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Rigidity of graphs in homogeneous and locally homogeneous spaces

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Abstract

This thesis studies rigidity of graphs and hypergraphs realised in homogeneous and locally homogeneous spaces. We develop an algebraic model for describing the motions of such realisations using group-theoretic methods. The resulting structures, which we call *graph-of-groups realisations*, provide a unified model capable of capturing a wide range of rigidity problems. Within this model, we recover foundational results, including a necessary condition for rigidity. Using homological methods, we show that, generically, this condition is also sufficient for rigidity problems in homogeneous spaces G/H, where H is a one-dimensional and self-normalising subgroup of a Lie group G. The resulting combinatorial conditions can be verified using the pebble game algorithm, which we analyse in novel settings. Beyond homogeneous spaces, we also study realisations of graphs in locally homogeneous spaces by interpreting these graphs as symmetric graphs in a homogeneous space. In particular, we study the minimal rigidity of symmetric graphs in the hyperbolic plane and obtain a combinatorial characterisation of minimal rigidity of graphs realised on compact orientable surfaces of genus $g \geq 2$.

Sammanfattning

Denna avhandling behandlar stelhet hos grafer och hypergrafer realiserade i homogena och lokalt homogena rum. Vi utvecklar en algebraisk modell för att beskriva rörelser hos sådana realiseringar med hjälp av gruppteoretiska metoder. De resulterande strukturerna, som vi kallar *graph-of-groups*-realiseringar, ger en enhetlig modell som kan fånga ett brett spektrum av stelhetsproblem. Inom denna modell återfår vi grundläggande resultat inom stelhetsteorin, inklusive ett nödvändigt villkor för stelhet. Med hjälp av homologiska metoder visar vi att detta villkor, generiskt, även är tillräckligt för stelhetsproblem i homogena rum G/H, där H är en endimensionell och självnormaliserande delgrupp av en Liegrupp G. De resulterande kombinatoriska villkoren kan verifieras med hjälp av en *pebble game*-algoritm, som vi analyserar i nya sammanhang. Utöver homogena rum studerar vi även realiseringar av grafer i lokalt homogena rum genom att tolka dem som symmetriska grafer i ett homogent rum. I synnerhet studerar vi minimal stelhet hos symmetriska grafer i det hyperboliska planet och erhåller ett kombinatoriskt villkor för minimal stelhet hos grafer realiserade på kompakta orienterbara ytor av genus $g \geq 2$.

Popular science summary

Rigidity theory asks: can a structure bend without breaking? It is the branch of mathematics that studies the rigidity or flexibility of articulated structures, which are structures consisting of various rigid parts connected at joints. A structure is considered rigid if its shape cannot be altered without breaking some of its components, while a flexible structure can be deformed continuously. This field has many applications throughout science and engineering, such as analysing materials and studying the stability of buildings.

The mathematical structure underlying such articulated structures is that of a graph. Graphs are mathematical models of networks. One can picture a graph as simply being a collection of points called vertices, and a collection of line segments called edges that join together the vertices. When drawing a graph in the plane, it could potentially admit a flex, meaning one can move its vertices without changing the edge lengths. The flex should be non-trivial, meaning that one does not simply rotate or translate the whole graph. Graphs that admit flexes are called flexible, while graphs that do not admit any flexes are called rigid.

This is illustrated in Figure 1 below. A graph with 4 nodes which looks like a square is flexible in the plane, since one can deform it by moving the two points on the diagonal closer together. However, if one adds an edge (i.e. a line segment) between two diagonally opposite points, it is no longer flexible, and it cannot be deformed, so it is rigid.

