-additive Measure and Integral Theories

The concept of a non-additive measure (a monotonic function on a set of sets, fuzzy measure, capacity, and also called non-additive probabilities, a term which is frequently used in game theory) can mathematically model the positive or negative synergies between objects in a decision-making problem.

Definition 3 ([28]) Let X be a finite set and let $P(X)$ be the power set of X. The function $\mu : P(X) \to [0, 1]$ is called a fuzzy measure on X, if

(i) $\mu(\varnothing) = 0$,
(ii) $\mu(X) = 1$,
(iii) $A \subseteq B$, $A, B \subseteq X$ implies $\mu(A) \leq \mu(B)$ (monotonicity).

When $A \cap B = \varnothing$, the function μ is called

- **(i)** additive, if $\mu(A \cup B) = \mu(A) + \mu(B)$,
- **(ii)** super-additive, if $\mu(A \cup B) \geq \mu(A) + \mu(B)$,
- **(iii)** sub-additive, if $\mu(A \cup B) \leq \mu(A) + \mu(B)$.

In the Lebesgue integral, it is assumed that there is no interaction. However, in practice, there often are positive or negative interactions. In such a case, non-additive measures and integrals are used instead of additive ones in decision-making problems. The Choquet integral is a type of non-additive integral that accounts for certain interactions as a generalization of the Lebesgue integral.

Definition 4 ([29,30]) Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let μ be a fuzzy measure on X. The Choquet integral of a function $f : X \to [0, 1]$ with respect to μ is defined by

$$(C)\int_X f \,\mathrm{d}\mu := \sum_{i=1}^{n} \left(f(x_{(i)}) - f(x_{(i-1)})\right)\mu(E_{(i)}), \tag{6}$$

where the sequence $\{x_{(i)}\}_{i=0}^{n}$ is the permutation of the sequence $\{x_i\}_{i=0}^{n}$ such that $0 =: f(x_{(0)}) \leq f(x_{(1)}) \leq f(x_{(2)}) \leq \cdots \leq f(x_{(n)})$ and $E_{(i)} := \{x_{(i)}, x_{(i+1)}, \ldots, x_{(n)}\}$. An equivalent formula is

$$(C)\int_X f \,\mathrm{d}\mu := \sum_{i=1}^{n} f(x_{(i)})\left[\mu(E_{(i)}) - \mu(E_{(i-1)})\right], \tag{7}$$

where the sequence $\{x_{(i)}\}_{i=1}^{n}$ is the permutation of the sequence $\{x_i\}_{i=1}^{n}$ such that $f(x_{(i-1)}) \geq f(x_{(i)})$ for all $i = 2, \ldots, n$, $E_{(k)} = \{x_{(i)} | i \leq k\}$ for $k \geq 1$ and $E_{(0)} = \varnothing$.

The Shapley value, an important concept of cooperative games, represents the average contribution of each player to the coalition, and in this study, the concept with fuzzy characteristics is used to calculate the investment ratios of stocks in the portfolio.

Definition 5 ([28]) Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let ϕ be a fuzzy measure on X. The Shapley value for each $x_i \in X$ is defined by

$$\nu^{\phi}(x_i) := \sum_{k=0}^{n-1} \gamma_k \sum_{\substack{K \subset X \setminus \{x_i\} \\ |K| = k}} (\phi(x_i \cup K) - \phi(K)), \tag{8}$$

with

$$\gamma_k = \frac{1}{n}\binom{n-1}{k}^{-1} = \frac{(n-k-1)!k!}{n!}$$

such as $\sum_{i=1}^{n} \nu^{\phi}(x_i) = 1$ and $\nu^{\phi}(x_i) \geq 0$ for any $x_i \in X$.

Wei et al. [17] introduce an aggregation operator for HFSs by motivating the definition of the Choquet integral:

Definition 6 ([17]) Let X be a finite set and h_i, $i = 1, 2, \ldots, n$ be a collection of HFEs on X, and μ be a fuzzy measure on X. Then, the hesitant fuzzy Choquet-ordered averaging operator is given as

$$\begin{aligned}
&\mathrm{HFCOA}_{\mu}(h_1, \ldots, h_n) \\
&= \bigoplus_{i=1}^{n} (h_{\sigma(i)}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))) \\
&= \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \ldots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{1 - \prod_{i=1}^{n} (1 - \gamma_i)^{\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})}\right\},
\end{aligned} \tag{9}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $h_{\sigma(i-1)} \geq h_{\sigma(i)}$ for all $i = 2, \ldots, n$, $A_{\sigma(k)} = \{h_{\sigma(i)} | i \leq k\}$ for $k \geq 1$ and $A_{\sigma(0)} = \varnothing$.

NOTE: We use the notation $\dot{h_{\mu}} = \mathrm{HFCOA}_{\mu}(h_1, \ldots, h_n)$ to mention the aggregated version of HFEs throughout this study for simplicity.

When calculating the portfolio risk, the following fuzzy entropy measure is used to calculate the amount of uncertain and inconsistent information each HFE has, and the obtained values are aggregated with the help of the Choquet integral.

Definition 7 ([31]) Suppose h is an arbitrary HFE. The comprehensive hesitant fuzzy entropy of h is defined by

$$E_{\mathrm{new}}(h) = \frac{\alpha\theta(h) + (1 - \alpha)\eta(h)}{\alpha + (1 - \alpha)\eta(h)}, \tag{10}$$

where $\theta(h) = \frac{1}{\#h}\sum_{\gamma\in h}(1-2|\gamma-0.5|)$ and $\eta(h) = |\gamma_{\max} - \gamma_{\min}|$ are the fuzziness degree and unclarity degree of the HFE h with balance parameter $\alpha \in (0, 1)$, respectively.

2.3 Some Recent Portfolio Models based within a Hesitant Fuzzy Environment

This section reviews the two qualitative portfolio optimization approaches by Zhou and Xu [15] and Deng et al. [16]. They propose their methods for n newly listed stocks based on m qualitative criteria in the hesitant fuzzy environment as an MCDM problem due to limited financial data.

They use a hesitant fuzzy weighted averaging (HFWA) operator which is proposed in [13], to find portfolio weights in terms of return and risk. The former uses the deviation function to measure expected risk, while the latter uses the concept of entropy in the hesitant fuzzy environments. In the following, $s(h)$ and $d(h)$ are the score function and the deviation of the HFE h, respectively. Moreover, $\bar{h}$ denotes the aggregated HFE using $\bar{h}_i = \bigoplus_{j=1}^{m} h_{ij}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

- Zhou and Xu [15] propose the following two qualitative portfolio optimization models based on three types of investors' profiles by defining the notions of D max-deviation degree and S min-score degree.

$$\text{Model 1}: \begin{cases} F(W) = \min d\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right), \\ \text{s.t. } s\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right) \geq S, \\ \sum_{i=1}^{n} \omega_i = 1, \omega_i \in [0, 1], \end{cases}$$

$$\text{Model 2}: \begin{cases} F(W) = \max s\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right), \\ \text{s.t. } d\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right) \leq D, \\ \sum_{i=1}^{n} \omega_i = 1, \omega_i \in [0, 1], \end{cases}$$

where the S min-score degree and D max-deviation degree depend on their range interval given as follows:

$$\left[\min s\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right), \max s\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right)\right] \quad \text{and}$$
$$\left[\min d\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right), \max d\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right)\right].$$

Then, the investor profile is separated into three parts: "risk seeker," "risk neutral," and "risk averse" in terms of the range of S and D. More specifically S and D are calculated as follows:

- **Case 1.** For a risk seeker, $S = \max s - \frac{1}{3}(\max s - \min s)$,
- **Case 2.** For a risk neutral, $S = \max s - \frac{2}{3}(\max s - \min s)$,
- **Case 3.** For a risk averse, $S = \min s$,

and

- **Case 1.** For a risk seeker, $D = \max d$,
- **Case 2.** For a risk neutral, $D = \min d + \frac{2}{3}(\max d - \min d)$,
- **Case 3.** For a risk averse, $D = \min d + \frac{1}{3}(\max d - \min d)$.

- Deng et al. [16] have proposed measuring risk with the concept of parametric entropy, as provided by [32], within the hesitant fuzzy environment and quantifying the return value with the score function in the same way as in [15] as follows:

$$\text{Model 3}: \begin{cases} F(W) = \min E_{\alpha}^{\beta}\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right) \\ \text{s.t. } s\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right) \geq \bar{S} \\ E_{\alpha}^{\beta}(h) = \frac{2-\beta}{2-\beta-\alpha} lb\left[\frac{1}{\#h}\sum_{\gamma\in h}\left((\gamma)^{\frac{\alpha}{2-\beta}} + (1-\gamma)^{\frac{\alpha}{2-\beta}}\right)\right] \\ \alpha > 0, \beta \in [0, 1], \alpha + \beta \neq 2 \\ \sum_{i=1}^{n} \omega_i \leq 1, l_i \leq \omega_i \leq u_i, \text{and } l_i, u_i \in [0, 1], \end{cases}$$

$$\text{Model 4}: \begin{cases} F(W) = \max s\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right) \\ \text{s.t. } E_{\alpha}^{\beta}\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right) \leq \bar{E} \\ E_{\alpha}^{\beta}(h) = \frac{2-\beta}{2-\beta-\alpha} lb\left[\frac{1}{\#h}\sum_{\gamma\in h}\left((\gamma)^{\frac{\alpha}{2-\beta}} + (1-\gamma)^{\frac{\alpha}{2-\beta}}\right)\right] \\ \alpha > 0, \beta \in [0, 1], \alpha + \beta \neq 2 \\ s(h) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma \\ \sum_{i=1}^{n} \omega_i \leq 1, l_i \leq \omega_i \leq u_i, \text{and } l_i, u_i \in [0, 1], \end{cases}$$

where $\bar{S}$ is the min-score degree as in [15] and $\bar{E}$ is the max-entropy degree and it depends on its range $\left[\min E_{\alpha}^{\beta}\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right), \max E_{\alpha}^{\beta}\left(\bigoplus_{i=1}^{n} \omega_i \bar{h}_i\right)\right]$. $\bar{E}$ is calculated as follows:

- **Case 1.** For a risk seeker, $\bar{E} = \max E_{\alpha}^{\beta}$,
- **Case 2.** For a risk neutral, $\bar{E} = \min E_{\alpha}^{\beta} + \frac{2}{3}(\max E_{\alpha}^{\beta} - \min E_{\alpha}^{\beta})$,
- **Case 3.** For a risk averse, $\bar{E} = \min E_{\alpha}^{\beta} + \frac{1}{3}(\max E_{\alpha}^{\beta} - \min E_{\alpha}^{\beta})$.

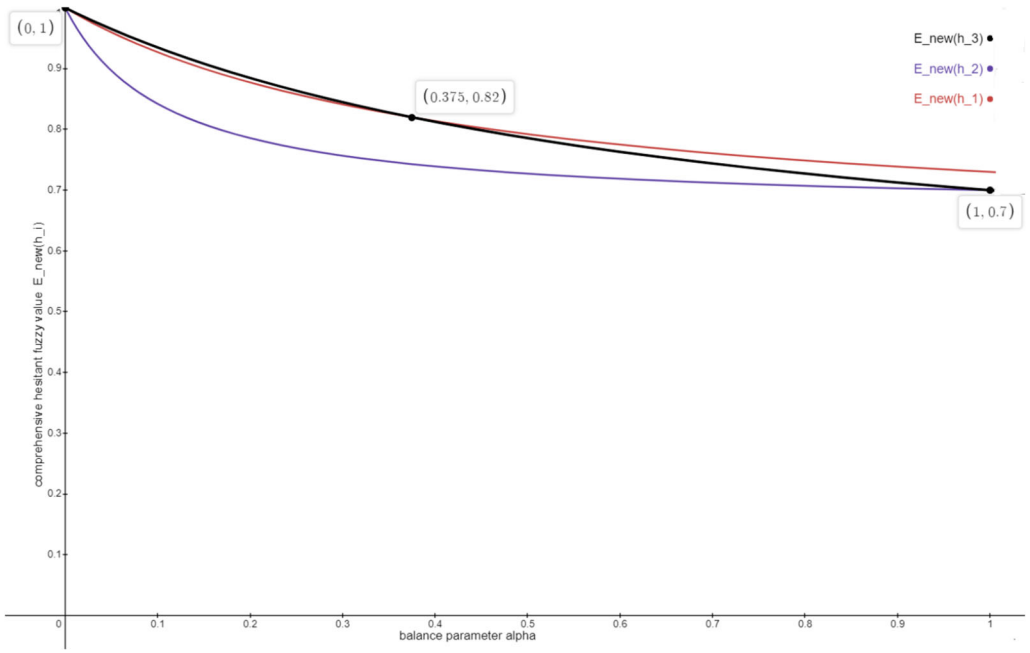

Fig. 1 Comparison between α and $E_{\text{new}}(h_i)$ for $i = 1, 2, 3$

3 A new Hesitant Fuzzy Entropy Based on the Choquet Integral

In this section, we introduce a new parametric entropy measure based on the Choquet integral for HFSs:

Definition 8 Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let μ be a fuzzy measure on X. A parametric Choquet entropy measure of the HFS A in X is given by

$$E^{\alpha}_{\text{CI}_\mu}(A) := (C)\int_X E_{\text{new}}(h_i)\,\mathrm{d}\mu, \tag{11}$$

where $E_{\text{new}}(h_i)$ is the comprehensive hesitant fuzzy function with balance parameter $\alpha \in (0, 1)$ of h_i corresponding to x_i for each $i = 1, 2, \ldots, n$.

Proposition 1 *Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let A be a HFS in X. The parametric Choquet entropy measure $E^{\alpha}_{\text{CI}_\mu}$ satisfies the following properties*:

(**P**$_1$) $0 \le E^{\alpha}_{\text{CI}_\mu}(A) \le 1$,

(**P**$_2$) $E^{\alpha}_{\text{CI}_\mu}(A) = 0$ *if and only if A is a crisp set, i.e.,*, $h_i = \left\{\gamma_i^j = 0 | j = 1, 2, \ldots, l_i\right\}$ *or* $h_i = \left\{\gamma_i^j = 1 | j = 1, 2, \ldots, l_i\right\}$ *for* $i = 1, 2, \ldots, n$.

(**P**$_3$) $E^{\alpha}_{\text{CI}_\mu}(A) = 1$, *if and only if* $h_i = \left\{\gamma_i^j = 0.5 | j = 1, 2, \ldots, l_i\right\}$ *for* $i = 1, 2, \ldots, n$.

(**P**$_4$) $E^{\alpha}_{\text{CI}_\mu}(A) = E^{\alpha}_{\text{CI}_\mu}(A^{\text{c}})$, *where A^{c} is the complement of A.*

(**P**$_5$) *If* $\theta(h_i) \le \theta(h_j)$ *and* $\eta(h_i) \le \eta(h_j)$ *then* $E^{\alpha}_{\text{CI}_\mu}(A)_i = (C)\int_X E_{new}(h_i)\,\mathrm{d}\mu \le (C)\int_X E_{new}(h_j)\,\mathrm{d}\mu = E^{\alpha}_{\text{CI}_\mu}(A)_j$ *for* $i, j = 1, 2, \ldots, n$.

Proof(**P**$_1$) Since $E_{\text{new}}(h_i) \in [0, 1]$ for any $i = 1, \ldots, n$ and the Choquet integral is monotone, we have $0 \le E^{\alpha}_{\text{CI}_\mu}(A) \le 1$.

(**P**$_2$) if A is a crisp set then $E_{\text{new}}(h_i) = 0$ for any $i = 1, \ldots, n$. So, $E^{\alpha}_{\text{CI}_\mu}(A) = 0$. Conversely, let $E^{\alpha}_{\text{CI}_\mu}(A) = 0$. Then $E_{\text{new}}(h_i) = 0$ because of the definition of the Choquet integral. Therefore, $\theta(h_i) = \eta(h_i) = 0$ and this implies $h_i = \{0\}$ or $h_i = \{1\}$ for each $i = 1, \ldots, n$.

(**P**$_3$) Let $h_i = \{0.5\}$. It implies that $E_{\text{new}}(h_i) = 1$ for $i = 1, 2, \ldots, n$. Then $E^{\alpha}_{\text{CI}_\mu}(A) = 1$. Conversely, if $E^{\alpha}_{\text{CI}_\mu}(A) = 1$, then $E_{\text{new}}(h_i) = 1$ which yields that $\theta(h_i) = 1$ and , $\eta(h_i) = 0$ and so we have $h_i = \{0.5\}$ for $i = 1, 2, \ldots, n$.

(**P**$_4$) Since $E_{\text{new}}(h_i) = E_{\text{new}}(h_i^{\text{c}})$ for $i = 1, 2, \ldots, n$, we get $E^{\alpha}_{\text{CI}_\mu}(A) = E^{\alpha}_{\text{CI}_\mu}(A^{\text{c}})$.

(**P**$_5$) If $\theta(h_i) \le \theta(h_j)$ and $\eta(h_i) \le \eta(h_j)$, $E_{\text{new}}(h_i) \le E_{\text{new}}(h_j)$ for $i = 1, 2, \ldots, n$. Then we obtain $E^{\alpha}_{\text{CI}_\mu}(A)_i \le E^{\alpha}_{\text{CI}_\mu}(A)_j$ because of the monotonicity property of the Choquet integral. Thus, the proof is completed. □

Let us illustrate this concept. Let $X = \{x_1, x_2, x_3\}$ be a fixed set, A is a HFS on X with the HFEs

$$h_A(x_1) = \{0.2, 0.4, 0.5\}, \quad h_A(x_2) = \{0.3, 0.4\},$$
$$h_A(x_3) = \{0.3, 0.2, 0.5, 0.6\}$$

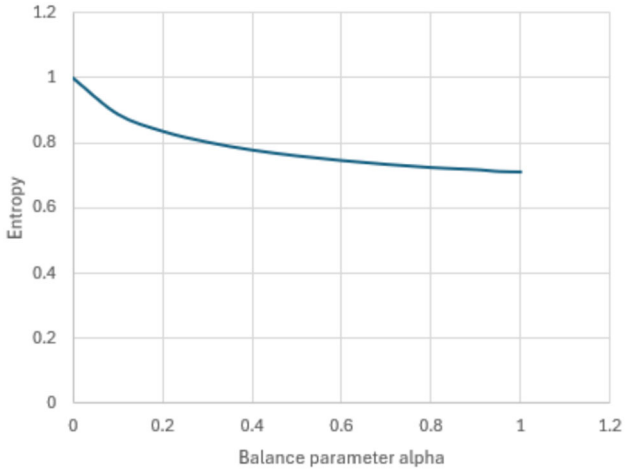

Fig. 2 Comparison between α and $E^{\alpha}_{\mathrm{CI}_\mu}(A)$

and μ be the fixed fuzzy measure on X such that

$$\begin{aligned}&\mu(\varnothing)=0 \quad (\{x_1\})=0.2 \quad \mu(\{x_2\})=0.6 \quad \mu(\{x_3\})=0.4,\\ &\mu(\{x_1,x_2\})=0.7 \quad \mu(\{x_1,x_3\})=0.5\\ &\mu(\{x_2,x_3\})=0.6 \quad \mu(X)=1.\end{aligned} \tag{12}$$

Figure 1 shows the order of the function $E_{\mathrm{new}}(h_i)$ for $i=1,2,3$ as follows:

$$\begin{cases} E_{\mathrm{new}}(h_3) > E_{\mathrm{new}}(h_1) > E_{\mathrm{new}}(h_2); & \text{if } \alpha \in (0, 0.375)\\ E_{\mathrm{new}}(h_3) = E_{\mathrm{new}}(h_1) > E_{\mathrm{new}}(h_2); & \text{if } \alpha = 0.375\\ E_{\mathrm{new}}(h_1) > E_{\mathrm{new}}(h_3) > E_{\mathrm{new}}(h_2); & \text{if } \alpha \in (0.375, 0.9996)\\ E_{\mathrm{new}}(h_3) > E_{\mathrm{new}}(h_1) = E_{\mathrm{new}}(h_2); & \text{if } \alpha \in (0.9996, 1)\end{cases}$$

We can compute the entropy of set HFS A and we can examine the entropy change according to α by using Eq. 11. Figure 2 shows that as the values of α increase, the entropy of the hesitant fuzzy set A decreases.

4 Model Design with Fuzzy Concepts in the Hesitant Fuzzy Set Setting

In this section, we provide two methods for risk-sensitive investors in the context of the portfolio optimization process.

We assume that the target investor is risk-sensitive and focuses on minimizing risk under the constraint of an acceptable return or maximizing return under the constraint of acceptable risk. Assume that an investor wants to invest into m stocks $\{y_1,\ldots,y_m\}$ under consideration of n qualitative criteria $\{x_1,\ldots,x_n\}$. The optimal portfolio ratios can be obtained based on the objective function, such as risk minimization and return maximization as follows:

- If the target investor minimizes risk with acceptable return, the following model is given:

$$\mathrm{Model\ 15}: \begin{cases} \min\limits_{z(\mu,\phi)} \sum_{i=1}^{m} v_i^{\phi}\left(E^{\alpha}_{\mathrm{CI}_\mu}(y_i)\right)\\ \text{s.t. } \sum_{i=1}^{m} v_i^{\phi} s(\dot{h}_{i\mu}) \geq f^{\delta}_{\text{min-score}}\\ E^{\alpha}_{\mathrm{CI}_\mu}(y_i) = (C)\int_X E_{\mathrm{new}}(h_{ij})\,\mathrm{d}\mu\\ \sum_{i=1}^{m} v_i^{\phi} = 1,\ \ v_i^{\phi} \in [0,1],\end{cases} \tag{13}$$

where $s(\dot{h}_i)$ is the score function of aggregated HFEs in the ith stock and v_i is the Shapley value of the ith stock for $i=1,2,\ldots,m$ and $f^{\delta}_{\text{min-score}}$ is the minimum score degree, defined by

$$\begin{aligned}&f^{\delta}_{\text{min-score}} : (0,1] \longrightarrow [S_{\min}, S_{\max}),\\ &f^{\delta}_{\text{min-score}} = \delta S_{\min} + (1-\delta) S_{\max},\end{aligned} \tag{14}$$

with control parameter $\delta \in (0,1]$. This function allows for customization of the expected minimum score degree of the target investor. As $\delta \to 0$, $f^{\delta}_{\text{min-score}} \to S_{\max}$ and the lower bound of this function is $S_{\min}$ for $\delta = 1$. The range of $f_{\text{min-score}}$ can be obtained as follows:

$$\begin{cases} S_{\max} = \max\limits_{z(\mu,\phi)} \sum_{i=1}^{m} v_i^{\phi} s(\dot{h}_{i\mu}) \text{ or } S_{\min} = \min\limits_{z(\mu,\phi)}\\ \sum_{i=1}^{m} v_i^{\phi} s(\dot{h}_{i\mu}),\\ \text{s.t. } \sum_{i=1}^{m} v_i^{\phi} = 1,\ \ v_i^{\phi} \in [0,1].\end{cases} \tag{15}$$

- If the target investor maximizes return with limited risk, the following model is given:

$$\mathrm{Model\ 16}: \begin{cases} \max\limits_{z(\mu,\phi)} \sum_{i=1}^{m} v_i^{\phi} s(\dot{h}_{i\mu})\\ \text{s.t. } \sum_{i=1}^{m} v_i^{\phi}\left(E^{\alpha}_{\mathrm{CI}_\mu}(y_i)\right) \leq f^{\rho}_{\text{max-risk}}\\ E^{\alpha}_{\mathrm{CI}_\mu}(y_i) = (C)\int_X E_{\mathrm{new}}(h_{ij})\,\mathrm{d}\mu\\ \sum_{i=1}^{m} v_i^{\phi} = 1,\ \ v_i^{\phi} \in [0,1],\end{cases} \tag{16}$$

where $f^{\rho}_{\text{max-risk}}$ is the maximum tolerable risk level of the investor defined by

$$\begin{aligned}&f^{\rho}_{\text{max-risk}} : (0,1] \longrightarrow (E_{\min}, E_{\max}],\\ &f^{\rho}_{\text{max-risk}} = \rho E_{\max} + (1-\rho) E_{\min},\end{aligned} \tag{17}$$

with a control parameter $\rho \in (0,1]$. This function allows for customization of the expected risk degree of the target

investor. As $\rho \to 0$, then $f^{\rho}_{\text{max-risk}} \to E_{\min}$. The upper bound of this function is $E_{\min}$ for $\rho = 1$. The range of $f^{\rho}_{\text{max-risk}}$ can be obtained as follows:

$$\begin{cases} E_{\max} = \max\limits_{z(\mu,\phi)} \sum_{i=1}^{m} v_i^{\phi} \left(E^{\alpha}_{\text{CI}_{\mu}}(y_i) \right) \text{ or} \\ E_{\min} = \min_{z(\mu,\phi)} \sum_{i=1}^{m} v_i^{\phi} \left(E^{\alpha}_{\text{CI}_{\mu}}(y_i) \right), \\ \text{s.t. } \sum_{i=1}^{m} v_i^{\phi} = 1, \ \ v_i^{\phi} \in [0,1]. \end{cases} \tag{18}$$

The construction of Models 5 and 6 is as follows:
Presume that an investor wants to invest into m stocks $Y = \{y_1, \ldots, y_m\}$ with respect to n qualitative criteria $X = \{x_1, \ldots, x_n\}$. Below, we present the steps for implementing the proposed Choquet integral model.

- **Step 1:** The function $E_{\text{new}}(h_{ij})$ is calculated by using Definition 7 for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. So, an entropy matrix $E = \left[h_{ij}\right]_{m\times n}$ is obtained as follows:

$$E = \begin{bmatrix} E_{\text{new}}(h_{11}) & E_{\text{new}}(h_{12}) & \ldots\ldots & E_{\text{new}}(h_{1n}) \\ E_{\text{new}}(h_{21}) & E_{\text{new}}(h_{22}) & \ldots\ldots & E_{\text{new}}(h_{2n}) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ E_{\text{new}}(h_{m1}) & E_{\text{new}}(h_{m2}) & \ldots\ldots & E_{\text{new}}(h_{mn}) \end{bmatrix}_{m\times n}.$$

  This step aims to evaluate the entropy of each HFE h_{ij} in m alternative stocks under n qualitative criteria.
- **Step 2:** Convert the entropy matrix $E = \left[h_{ij}\right]_{m\times n}$ $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ to the column vector $\bar{E} = \left[E^{\alpha}_{\text{CI}_{\mu}}(y_i)\right]_{m\times 1} = \left[E^{\alpha}_{\text{CI}_{\mu}}(y_1), E^{\alpha}_{\text{CI}_{\mu}}(y_2), \ldots, E^{\alpha}_{\text{CI}_{\mu}}(y_m)\right]^{t}_{m\times 1}$ by aggregating all entropy values of entries of the entropy matrix E with the help of Definition 8 for each $i = 1, 2, \ldots, m$ and $\alpha \in (0, 1)$. This aggregation is a fuzzy measure that considers the interactions between decision criteria. All of these fuzzy measures are determined by software (MATLAB) and depend on objectives such as return maximization or risk minimization. During the fuzzy measure identification process, we use a lexicographical order on X:

  Let $\mathcal{P}(X) = \{A_1, A_2, \ldots, A_{2^n}\}$ be the power set of X ordered lexicographically.
  Then, define $z_j := \mu(A_j)$, for $j = 1, \ldots, 2^n$.