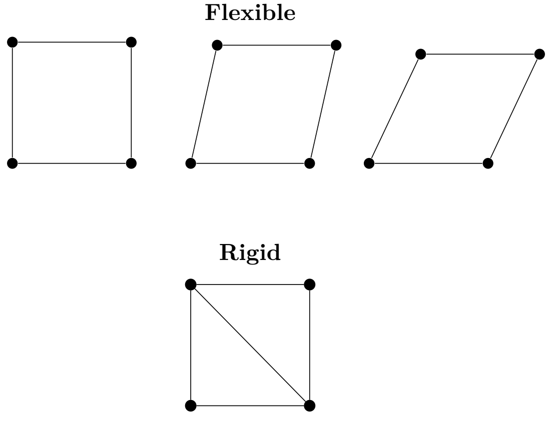

Figure 1: A flexible graph is pictured in the top row, while a rigid graph is pictured in the bottom row.

Rigidity is well understood for graphs in the plane, but it is poorly understood for three dimensions. In the plane, the Geiringer-Laman theorem states that there is a simple counting condition for the vertices and edges which tells you whether a graph is rigid. It is, however, a difficult open problem to determine rigidity of structures in three dimensional space and higher. If resolved, this could potentially have benefits in engineering and science as well. The problem of rigidity in three dimensions remains out of scope of our current methods, and new tools need to be developed to address it. To better understand rigidity, many researchers have thus considered variants of rigidity to develop better tools for understanding rigidity.

In this thesis, a general model for variants of rigidity is developed. This model is heavily based on the mathematics of symmetries: group theory. For graphs in the plane, one can describe the motions of the graphs using translations and rotations, which are precisely the symmetries of the plane. By describing the motions of graphs using only the symmetries, one obtains a model that very cleanly generalises to other contexts, as groups are general objects.

Groups have had an enormous influence on geometry. In modern geometry, this is perhaps most obvious in the study of homogeneous spaces. These are geometric spaces that, in some sense, look the same at all points. The model developed in this thesis, called the graph-of-groups model, can be considered a general model for studying realisations of graphs in homogeneous spaces.

The graph-of-groups model does not, however, cover all possible variants of rigidity. One such variant is that of rigidity of graphs on surfaces. On the sphere and on the torus (the surface of a doughnut), rigidity of graphs is understood. We consider here the case of surfaces with 2 or more holes. Surprisingly, on all of these surfaces, rigidity can have slightly different characterisations. Group-theoretic tools are also vital to the study of these graphs on surfaces. From a general perspective, the closed surfaces mentioned above can be seen as locally homogeneous spaces (aside from the sphere, which is a homogeneous space). These are spaces that are like homogeneous spaces, but which can have a more complicated global structure.

By studying rigidity through the lens of symmetry, this thesis contributes to a broader understanding of rigidity. The methods in this thesis provide new ways to study rigidity in general homogeneous spaces and we characterise the rigidity of graphs on more complicated surfaces than previously studied.

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1 Introduction

The goal at the outset of this project was to find an algebraic structure underlying the motions of articulated structures. A natural object of interest is that of a *bar-joint framework*, also called a *linkage*. These are planar or spatial structures consisting of rigid bars and joints around which the bars may rotate.

These objects have a rich mathematical history. During the industrial revolution, engineers and mathematicians were faced with the problem of turning linear motion, coming from a piston, into circular motion. This problem was resolved by the so-called Peaucellier-Lipkin linkage, see Figure 2. Remarkably, it was shown already in 1875 by Kempe [49] that for any bounded segment of an arbitrary algebraic curve, there exists a linkage that draws it.

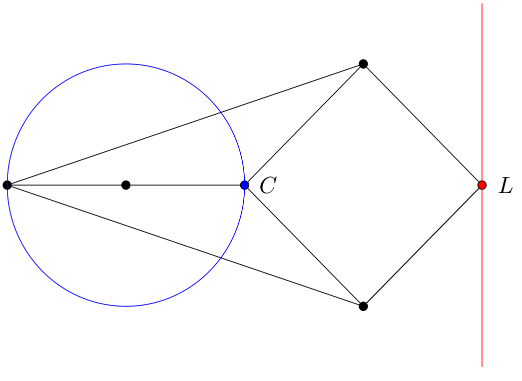

Figure 2: The Peaucellier-Lipkin linkage. Moving the point C on the circle corresponds to moving the point L on the line.

For such structures, we will develop a description of the motions from a group-theoretical perspective in Paper I. Any rigid motion of a bar in Euclidean space can be described using elements of the Euclidean group. For the group elements on the bars to define a motion on the entire structure, a simple condition on the group elements needs to hold. We show that this condition is sufficient to describe the motions. The resulting algebraic structure obtained in this way is that of a groupoid. We call the resulting structures *graph-of-groups realisations*.

With the algebraic structure defined, the next step would be to formulate questions about the articulated structures. Perhaps the first question one asks is: "What are the conditions for a non-trivial motion to exist at all?" This question is already a deep and difficult mathematical question, as it lies at the heart of the field of mathematics called *rigidity theory*. The remainder of the work in this thesis essentially focuses on this question and its variants.

We will give an introduction to rigidity theory in Section 2. Classically, rigidity theory seeks to study whether a graph admits deformations when realised in a Euclidean space. If a graph admits a deformation, then it is called flexible, while if it does not admit any deformations, it is called rigid.

In general, rigidity or flexibility depends on both the realisation and the graph. However, if one takes a generic realisation of a graph, then rigidity or flexibility depends only on the graph. This leads to the study of rigidity as a combinatorial property — i.e., a graph-theoretic property.

In the plane, rigidity is well-understood combinatorially, but in dimension 3 and higher, rigidity is still poorly understood.

We will discuss rigidity as a mathematical theory, but the applications of this field are too important to overlook. Rigidity is clearly useful to mechanical engineers, but it also has wide-ranging applications: from material science [44], to the steering of a flock of robots [2], and to recommendation systems in machine learning [74].

In dimension 2, rigidity is characterised by a simple counting condition. This counting condition can also be checked algorithmically, using the pebble game algorithm. We will give some background on the pebble game algorithm and other algorithms for checking rigidity. Some novel properties of the pebble game will be studied in Paper II.

Since rigidity is poorly understood in dimensions 3 and higher, researchers have looked towards other structures to better understand rigidity, such as rigidity of graphs on surfaces, frameworks consisting of rigid spatial bodies connected by bars, the study of drawings of graphs with fixed edge directions, and many others.

It turns out that many of these distinct problems can also be modelled using graph-of-groups realisations, as the model is general and based on group theory. We show that one can vary the groups defining the graph-of-groups realisation, and one obtains a different problem. This approach thus unifies the various analogous structures into a single model and allows one to find commonalities behind the problems that are easy to solve.

A more geometric interpretation is that graph-of-groups realisations are realisations of graphs in homogeneous spaces. Homogeneous spaces are, informally speaking, spaces that look the same everywhere. More formally, one definition of a homogeneous space that goes back to Klein's Erlangen programme is that it is a coset space G/H, where G is a group and H is a subgroup. In Paper III, we show that simple counting conditions characterise rigidity under precise conditions on G and H, from which one recovers the characterisation of rigidity in the plane.

Though the model is general, graph-of-groups realisations do not describe all conceivable rigidity theoretic problems. One broad class of exceptions is that of symmetric rigidity problems. In symmetric-forced rigidity, one considers only motions of a framework such that a certain symmetry of the framework is maintained. See Figure 3 for an example.

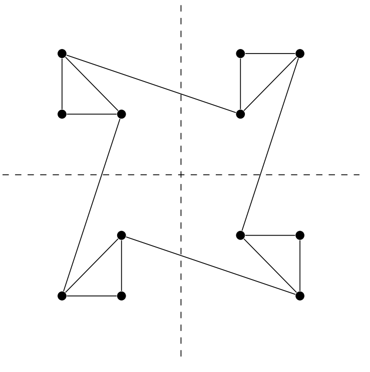

Figure 3: A framework which has a 4-fold rotational symmetry.