- **Step 3:** Aggregate each row of matrix H to one HFE by using Eq. (9) to obtain column vector $\dot{H} = [\dot{h}_{1\mu}, \dot{h}_{2\mu}, \ldots, \dot{h}_{m\mu}]^t$. We use the same fuzzy measure μ for this aggregation as well because it is a row aggregation based on criteria interaction, as well. Then calculate a score of each component by using Eq. (4) to obtain a column vector $S = [s(\dot{h}_{1\mu}), s(\dot{h}_{2\mu}), \ldots, s(\dot{h}_{m\mu})]^t$. Now we obtain $s(\dot{h}_{i\mu})$ for each $i = 1, 2, \ldots, m$.
- **Step 4:** The Shapley value v_i is computed for each stock $(i = 1, 2, \ldots, m)$ with the help of Definition 5 to obtain portfolio weights. This step aims to calculate the contribution of each stock to the system with the help of the Shapley value. The calculation of the Shapley value also depends on the fuzzy measure, and these values are calculated automatically with MATLAB, considering the interaction between assets. During the fuzzy measure identification process, we use a lexicographical order on Y:

  Let $\mathcal{P}(Y) = \{B_1, B_2, \ldots, B_{2^m}\}$ be the power set of Y ordered lexicographically.
  Then, define $z_{2^n+i} := \phi(B_i)$, for $i = 1, \ldots, 2^m$.

  We concatenate the fuzzy measures on X and Y by lexicographical order into a unified vector z, where the first 2^n values are based on μ and the next 2^m values on ϕ.
- **Step 5a:** If Model 5 is chosen, we calculate the range of the function $f^{\delta}_{\text{min-score}}$ with the assistance of Model (15) and find minimum score value by using Eq. (14).
- **Step 5b:** If Model 6 is selected, calculate the range of the function $f^{\rho}_{\text{max-risk}}$ utilizing Model (18) and find the maximum-risk value by using Eq. (17). This step aims to allow investors to identify their investor profile.
- **Step 6:** Solve the non-linear programming given in Model 5 or Model 6 concerning $z(\mu, \phi) = [z_1, \ldots, z_{2^n}, z_{2^n+1}, \ldots, z_{2^m}]$. Then, calculate the Shapley value v_i^{ϕ} for $i = 1, 2, \ldots, m$.
  Figure 3 summarizes the entropy-based portfolio optimization approaches:

5 A Portfolio Optimization Problem Within the Hesitant Fuzzy Set Setting

The concept of HFS has become one of the preferred toolsets by researchers in portfolio optimization problems because it allows mathematical modeling of the difficulty experienced by decision makers due to hesitation, when faced with difficult and complex information (see, e.g., Table 1). In this optimization process, investors can seek a professional consultation from a financial analyst or experts to obtain qualitative information about future development potential, etc., due to a lack of financial reports and missing trading data of IPOs (see, e.g., [10,11,14]). In our case study, we adopt an example from [15,16]. In this example, the authors do not mention any expert knowledge elicitation method or expert number when they were constructing the hesitant fuzzy deci-

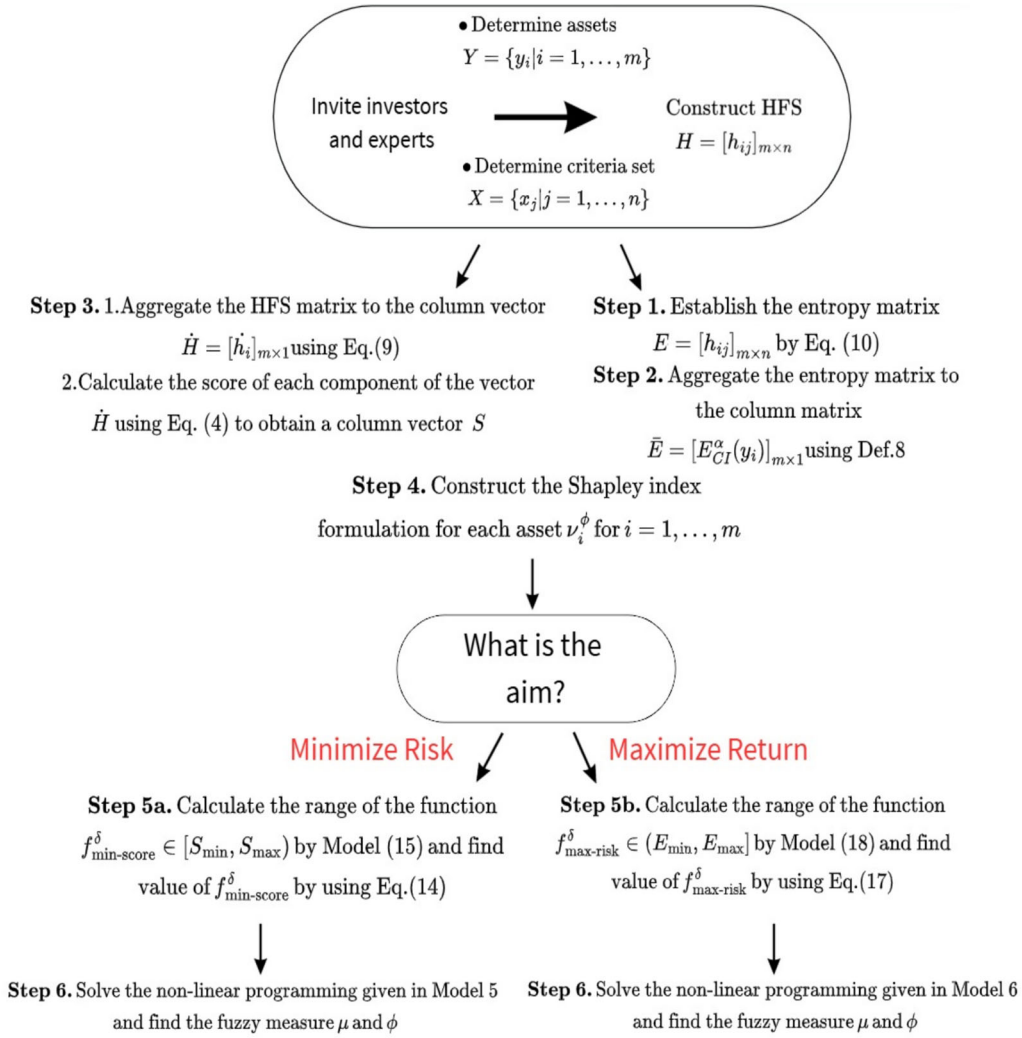

Fig. 3 Flowchart of the entropy-based portfolio optimization procedures

sion matrix in their case studies. Example 1 can illustrate a conjectural hesitant fuzzy decision matrix construction.

Example 2 An investor intends to invest a long-term investment of $\$1,000,000$ from an idle fund into the New Tertiary Board market, a financing platform for small and medium-sized enterprises (SMEs) in China. The newly listed companies on this stock market, considered to be promising by investors, which respectively represent four emerging industries, and their evaluation criteria are given in Table 1.

The investor or an expert analyzes the qualities of the stocks y_i $(i = 1, 2, 3, 4)$ under each criterion x_j $(j = 1, 2, 3)$ in the hesitant fuzzy environment and sets up the hesitant fuzzy‘ decision matrix $H = \left[h_{ij}\right]_{4\times 3}$ as follows:

$$H = \begin{bmatrix} \{0.45\} & \{0.35, 0.95\} & \{0.15\} \\ \{0.35, 0.65, 0.90\} & \{0.10\} & \{0.55\} \\ \{0.75\} & \{0.15\} & \{0.35\} \\ \{0.10, 0.70\} & \{0.30\} & \{0.65\} \end{bmatrix}.$$

Step 1: Construct the entropy matrix $E = \left[h_{ij}\right]_{4\times 3}$ by using the definition of $E_{\text{new}}(h_{ij})$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ as follows:

$$E = \begin{bmatrix} 0.9 & 0.625 & 0.3 \\ 0.7118 & 0.2 & 0.9 \\ 0.5 & 0.3 & 0.7 \\ 0.625 & 0.6 & 0.7 \end{bmatrix}.$$

During the matrix construction process, we fix $\alpha = 0.5$.
Step 2, 3: Obtain the column vectors $\bar{E}$ and S based on the vector z with the lexicographical ordered subsets of X and Y by using Definitions 8 and 6, respectively, as below:

$$z(\mu, \phi) = \Big[\underbrace{z_1, \ldots, z_{2^n}}_{z_j = \mu(A_j), \text{ for } j=1,\ldots,2^n} , \underbrace{z_{2^n+1}, \ldots, z_{2^m}}_{z_{2^n+i} = \phi(B_i), \text{ for } i=1,\ldots,2^m} \Big],$$

Table 2 Fuzzy measure μ in Model 5

	$\emptyset$	$\{x_2\}$	$\{x_3\}$	$\{x_2, x_3\}$
$\emptyset$	$z_1 = 0.0000$	$z_3 = 0.3387$	$z_4 = 0.1024$	$z_7 = 0.6758$
$\{x_1\}$	$z_2 = 0.3339$	$z_5 = 0.6398$	$z_6 = 0.5469$	$z_8 = 1.0000$

$$\bar{E} = \begin{bmatrix} 0.3z_8 + 0.325z_5 + 0.275z_2 \\ 0.2z_8 + 0.5118z_6 + 0.1882z_4 \\ 0.3z_8 + 0.2z_6 + 0.2z_4 \\ 0.6z_8 + 0.025z_6 + 0.075z_4 \end{bmatrix},$$

$$S = \begin{bmatrix} 1 - \frac{1}{2}(0.65^{z_3-z_1}0.55^{z_5-z_3}0.85^{z_8-z_5} + 0.05^{z_3-z_1}0.55^{z_5-z_3}0.85^{z_8-z_5}) \\ 1 - \frac{1}{3}(0.65^{z_2-z_1}0.45^{z_6-z_2}0.90^{z_8-z_6} + 0.35^{z_2-z_1}0.45^{z_6-z_2}0.90^{z_8-z_6} \\ +0.10^{z_2-z_1}0.45^{z_6-z_2}0.90^{z_8-z_6}) \\ 1 - (0.25^{z_2-z_1}0.65^{z_6-z_2}0.85^{z_8-z_6}) \\ 1 - \frac{1}{2}(0.35^{z_4-z_1}0.90^{z_6-z_4}0.70^{z_8-z_6} + 0.35^{z_4-z_1}0.30^{z_6-z_4}0.70^{z_8-z_6}. \end{bmatrix} \tag{19}$$

Step 4: Calculate the Shapley value formulations of each row of matrix H concerning the fuzzy measure ϕ on Y as $v^{\phi} = [v_1^{\phi}, v_2^{\phi}, v_3^{\phi}, v_4^{\phi}]^t$ by using Definition 5.

Step 5: Calculate the range of the function $f^{\delta}_{\text{min-score}}$ using Model 15. Then we have $f^{\delta}_{\text{min-score}} \in [0.1676, 0.7314]$. At this point, the investor should determine her/his investor profile by using the function $f^{\delta}_{\text{min-score}}$. Throughout the solution process, we fix the control parameter $\delta = 0.5$.

Moreover, compute the range of the function $f^{\rho}_{\text{max-risk}}$ based on the Model 18, then we have, $f^{\rho}_{\text{max-risk}} \in [0.2683, 0.8616]$. At this point, the investor should use the function $f^{\rho}_{\text{max-risk}}$ to determine their investor profile. We fix the control parameter $\rho = 0.5$ during the solution process.

Step 6: Construct the minimum entropy model based on Model 13 and we get the optimal investment ratios v_i^{ϕ} are obtained as $v_1^{\phi} = 0.2446$, $v_2^{\phi} = 0.2573$, $v_3^{\phi} = 0.2763$ and $v_4^{\phi} = 0.2218$ with the risk value 0.5317 and with the minimum score degree $f^{0.5}_{\text{min-score}} = 0.4495$ for Model 5. Table 2 indicates the values of fuzzy measure μ that are automatically identified by MATLAB for Model 5, where the intersection value of a row with a column indicates the union of their corresponding sets. For example, 0.6398 represents the fuzzy measure value of $\{x_1, x_2\}$.

So, we introduce a local weighting scheme in which criterion weights are dynamically adjusted for each alternative. That is, for a given criterion, each alternative receives a distinct weight, reflecting the context-dependent importance of the criterion in that specific evaluation. These weights are not elicited directly from the decision-maker but are instead inferred from the expert-provided decision matrix using a custom MATLAB-based optimization routine. Now, we can calculate the weights of each criterion in the universal set $X = \{x_1, x_2, x_3\}$ in terms of each alternative by using Definition 6 as below:

$$\omega_j = \mu(A_{\sigma(j)}) - \mu(A_{\sigma(j-1)}) \quad \text{for } j = 1, 2, 3.$$

For the first alternative ($i = 1$):

(1) for $j = 1$, $\omega_1 = \mu(A_{\sigma(1)}) - \mu(A_{\sigma(0)}) = \mu(\{x_2\}) - \mu(\varnothing) = z_3 - z_1 = 0.3387$,
(2) for $j = 2$, $\omega_2 = \mu(A_{\sigma(2)}) - \mu(A_{\sigma(1)}) = \mu(\{x_1, x_2\}) - \mu(\{x_2\}) = z_5 - z_3 = 0.3011$,
(3) for $j = 3$, $\omega_3 = \mu(A_{\sigma(3)}) - \mu(A_{\sigma(2)}) = \mu(\{x_1, x_2, x_3\}) - \mu(\{x_1, x_2\}) = z_8 - z_5 = 0.3602$.

For the second and third alternatives ($i = 2, 3$):

(1) for $j = 1$, $\omega_1 = \mu(A_{\sigma(1)}) - \mu(A_{\sigma(0)}) = \mu(\{x_1\}) - \mu(\varnothing) = z_2 - z_1 = 0.3339$,
(2) for $j = 2$, $\omega_2 = \mu(A_{\sigma(2)}) - \mu(A_{\sigma(1)}) = \mu(\{x_1, x_3\}) - \mu(\{x_1\}) = z_6 - z_2 = 0.2130$,
(3) for $j = 3$, $\omega_3 = \mu(A_{\sigma(3)}) - \mu(A_{\sigma(2)}) = \mu(\{x_1, x_2, x_3\}) - \mu(\{x_1, x_3\}) = z_8 - z_6 = 0.4531$.