Symmetry-forced rigidity can be used to study frameworks on surfaces. An example that has been studied in this way is that of graphs on the torus. The torus may be seen as a quotient of $\mathbb{R}^2$ by the $\mathbb{Z}^2$ action. Using this construction of the torus, $\mathbb{Z}^2$-symmetric graphs correspond to graphs on the torus, see Figure 4.

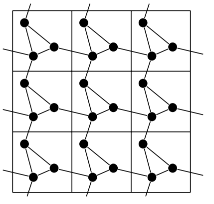

Figure 4: A $\mathbb{Z}^2$-symmetric framework. This corresponds to a graph with 3 vertices on a torus.

The torus is thus the quotient of $\mathbb{R}^2$ by a discrete group of isometries. This shows that the torus is an example of what is generally called a locally homogeneous space: a quotient of a homogeneous space G/H by a discrete group $\Lambda \subseteq G$. We will not study rigidity in locally homogeneous spaces in full generality, but we will study the remaining compact 2-dimensional case in particular detail; the rigidity of graphs on surfaces of genus greater than or equal to 2 that arise as quotients of the hyperbolic plane by a discrete group.

In the remainder of this thesis, we will give an overview of rigidity theory in Section 2, where we will treat the case of bar-joint frameworks in some detail, provide background for some variants of rigidity, and describe some of the other work providing general models for rigidity-theoretic problems. We then give an overview of the papers which are included in this thesis in Section 3, and in Section 4 we describe potential future research directions.

2 Rigidity of graphs

In this section, we will give an overview of rigidity theory, emphasising the aspects that will be relevant to this thesis. See [20, 36, 71, 90], for more information. In Sections 2.1-2.4, we will give an overview of the basic theory of bar-joint frameworks, which is fundamental to rigidity theory. We will discuss both global and local rigidity, even though in the context of this thesis, local rigidity plays a much more important role.

2.1 Framework rigidity

The main object of study in rigidity theory is the bar-joint framework.

Definition 2.1. Let $\Gamma = (V, E)$ be a graph. A d-dimensional bar-joint framework is a pair (Γ, p), where $p : V \to \mathbb{R}^d$ is a function.

By considering the set of functions $p : V \to \mathbb{R}^d$ as $(\mathbb{R}^d)^{|V|}$, fixing some order on the vertex set, we can equip the set of d-dimensional bar-joint frameworks with a topological space, equipping $\mathbb{R}^d$ with the Euclidean topology. One then defines the rigidity map

$$f_\Gamma : (\mathbb{R}^d)^{|V|} \to \mathbb{R}^{|E|} : p \mapsto \|p(v) - p(w)\|^2_{vw \in E}.$$

We denote by $E(d)$ the group of Euclidean transformations. This is a matrix group, consisting of the matrices of the form

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$$

such that $AA^t = \mathrm{Id}$, and $b \in \mathbb{R}^d$. This group acts on $\mathbb{R}^d$, by

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \cdot (x_1, \ldots x_d) = A(x_1, \ldots x_d) + b.$$

$E(d)$ then also acts on $\mathbb{R}^{d|V|}$, by $(\sigma, p) \mapsto \sigma \circ p$, considering $\sigma \in E(d)$ as a map $\sigma : \mathbb{R}^d \to \mathbb{R}^d$.

Definition 2.2. Two d-dimensional bar-joint frameworks of the same graph $(\Gamma, p), (\Gamma, q)$ are said to be equivalent if $f_\Gamma(p) = f_\Gamma(q)$. The frameworks are said to be congruent if there exists some $\sigma \in E(d)$ such that $\sigma \circ p = q$. A d-dimensional bar-joint framework (Γ, p) is said to be:

- globally rigid if every framework (Γ, q) that is equivalent to (Γ, p) is also congruent to (Γ, p);
- locally rigid if there exists a nonempty open subset $U \subseteq \mathbb{R}^{d|V|}$ with $p \in U$, such that every framework (Γ, q) with $q \in U$ that is equivalent to (Γ, p) is also congruent to (Γ, p).