For the fourth alternative ($i = 4$):

1. for $j = 1$, $\omega_1 = \mu(A_{\sigma(1)}) - \mu(A_{\sigma(0)}) = \mu(\{x_3\}) - \mu(\varnothing) = z_4 - z_1 = 0.1024$,
2. for $j = 2$, $\omega_2 = \mu(A_{\sigma(2)}) - \mu(A_{\sigma(1)}) = \mu(\{x_1, x_3\}) - \mu(\{x_3\}) = z_6 - z_4 = 0.4445$,
3. for $j = 3$, $\omega_3 = \mu(A_{\sigma(3)}) - \mu(A_{\sigma(2)}) = \mu(\{x_1, x_2, x_3\}) - \mu(\{x_1, x_3\}) = z_8 - z_6 = 0.4531$.

Determining the weights of criteria is often a time-consuming and cognitively demanding task for decision makers. To overcome this limitation, we propose a data-driven approach that infers optimal weights directly from the decision matrix provided by the decision maker. The method identifies the set of criterion weights that optimize the overall decision quality, based solely on the expert-supplied evaluation data. This not only alleviates the cognitive burden on decision makers but also ensures internal consistency between the decision matrix and the resulting ranking. The one of the goal of the proposed Model 5 and Model 6 is to optimize overall decision consistency by identifying the most coherent weight distribution embedded in the evaluation data. This data-driven local weighting strategy allows for a more nuanced and adaptive decision model, reducing subjectivity while enhancing model fidelity. Moreover, Table 3 indicates the fuzzy measure values ϕ as below:

To derive the investment ratio for each alternative, we use a Shapley-value-based allocation. The score of each alternative is weighted by its Shapley value, calculated with respect to a fuzzy measure obtained from the decision matrix. The fuzzy measure is learned using a MATLAB routine, allowing

Table 3 Fuzzy measure ϕ in Model 5

	$\emptyset$	$\{y_3\}$	$\{y_4\}$	$\{y_3, y_4\}$
$\emptyset$	$z_9 = 0.0000$	$z_{12} = 0.1184$	$z_{13} = 0.0737$	$z_{19} = 0.4396$
$\{y_1\}$	$z_{10} = 0.0943$	$z_{15} = 0.4414$	$z_{16} = 0.4018$	$z_{22} = 0.7806$
$\{y_2\}$	$z_{11} = 0.0980$	$z_{17} = 0.4742$	$z_{18} = 0.4075$	$z_{23} = 0.7956$
$\{y_1, y_2\}$	$z_{14} = 0.4384$	$z_{20} = 0.8269$	$z_{21} = 0.7615$	$z_{24} = 1.0000$

Table 4 Fuzzy measure μ in Model 6

	$\emptyset$	$\{x_2\}$	$\{x_3\}$	$\{x_2, x_3\}$
$\emptyset$	$z_1 = 0.0000$	$z_3 = 0.5694$	$z_4 = 0.0094$	$z_7 = 0.6906$
$\{x_1\}$	$z_2 = 0.8599$	$z_5 = 0.8984$	$z_6 = 0.8672$	$z_8 = 1.0000$

the model to capture the interaction among the criteria and provide data-driven, interpretable investment guidance.
Moreover, establish a return maximization Model 16 and we get a return value 0.6310 with the acceptable risk value of $f^{0.5}_{\text{max-risk}} = 0.5650$ and optimal investment ratios v_i^{ϕ} as $v_1^{\phi} = 0.0845$, $v_2^{\phi} = 0.2333$, $v_3^{\phi} = 0.5504$, and $v_4^{\phi} = 0.1318$. Table 4 indicates the values of the fuzzy measure obtained in Model 6 as follows:

The weight of each criterion can be obtained by the same method in Model 5. Here, the local weighting approach is used, where the importance of a criterion varies across alternatives. The weights are not manually set but automatically derived from the decision matrix, using an optimization process that ensures consistency and decision quality. This allows for adaptive and context-sensitive evaluation without requiring explicit input from the decision-maker. Moreover, Table 5 indicates the fuzzy measure values to calculate investment ratios by using the Shapley values from Definition 5 as follows:
The resulting investment ratios are derived by considering both criteria interactions and asset dependencies through a fuzzy measure-based modeling framework. While the inclusion of criteria interactions primarily enhances the model's sensitivity and robustness, the interdependencies among assets offer actionable insights that can inform practical investment strategies. Therefore, the subsequent analysis in this study focuses on asset-level interactions and their implications for portfolio adjustment and optimization.

6 Comparison Analysis

In this section, we compare the results of our proposed new methods with those of previous ones. Zhou and Xu [15] and Deng et al. [16] solved Example 2 in terms of investors' risk perception, such as risk-averse, risk-neutral, and risk-seeker investors. Zhou and Xu [15] proposed two models (Models 1 and 2). Moreover, Deng et al. [16] developed two portfolio optimization models (Models 3 and 4) by replacing the deviation function with the entropy to measure risk in a portfolio.

What are the differences between Models 1, 3, and the proposed 5?

(1) While Models 1 and 3 collect the HFEs in the decision matrix into a single HFE, they ignore the weight of the criteria and the interaction between criteria. However, Model 5 uses an aggregation operator that considers the interaction between criteria for model sensitivity.
(2) Model 1 uses the deviation function to measure risk, while Model 3 uses parametric entropy. However, both models apply standard deviation and entropy formulas to the aggregated column matrix after using an aggregation operator that does not consider the interaction between the criteria for both the deviation function and entropy. Model 5 first calculates the entropy of each HFE in the decision matrix on a criterion basis and then creates the aggregated column matrix with the help of the Choquet integral, which considers the interaction between the criteria.
(3) While Models 1 and 3 offer the investor three different investor profiles, such as risk seeker, risk neutral, and risk averse, Model 5 does not label the investor with a restricted profile scheme and gives them the freedom to choose her/his own investor profile.
(4) While Models 1 and 3 use the concept of weighted average when calculating investment ratios, Model 5 calculates the contribution of each stock to the portfolio using the Shapley value, which accounts for the correlation between stocks.

The following Table 6 provides further comparison:
Based on Table 6, we can conclude the following:

(1) The results of Model 1 for risk-neutral and risk-averse investors are the same. This shows us that the investor profile regulation does not sufficiently reflect investors' preferences for Model 1. Moreover, in investment ratios for stocks, both Models 1 and 3 are generally eager to assign the same investment ratio after selecting the most

Table 5 Fuzzy measure ϕ in Model 6

	$\emptyset$	$\{y_3\}$	$\{y_4\}$	$\{y_3, y_4\}$
$\emptyset$	$z_9 = 0.0000$	$z_{12} = 0.5326$	$z_{13} = 0.0218$	$z_{19} = 0.6356$
$\{y_1\}$	$z_{10} = 0.0454$	$z_{15} = 0.5331$	$z_{16} = 0.3827$	$z_{22} = 0.7869$
$\{y_2\}$	$z_{11} = 0.1714$	$z_{17} = 0.7544$	$z_{18} = 0.4252$	$z_{23} = 0.9754$
$\{y_1, y_2\}$	$z_{14} = 0.2327$	$z_{20} = 0.9571$	$z_{21} = 0.4522$	$z_{24} = 1.0000$

Table 6 Comparison of the results of Models 1 and 3

Investment ratios (weighted mean)	Risk-seeking		Risk-neutral		Risk-averse	
	Model 1	Model 3	Model 1	Model 3	Model 1	Model 3
ω_1	0.1170	0.5527	0.0500	0.6669	0.0500	0.7464
ω_2	0.1522	0.3423	0.0500	0.2331	0.0500	0.1536
ω_3	0.6531	0.0500	0.8500	0.0500	0.8500	0.0500
ω_4	0.0777	0.0500	0.0500	0.0500	0.0500	0.0500
Returns	0.8717	0.8716	0.8666	0.8662	0.8666	0.8608

invested stock. So, it caused low diversity among stocks. Also, the returns they obtained are extremely similar.

(2) Model 1 tends to invest more funds in stock three (y_3) and generally invests equally in the remaining stocks, while Model 3 tends to invest more funds in stock one and invests equally in stocks three and four.

(3) The values for the returns of both models are very close. In this case, the deviation function and entropy do not differ much from each other in measuring risk.

The results obtained for Model 5 facilitate the function $f^{\delta}_{\text{min-score}}$ with the control parameter $\delta = 0.5$, and it is then solved by applying the investor profiles suggested in Models 1 and 3. The results are given in Table 7:

Based on Table 7, we can conclude the following:

(1) The investment ratios and returns obtained for each of the four cases differ. This shows that there is diversity in the results.

(2) While we observe that the highest investment is made in the second stock for the risk-neutral and risk-averse investors, risk seekers invest most in the third stock. The investment ratios and returns given in Tables 7 and 6 differ substantially. This shows that our method is significantly different from the other two methods. Moreover, the risk-neutral and the risk-averse profiles invest most in the second stock (y_2), while the risk seeking chooses the third stock (y_3).

Based on Tables 6 and 7, we can conclude the following:

(1) The most obvious thing seen in Models 1 and 3 is the lack of diversity between investment ratios and returns. In Model 5, this is not the case.

(2) Models 1 and 3 don't reflect investor preferences well; although the investor profile has changed, investors' returns and ratios are the same. Moreover, in some cases, although the method and process have changed between the two models, the investment ratios obtained by investors with different profiles have not changed. In contrast, according to investor profiles, Model 5 makes the least investment.

(3) While Models 1 and 3 always invest the most in the third and first stocks, respectively, in Model 5, this situation varies depending on the investor's profile.

These results demonstrate that accounting for the interaction between the different criteria does have a significant effect on the investment ratio, and hence if the interaction between criteria underlying the investment decision problem is relevant in the real world, which would be a natural assumption to make, then models such as Model 5 are required for analysis.

The following table presents the results of Models 2 and 4:

Based on Table 8, we thus conclude the following:

(1) The return values obtained for these two models are quite close to each other. This means no significant difference exists between using the deviation function and entropy, especially for risk-seeking investors. Moreover, $\omega_3 = 0.0085 \leq 0.05$ investment ratios of Model 4 for risk seeking and risk neutral. So, their model does not satisfy optimization restrictions (see, e.g., [16, Sect. 5.1.2, Step 2]).

(2) Model 4 obtained the same expected return value for all three investor profiles. This shows that the investor profile selection technique is not functioning correctly.

Table 7 Comparison of the results of Model 5

Investment ratios (Shapley value)	$f^{0.5}_{\text{min-score}}$ Model 5	Risk-seeking Model 5	Risk-neutral Model 5	Risk-averse Model 5
ν_1^{ϕ}	0.2446	0.1719	0.1519	0.1994
ν_2^{ϕ}	0.2573	0.2655	0.4008	0.4932
ν_3^{ϕ}	0.2763	0.3759	0.3892	0.2611
ν_4^{ϕ}	0.2218	0.1867	0.0581	0.0463
Returns	0.4636	0.5073	0.3478	0.1878

Table 8 Comparison of the results of Models 2 and 4

Investment ratios (weighted mean)	Risk-seeking		Risk-neutral		Risk-averse	
	Model 2	Model 4	Model 2	Model 4	Model 2	Model 4
ω_1	0.3164	0.3164	0.3084	0.3164	0.2843	0.3182
ω_2	0.3256	0.3264	0.3220	0.3264	0.3114	0.3282
ω_3	0.0863	0.0085	0.1012	0.0085	0.1456	0.0804
ω_4	0.2717	0.2717	0.2684	0.2717	0.2587	0.2732
Returns	0.8771	0.8771	0.8770	0.8771	0.8769	0.8771

(3) While Models 2 and 4 invest the most funds in the second stock and the least in the third stock in all cases.

The results obtained for Model 6 facilitate the function $f^{\rho}_{\text{max-risk}}$ with the control parameter $\rho = 0.5$ and the model is solved by applying the investor profiles suggested in Models 2 and 4. The results obtained are as follows:

Based on Table 9, we can deduce the following:

(1) In all four cases, there is diversity in both investment ratios and expected returns.
(2) The most highly invested stock is the third stock (y_3), and the least invested stock is the fourth stock (y_4), except for the case $\rho = 0.5$ and for the risk-neutral investor.

Based on Tables 8 and 9, we conclude the following:

(1) While the most highly invested stock of Models 2 and 4 is stock 2 (y_2), the most highly invested stock of Model 6 is stock 3 (y_3). This is because the interaction can be modeled in Model 6.
(2) It is observed that Models 2 and 4 obtained the same returns despite using different methods. In addition, when Model 4 is solved according to different investor profiles, the same investment ratios are obtained, which does not make much sense from a financial perspective. In contrast, Model 6 is sensitive to investors' expectations and reflects investors' preferences in the results.

7 Sensitivity Analysis

In this section, we measure the effect of the balance parameter α and control parameters δ and ρ on the proposed models.

7.1 Investigating the Effect of α and δ in Model 5

This subsection examines how risk and return values are influenced by the balance (α) and control (δ) parameters in Model 5. The balance parameter α affects the entropy measure; hence, when α changes, the objective function of the optimization problem, which is underlying Model 5, also changes simultaneously. Then, each system with the corresponding new objective function is solved by changing the control parameter δ. When δ varies within the interval $(0, 1)$, the values of the function $f^{\delta}_{\text{min-score}}$ change in terms of δ. Hence, the parameter α has no effect on the range of $f^{\delta}_{\text{min-score}}$. Figure 4 shows the change in return and risk values as α and δ change.