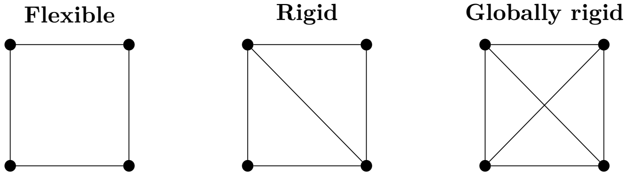

Figure 5: Three 2-dimensional bar-joint frameworks. The second graph is not globally rigid, as one can reflect one of the vertices across the diagonal to obtain an equivalent framework.

These concepts are illustrated in figure 5. The frameworks that are congruent to a given framework (Γ, p) can be shown to be given by $f_{K_n}^{-1}(f_{K_n}(p))$. In other words, the complete graph is globally rigid. The frameworks that are equivalent to a given framework (Γ, p) are given by $f_\Gamma^{-1}(f_\Gamma(p))$. The space $f_\Gamma^{-1}(f_\Gamma(p))$ is called the *configuration space*. Since $\|p(v) - p(w)\|^2$ is a polynomial, the configuration space is a real algebraic variety.

There are several equivalent formulations of local rigidity. We give the proof of one of the implications, which is standard but not readily found in the literature. In studying rigidity in other contexts, some formulations will generalise cleanly while others do not.

Definition 2.3. A (non-trivial) flex of a framework is a function $\gamma : [0,1] \to f_\Gamma^{-1}(f_\Gamma(p))$ with $\gamma(0) = p$ such that there exists some $\varepsilon > 0$ for which $\gamma(t) \in f_\Gamma^{-1}(f_\Gamma(p)) \setminus f_{K_n}^{-1}(f_{K_n}(p))$ holds for all $t \in (0, \varepsilon)$.

Lemma 2.4. *If (Γ, p) is locally rigid and $\sigma \in E(d)$, then $(\Gamma, \sigma \circ p)$ is locally rigid.*

Proof. We know there exists a $U \subseteq f_\Gamma^{-1}(f_\Gamma(p))$ such that any $q \in U$ is congruent to p. Note that $\sigma \cdot U \subseteq f_\Gamma^{-1}(f_\Gamma(\sigma \cdot p)) = f_\Gamma^{-1}(f_\Gamma(p))$ is an open subset, and it is clear that any $q \in \sigma \cdot U$ is congruent to $\sigma \circ p$, which completes the proof. □

Theorem 2.5. *The following are equivalent. Let* Γ *be a finite graph, and let* $p : V \to \mathbb{R}^d$.

1. *The bar-joint framework* (Γ, p) *is locally rigid.*
2. *There does not exist a non-trivial flex of* (Γ, p).
3. *The connected component of* $f_\Gamma^{-1}(f_\Gamma(p))$ *containing* p *is the same as the connected component of* $f_{K_n}^{-1}(f_{K_n}(p))$ *containing* p.
4. *The point* $[p]$ *is isolated in* $f_\Gamma^{-1}(f_\Gamma(p))/E(d)$.

Proof. The equivalence of 1 and 2 is proved in [3, Proposition 1]. It is obvious that 3 implies 1. We now show that 1 implies 3. Let C be the connected component of $f_\Gamma^{-1}(f_\Gamma(p))$ containing p. By Lemma 2.4, for every $x \in f_{K_n}^{-1}(f_{K_n}(p)) \cap C \subseteq f_\Gamma^{-1}(f_\Gamma(p))$, for all open neighbourhoods O_x of x in $f_\Gamma^{-1}(f_\Gamma(p))$ there exists an open subset $U_x \subseteq O_x$ such that $U_x \subseteq f_{K_n}^{-1}(f_{K_n}(p)) \cap C$. One then sees immediately that $\bigcup U_x = f_{K_n}^{-1}(f_{K_n}(p)) \cap C$ is open. On the other hand, it is clear that $f_{K_n}^{-1}(f_{K_n}(p))$ is closed, and thus $f_{K_n}^{-1}(f_{K_n}(p)) \cap C$ is open and closed. Since $f_{K_n}^{-1}(f_{K_n}(p)) \cap C \subseteq C$, and since $f_{K_n}^{-1}(f_{K_n}(p)) \cap C$ is open and closed, we must have $(f_{K_n}^{-1}(f_{K_n}(p)) \cap C = C$. The equivalence of 3 and 4 is clear, using the fact that $f_{K_n}^{-1}(f_{K_n}(p))$ is the set of congruent frameworks. □

In dimension 1, a bar-joint framework is rigid if and only if its underlying graph is connected. Deciding whether a given bar-joint framework is rigid in dimension ≥ 2 is a co-NP hard problem [1] (i.e. checking whether a framework is flexible is NP-hard). In practice, one can, however, test efficiently for infinitesimal rigidity. We shall see in the next section that infinitesimal rigidity and local rigidity are generically equivalent.

Definition 2.6. Let (Γ, p) be a d-dimensional bar-joint framework. We define the rigidity matrix to be half the Jacobian matrix of f_Γ at p, that is $R(\Gamma, p) := \frac{1}{2}(df_\Gamma)_p$.

More concretely, $R(\Gamma, p)$ is the $|E| \times d|V|$ matrix, with rows indexed by the edges and columns indexed by the coordinates of $(p(v))_{v \in V}$. The entries of the row indexed by the edge $e = uv$ are given by:

$$0 \quad \dots \quad 0 \quad \overbrace{p(u) - p(v)}^{\text{columns } u} \quad 0 \quad \dots \quad 0 \quad \overbrace{p(v) - p(u)}^{\text{columns } v} \quad 0 \quad \dots \quad 0$$

Definition 2.7. Let (Γ, p) be a d-dimensional bar-joint framework, fix some $v \in V$, and suppose that

$$\mathrm{Span}\{p(u) - p(v) \mid u \in V\}$$

has dimension k. Elements of $\ker(R(\Gamma, p))$ are called infinitesimal motions. We say that (Γ, p) is infinitesimally rigid if

$$\mathrm{Rank}(R(\Gamma, p)) = d|V(\Gamma)| - \left(\binom{d+1}{2} - \binom{d-k}{2}\right).$$

The rank can be computed numerically in cubic time, so infinitesimal rigidity is computationally much more tractable. Let us motivate the term $\left(\binom{d+1}{2} - \binom{d-k}{2}\right)$. As in the continuous case, frameworks (Γ, p) always admit certain infinitesimal motions, coming from the Lie algebra $\mathfrak{e}(d)$, which is the algebra of matrices

$$\begin{bmatrix} S & b \\ 0 & 1 \end{bmatrix} \text{ where } S \in \mathbb{R}^{d\times d}, b \in \mathbb{R}^d, S^t = -S$$

This Lie algebra $\mathfrak{e}(d)$ acts on $\mathbb{R}^d$, i.e. for all $X \in \mathfrak{e}(d)$, one obtains a vector field on $\mathbb{R}^d$ given by $x \mapsto Sx + b$. This is the infinitesimal action associated with the action of $E(d)$ on $\mathbb{R}^d$. This action always defines infinitesimal motions, and they span a space of dimension

$$\left(\binom{d+1}{2} - \binom{d-k}{2}\right).$$

One thus always has

$$\text{Rank}(R(\Gamma, p)) \leq d|V| - \left(\binom{d+1}{2} - \binom{d-k}{2}\right),$$

and (Γ, p) is infinitesimally rigid if this inequality is an equality. There is one more notion associated with the rigidity matrix, namely that of stresses. Abstractly, these can be thought of as elements of the cokernel of df_Γ.