Based on Fig. 4, we conclude that

- Returns strictly increase for each fixed α when δ decreases. Although the same does not strictly apply to all risk values, we can say that the risk generally increases for each fixed α when δ decreases.
- As δ decreases, both risk and return increase for fixed α. Figures 5 and 6 visualize these findings.

Based on Fig. 6, we can deduce that as δ increases, the return decreases, and as δ decreases, the return increases.

On the basis of Fig. 7, we conclude that

Table 9 Comparison of the results of Model 6

Investment ratios (Shapley value)	$f^{0.5}_{\text{max-risk}}$ Model 6	Risk-seeking Model 6	Risk-neutral Model 6	Risk-averse Model 6
ν_1^{ϕ}	0.0845	0.1996	0.1213	0.0301
ν_2^{ϕ}	0.2333	0.1898	0.2037	0.2040
ν_3^{ϕ}	0.5504	0.5201	0.5389	0.7363
ν_4^{ϕ}	0.1318	0.0905	0.1361	0.0296
Returns	0.6310	0.6887	0.6655	0.6328

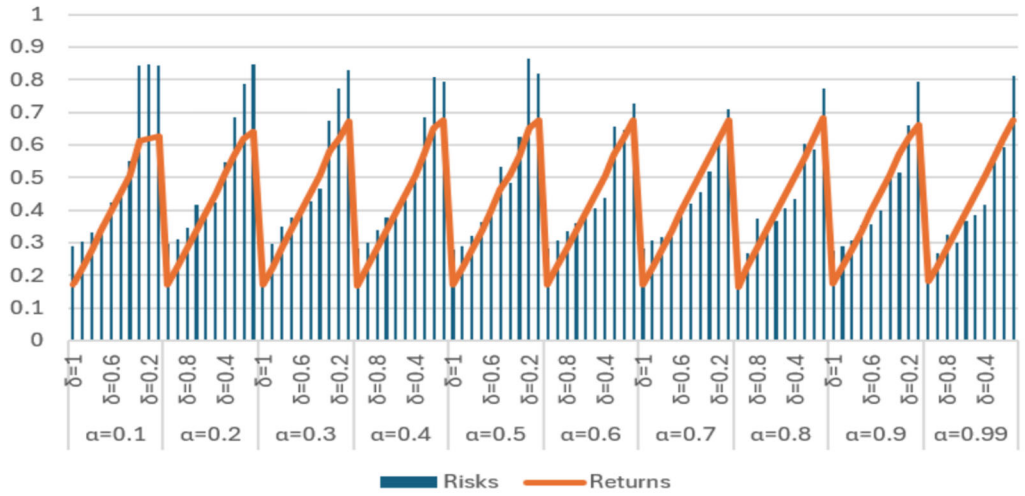

Fig. 4 Comparison of risk and returns in terms of α and δ

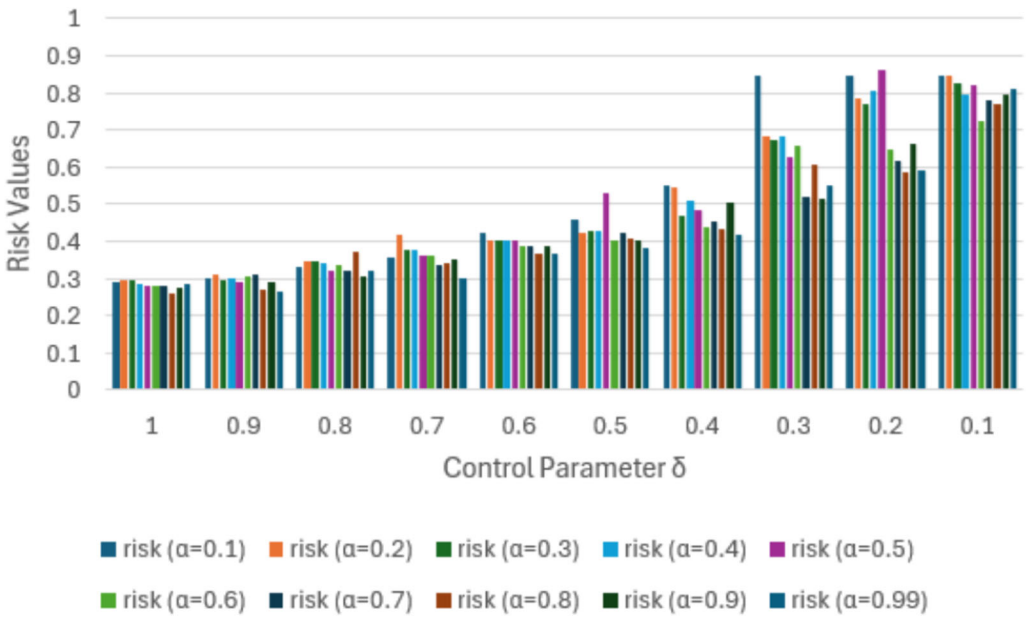

Fig. 5 Effect of α and δ on Risk

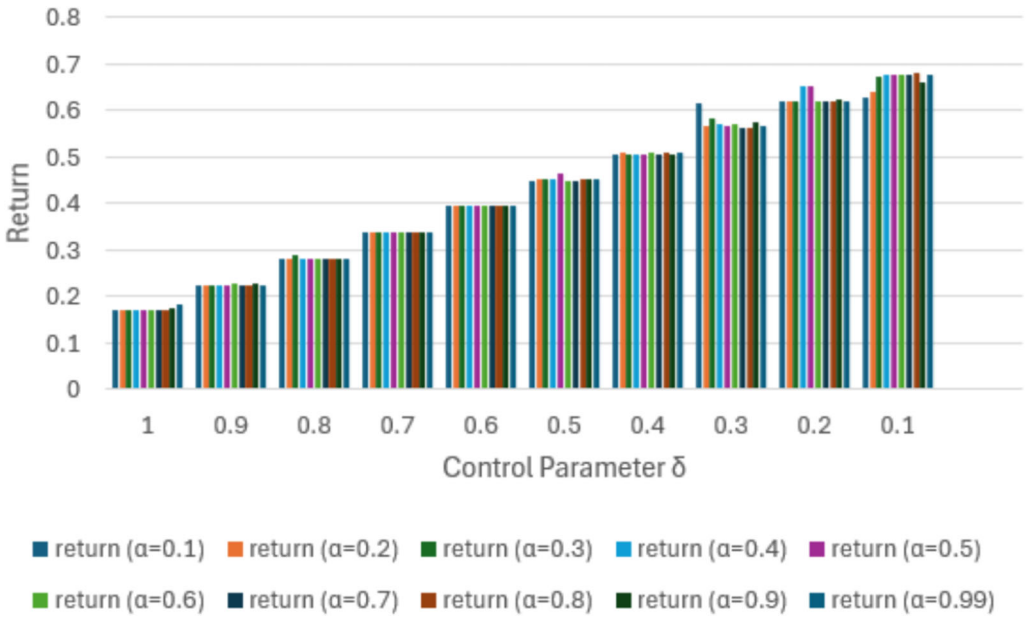

Fig. 6 Effect of α and δ on Return

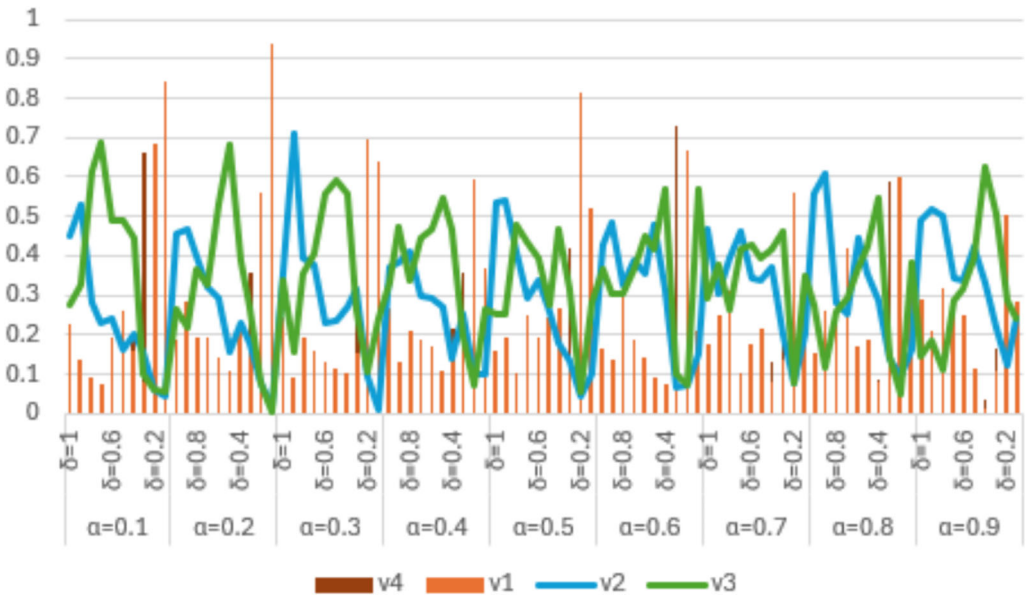

Fig. 7 Rank of investment ratios

- For each fixed α, the third stock is predominantly more invested, and the fourth stock is least invested as the δ changes.
- investment ratios are not strictly affected by the increase or decrease of α and δ.
- For each fixed α, Model 5 tends to invest the most in the first stock for $\delta = 0.1$ and 0.2 cases, except for $\alpha = 0.7$ and 0.6. But in this case, if it cannot invest the most in the first stock, it chooses the third stock for maximum investment.

7.2 Investigating the Effect of α and ρ in Model 6

This subsection investigates how balance (α) and control (ρ) parameters affect Model 6. Since the parameter α is the balance parameter of the entropy measure, it affects the range of entropy and thus affects the value of the function $f^{\rho}_{\text{max-risk}}$ together with ρ. Since these effects change the risk measure, the portfolio's return changes with each risk value change. Figure 8 shows the change in return and risk values as α and ρ change.

Based on Fig. 8, we can conclude that

Fig. 8 Comparison risk and return values in terms of α and ρ

- It can be seen that risk values strictly decrease for fixed $\alpha = 0.1, 0.2, 0.9$, and 0.99 when ρ decreases. Moreover, for other values of α, risk values strictly increase for $\rho = 1$ and $\rho = 0.9$ and strictly decrease other values of ρ and this strictly decreases break down only at $\alpha = 0.8$ because the risk values increase for $\rho = 0.7$ and $\rho = 0.6$. Although the same strict situation does not apply to all return values, we can say that the returns generally decrease for each fixed α when ρ decreases.
- In general, we can say that as ρ decreases, both risk and returns decrease for fixed α.

Figures 9 and 10 are given to make the obtained values more visible.

Based on Figs. 9 and 10, we conclude that

- As ρ increases, risk increases, and as ρ decreases, risk decreases for $\alpha = 0.1, 0.2, 0.9,$ and 0.99, ρ. For the other cases of α, the same holds except for $\rho = 1$ and 0.9 as well as $\rho = 1$ and 0.9 at $\alpha = 0.8$ (see, Fig. 9).
- As α increases, the risk sometimes increases and sometimes decreases depending on the value of α. However, two different groups of patterns are observed. The same pattern is observed in the increase and decrease of risk values when ρ is between 0.1 and 0.4. and for $\rho = 0.5$ and $\rho = 0.6$. The other values of ρ break these patterns. In the first pattern, there is a small difference of -0.0012 for $\rho = 0.1$ when $\alpha = 0.6$ and $\alpha = 0.7$ and further, there is a -0.0015 difference when $\alpha = 0.3$ and $\alpha = 0.4$ at $\rho = 0.6$ (see, Fig. 11).
- As α increases and ρ decreases simultaneously, the summation of risk values decreases. As α changes, it can be said that the maximum summation of the return value is at $\rho = 1$ and the minimum summation of the return value is at $\rho = 0.1$.
- When α is fixed in the return table and the values obtained from ρ are examined, or when ρ is fixed and the values of α are varied, no monotonous increase, decrease, or clustering is observed (see, Fig. 10).

  Based on Fig. 12, we conclude that the stock with the highest investment ratio is predominantly the third stock, and the stock to which the least funds are transferred is the fourth stock.

More detailed results are provided in the Appendices. As the entropy parameter α and the risk-appetite parameters δ and ρ vary, the proposed models yield different return risk intervals (see Tables 10, 11, 12) and investment ratios (see Tables 11, 12, 13). This demonstrates that the outputs are sensitive to parameter changes and robustly reflect investors' preferences: each shift in parameters results in a distinct portfolio allocation.

By contrast, the baseline models [15,16] do not capture this property. The models of Zhou and Xu [15] are non-parametric and rely on trisection of risk attitudes, which only covers three discrete investor types (risk averse, neutral, and risk seeking). So, we do not do further comparison of the outcome of their models and the outcome of our proposed models in Sect. 6. Deng et al. [16] introduced a parametric entropy model, but their results (Tables 10–11 in [16]) often assign identical ratios to different assets, limiting portfolio diversity.

Our models overcome these limitations in two ways. First, the entropy parameter α continuously adjusts both the risk measure and the tolerable risk range of the investor (e.g., $[0.2939, 0.7197]$ at $\alpha = 0.1$ versus $[0.2600, 0.8038]$ at $\alpha = 0.99$), ensuring dynamic reflection of preferences. Second, the functions $f^{\delta}_{\text{min-score}}$ and $f^{\rho}_{\text{max-risk}}$ generalize the trisection-based profiles, covering the full spectrum of risk

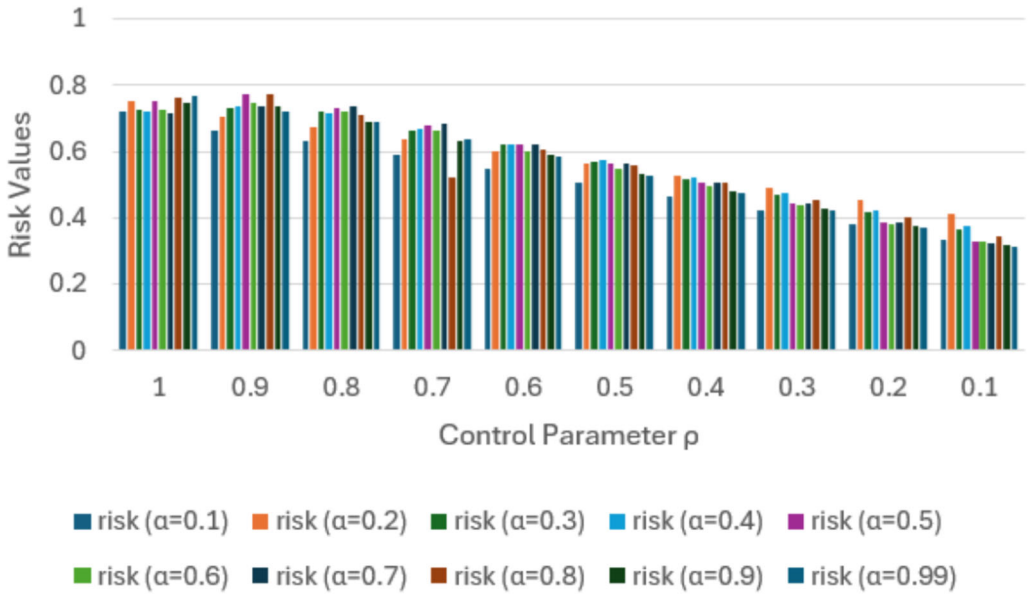

Fig. 9 Effect of α and ρ on Risk

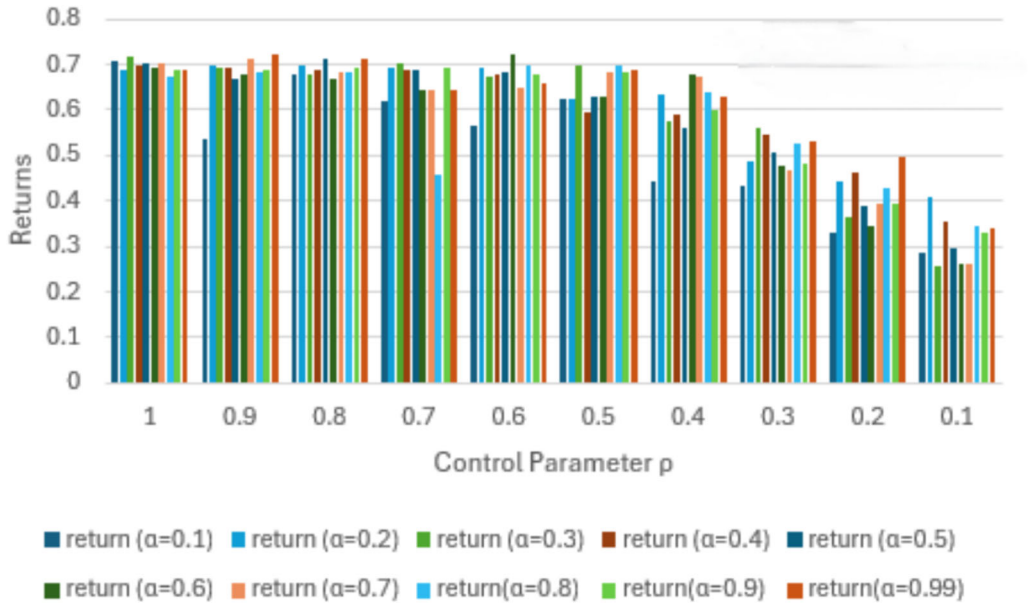

Fig. 10 Effect of α and ρ on Return

attitudes rather than only three points. If $\delta = 1, \frac{2}{3}$ and $\delta = \frac{1}{3}$ then the function $f^{\delta}_{\text{min-score}}$ intersects S in score-trisection model in [15,16] as risk averse, risk neutral, and risk seeker, respectively. Similarly, if $\rho = 1, \frac{2}{3}$ and $\rho = \frac{1}{3}$ function $f^{\rho}_{\text{max-risk}}$ intersect D and $\bar{E}$ in deviation trisection model in [15,16] as risk seeker, risk neutral and risk averse. The proposed models cover the investor profile of trisection models regarding their three types of investor profiles. As shown in Tables 11, 12, 13, investment ratios do not collapse into identical allocations, confirming both robustness and diversity in the proposed approach.

8 Conclusion

This study proposed two qualitative financial portfolio optimization models that integrate HFSs, parametric entropy, and Shapley value allocation to capture investor perceptions more realistically. HFSs allow decision makers to express hesitancy, while fuzzy measure-based aggregation and Choquet integral account for criterion interactions. A novel parametric entropy measure introduces the entropy of each hesitant fuzzy element in the decision matrix separately and then aggregates them using the Choquet integral. In this framework, fuzzy measures are determined automatically and

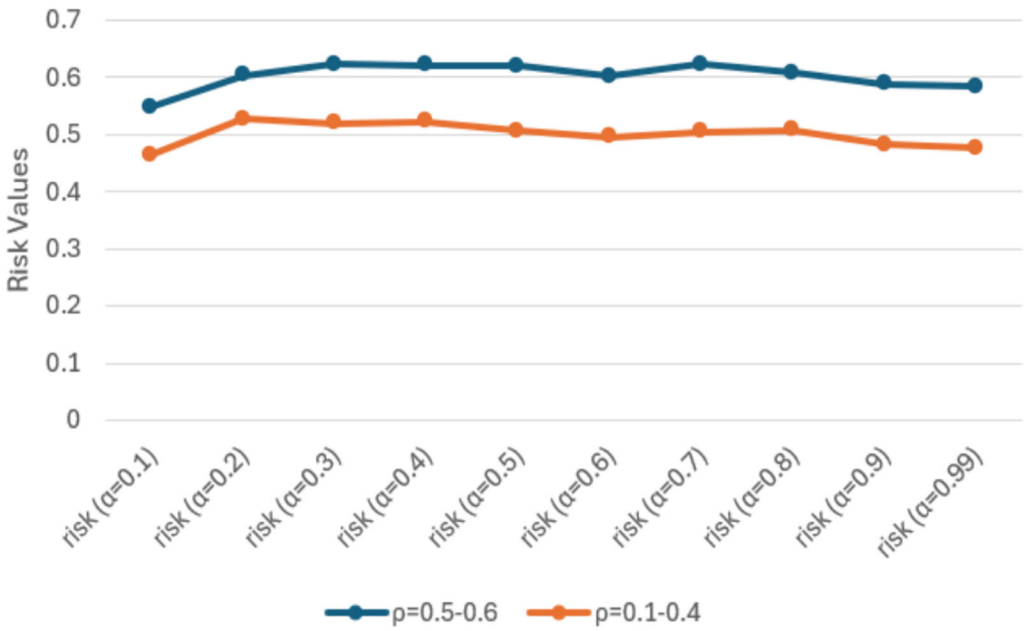

Fig. 11 Impact of α to risk values

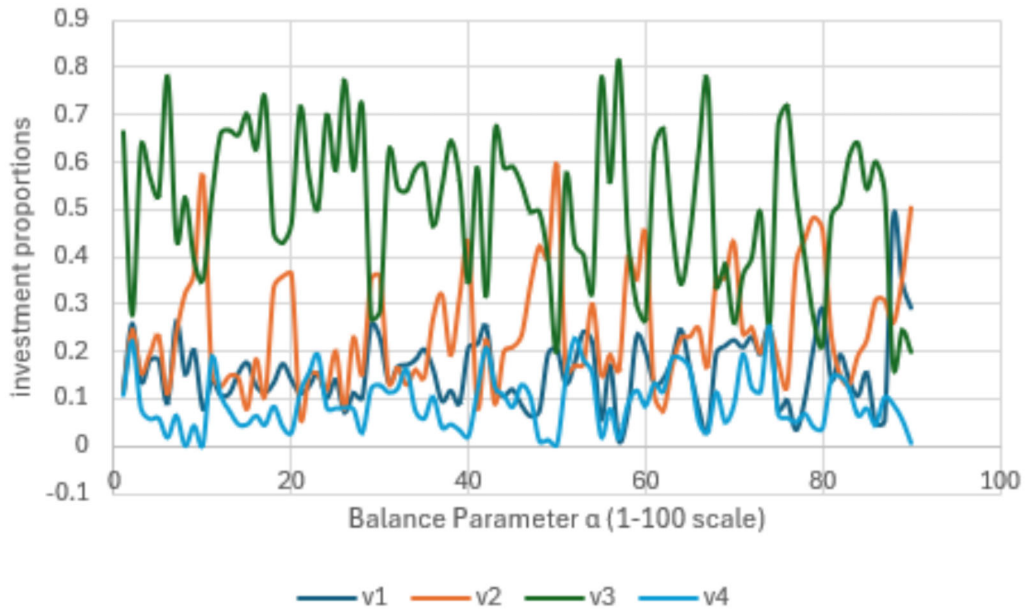

Fig. 12 Rank of investment ratios

dynamically through non-linear programming. The entropy parameter generates a new decision matrix and defines an updated acceptable risk tolerance range for the investor. Two parametric investor risk-appetite functions define a continuous spectrum of investor risk attitudes, and Shapley values ensure fair allocation of investment ratios beyond what weighted means can offer.

Comparative analysis shows that existing models often yield non-sensitive and homogeneous allocations, particularly under risk minimization, whereas the proposed approach produces more diverse and interaction-aware portfolios. The results demonstrate that criterion interaction must be considered seriously in portfolio optimization. The proposed multi-level optimization framework, which simultaneously determines criterion weights and investment ratios, enhances the sensitivity and responsiveness of the portfolio optimization system.

The methodology has practical value for multiple stakeholders in financial decision-making. Pre-IPO and newly listed IPO investors must base their investment decisions on qualitative information such as analyst reports, media news, and technological trends when historical data are unavailable.

Venture Capital (VC) and Private Equity (PE) Quantitative data are generally limited in start-up and scale-up investments. Individual investors are particularly valuable in their daily practice in expressing their risk appetite with intermediate values such as "I am not completely risk-averse, I am a bit moderate." Fund managers/portfolio managers can use them because, when optimizing risk-return, classical models generally work when data are available. However, uncertainty is high in emerging markets, frontier markets, or new financial instruments. The proposed methods provide the fund manager with the information that the criteria weights are not fixed but change according to the investor group's risk appetite, resulting in more flexible fund structures. Instead of relying on a single risk profile (risk averse, risk neutral, risk seeker) when making portfolio recommendations to their clients, financial advisors and analysts can use your method to advise on intermediate values. Furthermore, Shapley-based fair distribution provides a more definitive answer to the question, "Which stock contributes how much?", which increases client confidence. Moreover, beyond the private sector, regulators and policy-makers can utilize such a framework for stress testing and systemic risk assessment by modeling investor risk preferences along a continuous spectrum rather than in rigid categories.

A limitation of the approach is the computational burden: the number of fuzzy measures increases exponentially with more criteria and alternatives, making the calculation of the Shapley value and Choquet integral complex. Future research will explore sparsity conditions and computational refinements to address this challenge. Moreover, beyond finance, the same fairness interpretability paradigm has been emphasized in other domains such as healthcare and cybersecurity, where deep learning models are supported by SHAP and LIME for explainability (e.g., [33–35]). Inspired by these developments, a promising direction for future work is to integrate qualitative fuzzy decision-making with quantitative deep learning and SHAP-based explainability, thereby creating a unified framework for explainable decision-support across finance, healthcare, and smart city security.

Table 10 Return and risk values under different values of α and δ

Control parameter	Balance parameter $\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$		$\alpha = 0.5$	
	Risks	Returns	Risks	Returns	Risks	Returns	Risks	Returns	Risks	Returns
$\delta = 1.0$	0.2888	0.1696	0.2975	0.1700	0.2973	0.1705	0.2831	0.1682	0.2786	0.1705
$\delta = 0.9$	0.3023	0.2244	0.3121	0.2244	0.2969	0.2243	0.3016	0.2245	0.2895	0.2243
$\delta = 0.8$	0.3331	0.2809	0.3455	0.2819	0.3477	0.2871	0.3398	0.2820	0.3212	0.2816
$\delta = 0.7$	0.3550	0.3370	0.4159	0.3383	0.3763	0.3381	0.3772	0.3383	0.3628	0.3371
$\delta = 0.6$	0.4229	0.3944	0.4023	0.3945	0.4041	0.3935	0.4045	0.3947	0.4048	0.3944
$\delta = 0.5$	0.4609	0.4498	0.4231	0.4510	0.4271	0.4510	0.4306	0.4509	0.5317	0.4636
$\delta = 0.4$	0.5501	0.5067	0.5454	0.5070	0.4676	0.5062	0.5119	0.5065	0.4830	0.5062
$\delta = 0.3$	0.8444	0.6140	0.6838	0.5654	0.6745	0.5821	0.6835	0.5695	0.6252	0.5658
$\delta = 0.2$	0.8466	0.6193	0.7857	0.6186	0.7715	0.6211	0.8080	0.6501	0.8635	0.6502
$\delta = 0.1$	0.8450	0.6282	0.8456	0.6410	0.8281	0.6732	0.7941	0.6746	0.8202	0.6750
Control parameter	Balance parameter $\alpha = 0.6$		$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$		$\alpha = 0.99$	
	Risks	Returns	Risks	Returns	Risks	Returns	Risks	Returns	Risks	Returns
$\delta = 1.0$	0.2821	0.1711	0.2819	0.1719	0.2577	0.1682	0.2738	0.1748	0.2850	0.1809
$\delta = 0.9$	0.3076	0.2262	0.3086	0.2245	0.2689	0.2244	0.2888	0.2263	0.2667	0.2245
$\delta = 0.8$	0.3340	0.2808	0.3188	0.2808	0.3724	0.2816	0.3059	0.2821	0.3234	0.2809
$\delta = 0.7$	0.3599	0.3373	0.3386	0.3370	0.3426	0.3371	0.3539	0.3382	0.2990	0.3372
$\delta = 0.6$	0.3871	0.3947	0.3857	0.3948	0.3657	0.3934	0.3876	0.3945	0.3685	0.3943
$\delta = 0.5$	0.4043	0.4498	0.4217	0.4498	0.4063	0.4499	0.4005	0.4509	0.3831	0.4511
$\delta = 0.4$	0.4367	0.5072	0.4556	0.5062	0.4336	0.5072	0.5059	0.5062	0.4181	0.5072
$\delta = 0.3$	0.6561	0.5684	0.5196	0.5633	0.6051	0.5630	0.5152	0.5748	0.5483	0.5663
$\delta = 0.2$	0.6471	0.6187	0.6178	0.6186	0.5857	0.6194	0.6614	0.6224	0.5939	0.6198
$\delta = 0.1$	0.7259	0.6763	0.7809	0.6754	0.7720	0.6823	0.7938	0.6615	0.8116	0.6750