Definition 2.8. An equilibrium stress is an element $\omega \in \mathbb{R}^E$ such that for every $v \in V$

$$\sum_{e=vw:v\in e} \omega_e(p(v) - p(w)) = 0$$

We denote the space of stresses by $S(\Gamma, p)$.

The following essentially follows from applying the rank-nullity theorem to $R(\Gamma, p)$. It was first stated by Maxwell for the 3-dimensional case in 1864.

Theorem 2.9 (Maxwell rule [60]). *Let $m = \dim(\ker(R(\Gamma, p)))$ and let $s = \dim(S(\Gamma, p))$. Then*

$$m - s = d|V| - |E|. \tag{1}$$

In particular,

$$m \geq d|V| - |E|. \tag{2}$$

The Maxwell-rule leads to a necessary condition for infinitesimal rigidity, which is most clearly stated in terms of minimal infinitesimal rigidity.

Definition 2.10. A bar-joint framework (Γ, p) is said to be minimally infinitesimally rigid if (Γ, p) is infinitesimally rigid, and for any edge $e \in E(\Gamma)$, the bar-joint framework $(\Gamma \setminus e, p)$ is not infinitesimally rigid.

Theorem 2.11. *Let (Γ, p) be a minimally infinitesimally rigid d-dimensional bar-joint framework that spans $\mathbb{R}^d$. Then*

$$|E| = d|V| - \binom{d+1}{2},$$

and for any subset V' with $|V'| \geq d + 1$

$$|E(V')| \leq d|V'| - \binom{d+1}{2}.$$

2.2 Generic rigidity

Rigidity and infinitesimal rigidity, as defined in Section 2.1, are properties of both the graph and the underlying geometric realisation. However, for generic points, the rigidity or infinitesimal rigidity depends only on the underlying graph. This was proved for graphs which are the 2-skeleta of 3-dimensional simplicial complexes by Gluck [31] and by Asimow and Roth [3]. Essentially, this follows from the constant rank theorem, which we state below. We recall that the rank of a smooth map $F : M \to N$ at a point x is defined as $\mathrm{Rank}(dF_x)$, and that a point $x \in M$ is a regular point if $\mathrm{Rank}(dF_x)$ is maximal.

Theorem 2.12. *[54, Theorem 4.12] Let M and N be manifolds of dimensions m and n respectively. Let $F : M \to N$ be a map of constant rank r. Then for all $x \in M$, there is some open chart (U, φ) with $x \in U$ and $\varphi(x) = 0$ and (W, ψ) with $F(x) \in W$ and $\psi(x) = 0$ such that*

$$\psi \circ F \circ \varphi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0) \text{ if } r < n$$

and

$$\psi \circ F \circ \varphi^{-1}(x_1, \dots, x_n, \dots, x_m) = (x_1, \dots, x_n) \text{ if } n = r$$

The rigidity map f_Γ has a maximal rank on a dense open subset U of $\mathbb{R}^{d|V|}$, since the sets of points p where $\mathrm{Rank}(d(f_\Gamma)_p) < k$ for different values of k are always closed algebraic subsets of $\mathbb{R}^{d|V|}$, as they are defined by certain minors vanishing. Hence, there is a dense and open set of regular points.

Using the constant rank theorem on U, one can see that $f_\Gamma^{-1}(f_\Gamma(p))$ is a manifold in the neighbourhood of a regular point p, and one has $T_p(f_\Gamma^{-1}(f_\Gamma(p))) = \ker(d(f_\Gamma)_p)$. These facts can be used to show the following.