Table 11 Investment ratios different values of α and δ

Control parameter	Balance parameter $\alpha = 0.1$				$\alpha = 0.2$				$\alpha = 0.3$			
	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ
$\delta = 1.0$	0.2292	0.4490	0.2769	0.0449	0.1887	0.4573	0.2633	0.0907	0.2440	0.3560	0.3368	0.0632
$\delta = 0.9$	0.1381	0.5293	0.3268	0.0058	0.2841	0.4652	0.2165	0.0342	0.0929	0.7110	0.1586	0.0375
$\delta = 0.8$	0.0943	0.2808	0.6152	0.0097	0.1959	0.4023	0.3676	0.0342	0.1953	0.3917	0.3546	0.0584
$\delta = 0.7$	0.0745	0.2328	0.6862	0.0065	0.1911	0.3214	0.3274	0.1601	0.1572	0.3783	0.4054	0.0591
$\delta = \frac{2}{3}$	0.2013	0.2242	0.4827	0.0918	0.0774	0.3008	0.6141	0.0077	0.2009	0.4138	0.2243	0.1610
$\delta = 0.6$	0.1956	0.2431	0.4905	0.0708	0.1420	0.2931	0.5314	0.0335	0.1301	0.2295	0.5549	0.0855
$\delta = 0.5$	0.2638	0.1649	0.4923	0.0790	0.1089	0.1576	0.6836	0.0499	0.1167	0.2356	0.5909	0.0568
$\delta = 0.4$	0.1620	0.2012	0.4465	0.1903	0.2127	0.2290	0.3881	0.1702	0.1040	0.2690	0.5552	0.0718
$\delta = \frac{1}{3}$	0.2195	0.1650	0.2645	0.3510	0.1331	0.1702	0.6427	0.0540	0.1289	0.2301	0.4641	0.1769
$\delta = 0.3$	0.0808	0.1579	0.0990	0.6623	0.1953	0.1798	0.2694	0.3555	0.1558	0.3159	0.2650	0.2633
$\delta = 0.2$	0.6837	0.0593	0.0590	0.1977	0.5577	0.0802	0.0796	0.2825	0.6988	0.0966	0.1056	0.0990
$\delta = 0.1$	0.8400	0.0420	0.0570	0.0610	0.9378	0.0168	0.0024	0.0430	0.6380	0.0083	0.2429	0.1108
Control parameter	Balance parameter $\alpha = 0.4$				$\alpha = 0.5$				$\alpha = 0.6$			
	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ
$\delta = 1.0$	0.2678	0.3726	0.3314	0.0282	0.1618	0.5329	0.2526	0.0527	0.1640	0.4304	0.3658	0.0398
$\delta = 0.9$	0.1323	0.3823	0.4734	0.0120	0.1954	0.5430	0.2555	0.0061	0.1381	0.4823	0.3062	0.0734
$\delta = 0.8$	0.2116	0.4089	0.3385	0.0410	0.1015	0.4094	0.4794	0.0097	0.3152	0.3286	0.3034	0.0528
$\delta = 0.7$	0.1904	0.2951	0.4450	0.0695	0.2491	0.2900	0.4341	0.0268	0.1859	0.3899	0.3663	0.0579
$\delta = \frac{2}{3}$	0.1083	0.3485	0.5010	0.0422	0.2217	0.3421	0.3484	0.0878	0.1486	0.3853	0.4080	0.0581
$\delta = 0.6$	0.1723	0.2946	0.4685	0.0646	0.1961	0.3392	0.3945	0.0702	0.1447	0.3542	0.4502	0.0509
$\delta = 0.5$	0.1112	0.2693	0.5457	0.0738	0.2446	0.2573	0.2763	0.2218	0.0903	0.4804	0.4143	0.0150
$\delta = 0.4$	0.1783	0.1390	0.4657	0.2170	0.2676	0.1765	0.4666	0.0893	0.0746	0.3142	0.5664	0.0448
$\delta = \frac{1}{3}$	0.1588	0.1523	0.1547	0.5342	0.0989	0.3098	0.3378	0.2535	0.2832	0.2574	0.2490	0.2104
$\delta = 0.3$	0.1931	0.2508	0.1987	0.3574	0.1326	0.1350	0.3120	0.4204	0.1051	0.0689	0.0979	0.7281
$\delta = 0.2$	0.5961	0.0974	0.0713	0.2352	0.8138	0.0446	0.0548	0.0868	0.6695	0.0712	0.0725	0.1868
$\delta = 0.1$	0.3673	0.1009	0.2659	0.2659	0.5221	0.0990	0.2829	0.0960	0.2127	0.1489	0.5665	0.0719
Control parameter	Balance parameter $\alpha = 0.7$				$\alpha = 0.8$				$\alpha = 0.9$			
	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ
$\delta = 1.0$	0.1760	0.4670	0.2905	0.0665	0.1533	0.5550	0.2746	0.0171	0.2899	0.4875	0.1442	0.0784
$\delta = 0.9$	0.2496	0.3033	0.3792	0.0679	0.2598	0.6087	0.1171	0.0144	0.2127	0.5209	0.1824	0.0840
$\delta = 0.8$	0.3101	0.3934	0.2645	0.0320	0.2373	0.2831	0.2605	0.2191	0.3157	0.4987	0.1135	0.0721
$\delta = 0.7$	0.1039	0.4612	0.4146	0.0203	0.4210	0.2515	0.2914	0.0361	0.2798	0.3405	0.2844	0.0953
$\delta = \frac{2}{3}$	0.1637	0.4724	0.3504	0.0135	0.2727	0.3742	0.2951	0.0580	0.2576	0.3792	0.2780	0.0852
$\delta = 0.6$	0.1748	0.3416	0.4270	0.0566	0.1712	0.4465	0.3694	0.0129	0.2503	0.3363	0.3124	0.1010
$\delta = 0.5$	0.2148	0.3358	0.3916	0.0578	0.1860	0.3530	0.4241	0.0369	0.1129	0.4247	0.3922	0.0702
$\delta = 0.4$	0.0797	0.3716	0.4169	0.1318	0.0819	0.2849	0.5476	0.0856	0.0101	0.3296	0.6236	0.0367
$\delta = \frac{1}{3}$	0.1361	0.2571	0.4558	0.1510	0.1319	0.3709	0.4224	0.0748	0.1558	0.2595	0.3679	0.2168
$\delta = 0.3$	0.1344	0.2034	0.4605	0.2017	0.1206	0.1425	0.1474	0.5895	0.1108	0.2182	0.5055	0.1655
$\delta = 0.2$	0.5608	0.0778	0.0772	0.2842	0.5991	0.0956	0.0476	0.2577	0.5021	0.1231	0.2838	0.0910
$\delta = 0.1$	0.2631	0.2028	0.3459	0.1882	0.3816	0.1621	0.3830	0.0733	0.2820	0.2390	0.2395	0.2395

Appendix A: Data for Model 5

Table 10 shows the risk and return values obtained as α and δ change. Table 11 shows the investment ratios in Model 5.

Appendix B: Data for Model 6

Table 12 shows the risk and return values obtained as α and ρ change. Table 13 shows the investment ratios in Model 6.

Table 12 Return and risk values under different values of α and ρ

Control parameter	Balance parameter $\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$		$\alpha = 0.5$	
	Risks	Returns	Risks	Returns	Risks	Returns	Risks	Returns	Risks	Returns
$\rho = 1.0$	0.7187	0.7053	0.7501	0.6857	0.7265	0.7183	0.7230	0.6998	0.7500	0.7047
$\rho = 0.9$	0.6650	0.5351	0.7057	0.6997	0.7323	0.6943	0.7351	0.6912	0.7732	0.6673
$\rho = 0.8$	0.6342	0.6782	0.6762	0.6959	0.7221	0.6802	0.7165	0.6862	0.7333	0.7128
$\rho = 0.7$	0.5910	0.6202	0.6367	0.6954	0.6655	0.7031	0.6674	0.6871	0.6777	0.6886
$\rho = 0.6$	0.5485	0.5669	0.6026	0.6929	0.6211	0.6720	0.6206	0.6793	0.6205	0.6811
$\rho = 0.5$	0.5060	0.6230	0.5655	0.6268	0.5689	0.6977	0.5720	0.5974	0.5643	0.6310
$\rho = 0.4$	0.4636	0.5449	0.5278	0.6328	0.5195	0.5761	0.5228	0.5899	0.5054	0.5630
$\rho = 0.3$	0.4216	0.4324	0.4904	0.4890	0.4685	0.5598	0.4736	0.5475	0.4462	0.5088
$\rho = 0.2$	0.3789	0.3301	0.4522	0.4481	0.4165	0.3626	0.4243	0.4624	0.3869	0.3914
$\rho = 0.1$	0.3364	0.2854	0.4144	0.4082	0.3630	0.2571	0.3749	0.3537	0.3275	0.2967
Control parameter	Balance parameter $\alpha = 0.6$		$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$		$\alpha = 0.99$	
	Risks	Returns	Risks	Returns	Risks	Returns	Risks	Returns	Risks	Returns
$\rho = 1.0$	0.7282	0.6942	0.7159	0.7014	0.7606	0.6716	0.7473	0.6857	0.7690	0.6874
$\rho = 0.9$	0.7471	0.6793	0.7349	0.7135	0.7749	0.6816	0.7349	0.6879	0.7190	0.7215
$\rho = 0.8$	0.7196	0.6709	0.7342	0.6817	0.7090	0.6815	0.6888	0.6951	0.6918	0.7108
$\rho = 0.7$	0.6635	0.6425	0.6817	0.6447	0.5197	0.4591	0.6336	0.6954	0.6383	0.6461
$\rho = 0.6$	0.6024	0.7201	0.6225	0.6477	0.6075	0.6980	0.5890	0.6791	0.5832	0.6587
$\rho = 0.5$	0.5509	0.6273	0.5629	0.6849	0.5581	0.6974	0.5344	0.6834	0.5290	0.6888
$\rho = 0.4$	0.4954	0.6775	0.5048	0.6754	0.5079	0.6378	0.4814	0.6007	0.4749	0.6284
$\rho = 0.3$	0.4387	0.4792	0.4457	0.4669	0.4535	0.5257	0.4278	0.4809	0.4224	0.5320
$\rho = 0.2$	0.3827	0.3445	0.3857	0.3932	0.3987	0.4302	0.3740	0.3931	0.3683	0.4970
$\rho = 0.1$	0.3265	0.2595	0.3253	0.2613	0.3459	0.3429	0.3201	0.3292	3143	0.3412

Table 13 Investment ratios different values of α and ρ

Control parameter	Balance parameter $\alpha = 0.1$				$\alpha = 0.2$				$\alpha = 0.3$			
	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ
$\rho = 1.0$	0.1159	0.1118	0.6622	0.1101	0.1357	0.1512	0.5233	0.1898	0.1095	0.0598	0.7143	0.1164
$\rho = 0.9$	0.2576	0.2438	0.2757	0.2229	0.1057	0.1255	0.6590	0.1098	0.1255	0.1506	0.5665	0.1574
$\rho = 0.8$	0.1337	0.1538	0.6315	0.0810	0.1099	0.1479	0.6660	0.0762	0.1524	0.1511	0.5018	0.1947
$\rho = 0.7$	0.1799	0.1937	0.5674	0.0590	0.1527	0.1435	0.6559	0.0479	0.1009	0.1194	0.6977	0.0820
$\rho = \frac{2}{3}$	0.1635	0.1872	0.5994	0.0499	0.1295	0.2621	0.5402	0.0682	0.0992	0.1483	0.6796	0.0729
$\rho = 0.6$	0.1801	0.2300	0.5289	0.0610	0.1748	0.0784	0.6993	0.0475	0.1376	0.1988	0.5817	0.0819
$\rho = 0.5$	0.0906	0.1103	0.7795	0.0196	0.1260	0.1824	0.6256	0.0660	0.0695	0.0775	0.7715	0.0875
$\rho = 0.4$	0.2616	0.2375	0.4351	0.0658	0.1119	0.1055	0.7361	0.0465	0.1105	0.2277	0.5818	0.0800
$\rho = \frac{1}{3}$	0.1332	0.2571	0.5673	0.0424	0.1952	0.2317	0.5420	0.0311	0.0438	0.2184	0.7053	0.0325
$\rho = 0.3$	0.1517	0.3228	0.5231	0.0024	0.1322	0.3342	0.4479	0.0857	0.1002	0.1531	0.7161	0.0306
$\rho = 0.2$	0.2002	0.3636	0.3908	0.0454	0.1725	0.3573	0.4273	0.0429	0.2542	0.3526	0.2699	0.1233
$\rho = 0.1$	0.0774	0.5674	0.3502	0.0050	0.1382	0.3629	0.4685	0.0304	0.2277	0.3576	0.2837	0.1310
Control parameter	Balance parameter $\alpha = 0.4$				$\alpha = 0.5$				$\alpha = 0.6$			
	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ
$\rho = 1.0$	0.1330	0.1319	0.6199	0.1152	0.2136	0.0819	0.5866	0.1179	0.1279	0.1554	0.5695	0.1454
$\rho = 0.9$	0.1663	0.1650	0.5446	0.1241	0.2510	0.2233	0.3156	0.2101	0.1745	0.1701	0.4261	0.2293
$\rho = 0.8$	0.1698	0.1287	0.5378	0.1637	0.1193	0.0884	0.6657	0.1266	0.2415	0.1721	0.4022	0.1842
$\rho = 0.7$	0.1806	0.1603	0.5828	0.0763	0.1039	0.1989	0.5875	0.1097	0.2178	0.2994	0.3273	0.1555
$\rho = \frac{2}{3}$	0.2266	0.1963	0.5216	0.0555	0.0552	0.0355	0.8708	0.0385	0.1097	0.1903	0.6374	0.0626
$\rho = 0.6$	0.2030	0.1441	0.5930	0.0599	0.1179	0.2083	0.5893	0.0845	0.0541	0.1495	0.7750	0.0214
$\rho = 0.5$	0.1615	0.2690	0.4639	0.1056	0.0845	0.2333	0.5504	0.1318	0.1700	0.1923	0.5565	0.0812
$\rho = 0.4$	0.0938	0.3192	0.5457	0.0413	0.0614	0.3392	0.4935	0.1059	0.0085	0.1610	0.8135	0.0170
$\rho = \frac{1}{3}$	0.0420	0.1064	0.8007	0.0509	0.0163	0.2385	0.7410	0.0042	0.0450	0.2560	0.6250	0.0740
$\rho = 0.3$	0.1157	0.1928	0.6436	0.0479	0.0715	0.4214	0.4942	0.0129	0.0695	0.3946	0.4462	0.0897
$\rho = 0.2$	0.0904	0.3138	0.5609	0.0349	0.1943	0.3982	0.3925	0.0150	0.2304	0.3518	0.2978	0.1200
$\rho = 0.1$	0.2071	0.4251	0.3452	0.0226	0.2039	0.5899	0.2012	0.0050	0.1980	0.4469	0.2675	0.0876
Control parameter	Balance parameter $\alpha = 0.7$				$\alpha = 0.8$				$\alpha = 0.9$			
	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ	ν_1^ϕ	ν_2^ϕ	ν_3^ϕ	ν_4^ϕ
$\rho = 1.0$	0.1302	0.1067	0.6285	0.1346	0.2074	0.2375	0.3590	0.1961	0.1382	0.2318	0.4838	0.1462
$\rho = 0.9$	0.1407	0.0740	0.6677	0.1176	0.2294	0.2488	0.3965	0.1253	0.1924	0.1484	0.5102	0.1490
$\rho = 0.8$	0.1817	0.1664	0.4647	0.1872	0.1944	0.1959	0.4916	0.1181	0.1387	0.1283	0.6095	0.1235
$\rho = 0.7$	0.2447	0.2271	0.3406	0.1876	0.2452	0.2456	0.2544	0.2548	0.1046	0.1906	0.6385	0.0663
$\rho = \frac{2}{3}$	0.1603	0.1918	0.5699	0.0780	0.1688	0.1407	0.5967	0.0938	0.1098	0.1916	0.6280	0.0706
$\rho = 0.6$	0.1640	0.2299	0.4413	0.1648	0.0760	0.1780	0.6785	0.0675	0.1528	0.2235	0.5418	0.0819
$\rho = 0.5$	0.0787	0.2483	0.6184	0.0546	0.0953	0.1261	0.7169	0.0617	0.0450	0.3072	0.6003	0.0475
$\rho = 0.4$	0.0316	0.1687	0.7694	0.0303	0.0319	0.3855	0.5302	0.0524	0.0493	0.3075	0.5381	0.1051
$\rho = \frac{1}{3}$	0.0679	0.3065	0.5274	0.0982	0.1836	0.2809	0.4097	0.1258	0.1013	0.4070	0.4401	0.0516
$\rho = 0.3$	0.1950	0.3510	0.3385	0.1155	0.0972	0.4359	0.3972	0.0697	0.4857	0.2598	0.1649	0.0896
$\rho = 0.2$	0.2102	0.3534	0.3836	0.0528	0.2088	0.4826	0.2671	0.0415	0.3458	0.3532	0.2430	0.0580
$\rho = 0.1$	0.2229	0.4285	0.2592	0.0894	0.2890	0.4572	0.2148	0.0390	0.2910	0.5028	0.1979	0.0083

Author contributions Ezgi Türkarslan: Conceptualization, Methodology, Formal analysis, Software, Validation,Visualization, Investigation, Writing - original draft.Christian Oliver Ewald: Conceptualization, Methodology, Writing - review & editing, Supervision.

Funding Open access funding provided by Umea University. This research was funded by the Kempe Foundations, Sweden, (Grant No. JCSMK22-0082).

Data Availability No data were used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Informed Consent This study does not involve human participants or personal data.

References

1. Markowitz, H.M.: Portfolio selection. J. Finance **7**(1), 77–91 (1952). https://doi.org/10.2307/2975974
2. Xu, J., Li, Y., Liu, K., Tao, C.: Portfolio selection: from underdiversification to concentration. Empir. Econ. **64**, 1539–1557 (2023). https://doi.org/10.1007/s00181-022-02300-x
3. Dominguez, A.R.: Portfolio optimization based on neural networks sensitivities from assets dynamics respect common drivers. Mach. Learn. Appl. (2023). https://doi.org/10.1016/j.mlwa.2022.100447
4. Yin, A., Zhong, X., Wang, R., Huang, J.Z.: An asset subset-constrained minimax optimization framework for online portfolio selection. Expert Syst. Appl. (2024). https://doi.org/10.1016/j.eswa.2024.124299
5. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–353 (1965). https://doi.org/10.1016/S0019-9958(65)90241-X
6. Yang, X., Chen, J., Liu, W., Zhao, X.: A multi-period fuzzy portfolio optimization model with investors' loss aversion. Soft Comput. **27**, 18829–18842 (2023). https://doi.org/10.1007/s00500-023-09030-x
7. Kaczmarek, K., Dymova, L., Sevastjanov, P.: A simple view on the interval and fuzzy portfolio selection problems. Entropy (2020). https://doi.org/10.3390/e22090932
8. Zhou, W., Luoa, D., Xu, Z.: Review of fuzzy investment research considering modelling environment and element fusion. Int. J. Syst. Sci. **570**, 1958–1982 (2022). https://doi.org/10.1080/00207721.2022.2031340
9. Joshi, D.K., Awasthi, N., Chaube, S.: Probabilistic hesitant fuzzy set based MCDM method with applications in portfolio selection process. Mater. Today Proc. **57**(Part 5), 2270–2275 (2022). https://doi.org/10.1016/j.matpr.2021.12.528
10. Wu, Q., Liu, X., Qin, J., Zhou, L.: Multi-criteria group decision-making for portfolio allocation with consensus reaching process under interval type-2 fuzzy environment. Inf. Sci. **570**, 668–688 (2021). https://doi.org/10.1016/j.ins.2021.04.096
11. Zhou, W., Xu, Z.: Hesitant fuzzy linguistic portfolio model with variable risk appetite and its application in the investment ratio calculation. Appl. Soft Comput. (2019). https://doi.org/10.1016/j.asoc.2019.105719
12. Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**(6), 529–539 (2010). https://doi.org/10.1002/int.20418
13. Xia, M.M., Xu, Z.S.: Hesitant fuzzy information aggregation in decision making. Int. J. Approx. Reason. **52**(3), 395–407 (2011). https://doi.org/10.1016/j.ijar.2010.09.002
14. Deng, X., Li, X., Liu, Y.: Hesitant fuzzy portfolio selection model with score and novel hesitant semi-variance. Comput. Ind. Eng. (2022). https://doi.org/10.1016/j.cie.2021.107879
15. Zhou, W., Xu, Z.S.: Portfolio selection and risk investment under the hesitant fuzzy environment. Knowl. Based Syst. **144**, 21–31 (2018). https://doi.org/10.1016/j.knosys.2017.12.020
16. Deng, X., Zhao, J., Fu, Q.: Novel portfolio selection models with parametric entropy and sensitivity analysis under hesitant fuzzy environment (2022). https://doi.org/10.21203/rs.3.rs-1361202/v1
17. Wei, G., Zhao, X., Wang, H., Lin, R.: Hesitant fuzzy Choquet integral aggregation operators and their applications to multiple attribute decision making. Information **15**(11), 441–448 (2012)
18. Alcantud, J.C.R., Campión, M.J., Induráin, E., Munárriz, A.: Scores of hesitant fuzzy elements revisited: 'was sind und was sollen'. Inf. Sci. (2023). https://doi.org/10.1016/j.ins.2023.119500
19. Magoc, T., Wang, X., Modave, F., Ceberio, M.: Applications of fuzzy measures and intervals in finance. Reliab. Comput. **15**(4), 300–311 (2011)
20. Magoc, T., Wang, X., Modave, F.: Application of fuzzy measures and interval computation to financial portfolio selection. Int. J. Intell. Syst. **25**(7), 621–635 (2010). https://doi.org/10.1002/int.20415
21. Kasperski, A., Zareba, A.M.: Application of fuzzy measures to financial portfolio selection problem. In: Information Systems Architecture and Technology, p. 55. Oficyna Wydawnicza Politechniki Wrocławskiej, Wroclaw (2013)
22. Magoč, T., Modave, F.: The optimality of non-additive approaches for portfolio selection. Expert Syst. Appl. **38**(10), 12967–12973 (2011). https://doi.org/10.1016/j.eswa.2011.04.093
23. Beliakov, G.: Knapsack problems with dependencies through non-additive measures and Choquet integral. Eur. J. Oper. Res. **301**(1), 277–286 (2022). https://doi.org/10.1016/j.ejor.2021.11.004
24. Brunelli, M., Corrente, S.: Modeling criteria and project interactions in portfolio decision analysis with the Choquet integral. Omega (2024). https://doi.org/10.1016/j.omega.2024.103076
25. Shapley, L.S.: A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games: Annals of Mathematics Studies, vol. II. Princeton University Press, Princeton (1953). https://doi.org/10.1515/9781400881970-018
26. CoreWeave: CoreWeave Announces Pricing of Initial Public Offering. CoreWeave (2025). https://www.coreweave.com/news/coreweave-announces-pricing-of-initial-public-offering
27. Zhou, W., Xu, Z.S.: Optimal discrete fitting aggregation approach with hesitant fuzzy information. Knowl. Based Syst. **178**, 23–33 (2015). https://doi.org/10.1016/j.knosys.2015.01.011
28. Grabisch, M.: *k*-Order additive discrete fuzzy measures and their representation. Fuzzy Sets Syst. **92**(2), 167–189 (1997). https://doi.org/10.1016/S0165-0114(97)00168-1
29. Choquet, G.: Theory of capacities. Ann. L'Inst. Fourier **5**, 131–295 (1954)

30. Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: Aggregation functions. In: Encyclopedia of Mathematics and Its Applications, vol. 127. Cambridge University Press (2009). https://doi.org/10.1017/CBO9781139644150
31. Xin, G., Ying, L.: Multi-attribute decision-making based on comprehensive hesitant fuzzy entropy. Expert Syst. Appl. (2024). https://doi.org/10.1016/j.eswa.2023.121459
32. Parameterized hesitant fuzzy entropy and its application. Comput. Eng. Sci. **41**(12), 2202–2210 (2019)
33. Shaheen, I., Javaid, N., Ali, Z., Ahmed, I., Khan, F.A., Pamucar, D.: A trustworthy and patient privacy-conscious framework for early diabetes prediction using deep residual networks and proximity-based data. Biomed. Signal Process. Control (2026). https://doi.org/10.1016/j.bspc.2025.108361
34. Khan, H., Javaid, N., Bashir, T., Ali, Z., Khan, F.A., Pamucar, D.: A novel deep gated network model for explainable diabetes mellitus prediction at early stages based on trustworthy data. Knowl. Based Syst. (2025). https://doi.org/10.1016/j.knosys.2025.114178
35. Ullah, Z., Arif, F., Ali, Z., UlHaq, Q.M.U., Babar, M., Islam, M., Irshad, A., Alturki, N., Bashir, A.K.: SDN-assisted spatial encoded sequence enabled BLSTM-based zero-trust anomaly detection model for consumer electronics of smart cities. IEEE Trans. Consum. Electron. (2025). https://doi.org/10.1109/TCE.2025.3603331

Ezgi Türkarslan is a Postdoctoral Researcher at Umeå University, Department of Mathematics and Mathematical Statistics. She received her PhD in Mathematics from Ankara University in 2023. Her research focuses on decision modeling under uncertainty, including non-additive measure theory, entropy-based optimization, and multi-criteria evaluation, with applications in finance, sustainability, and environmental systems. Her work integrates mathematical modeling, behavioral decision processes, and data-driven approaches.

Christian Oliver Ewald is a Professor of Financial Mathematics at the Department of Mathematics and Mathematical Statistics, Umeå University, Sweden. He is also affiliated with HINN Business Analytic Group, Lillehammer (Norway), and Adam Smith Business School, University of Glasgow (UK). His areas of interest are financial mathematics, quantitative finance, risk management, and commodities. He is educated at the Universities of Mainz (MSc, Mathematics) and Heidelberg (PhD, Mathematics) and holds a Higher Doctoral Degree (Habilitation, Doctor of Science) from the University of Kaiserslautern. He has previously held a Professorship in Mathematics at the University of Sydney (2009-2011), and Associate Professorships at Nottingham University Business School (China), University College Cork (Ireland), University of St. Andrews (UK), and University of Leeds (UK).