Theorem 2.13. *[3] Let $p \in \mathbb{R}^{d|V|}$ be a regular point. Then (Γ, p) is infinitesimally rigid if and only if (Γ, p) is locally rigid. In particular, there is a dense open subset of points where either (Γ, p) is rigid or (Γ, p) is flexible.*

As stated, the points where $\mathrm{rank}(d(f_\Gamma)_p)$ does not reach the maximum are points where certain minors of the rigidity matrix vanish. These minors then define non-trivial polynomials belonging to $\mathbb{Q}[x_1, \dots, x_{d|V|}]$, and thus if $p \in \mathbb{R}^{d|V|}$ is a generic point, meaning its coordinates are algebraically independent over the rationals, then for this point, rigidity and infinitesimal rigidity are equivalent. This stronger notion is sometimes used instead of regular points, especially in the context of global rigidity [17, 33]. The non-generic points of $\mathbb{R}^n$ form a measure zero set (with respect to the Lebesgue measure), since they are given by the following countable union of measure zero sets:

$$\bigcup_{P \in \mathbb{Q}[x_1, \dots, x_n]} \{x \in \mathbb{R}^n \mid P(x) = 0\}.$$

Though we will be principally interested in local rigidity, we say a few words about global rigidity. Global rigidity is also a generic property. This is, however, a much deeper fact, proved by Gortler, Healy, and Thurston [33]. They proved that a sufficient condition for rigidity in terms of stress matrices, which was found by Connelly [17], is also necessary for global rigidity. From any stress $\omega \in \mathbb{R}^E$, one can define a stress matrix Ω, which is a $V \times V$ matrix analogous to the graph Laplacian, defined by

$$\begin{aligned} \Omega_{v,v} &= \sum_{e=uv:v\in e} \omega_{uv} && \text{for } v \in V \\ \Omega_{u,v} &= -\omega_{uv} && \text{if } u \neq v, uv \in E \\ \Omega_{u,v} &= 0 && \text{otherwise.} \end{aligned}$$

It is not too hard to show that if ω is a stress coming from an affinely spanning d-dimensional bar-joint framework, then $\mathrm{Rank}(\Omega) \leq |V| - (d+1)$. It turns out that the converse characterises global rigidity:

Theorem 2.14. *[33] Let (Γ, p) be a generic d-dimensional bar-joint framework, and suppose $|V(\Gamma)| \geq d+2$. Then (Γ, p) is globally rigid if and only if there exists an equilibrium stress such that the associated stress matrix Ω has rank $|V| - (d+1)$.*

For generic points, either all admit such a stress matrix, or none of them do. Hence, global rigidity is a property that is independent of the bar-joint framework for almost all bar-joint frameworks.

Recently, a similar algebraic condition has been developed by Garamvölgyi [30] for so-called globally linked pairs. A pair of vertices $\{u, v\}$ in a graph Γ is said to be globally d-linked if, for all generic frameworks $p \in \mathbb{R}^{d|V|}$ and any framework $q \in f_\Gamma^{-1}(f_\Gamma(p))$, one has

$$d(p(u), p(v)) = d(q(u), q(v)).$$

We will need some terminology to state the theorem. First of all, a pair of vertices $\{u, v\}$ is d-linked if generically $\mathrm{Rank}(R(\Gamma + uv, p)) = \mathrm{Rank}(R(\Gamma, p))$. This means that locally around p, the length $d(p(u), p(v))$ is determined, even if there is no edge between u and v. Let p be a generic framework. One says that two vertices $\{u, v\}$ are d-stress linked if $\{u, v\}$ are d-linked, and if the following holds

$$S(\Gamma, q) \subseteq S(\Gamma, p) \implies S(\Gamma + uv, q) \subseteq S(\Gamma + uv, p).$$

In words, this means that if all stresses of q are stresses of p, then all stresses of q for the graph with uv added are stresses of p for that graph.

Theorem 2.15. *[30] If (Γ, p) is a d-dimensional bar-joint framework and if $\{u, v\}$ are d-stress linked, then $\{u, v\}$ are globally linked.*

Garamvölgyi furthermore conjectured that the sufficient condition (d-stress linkedness) is equivalent to global linkedness.

2.2.1 Special position frameworks

The results so far in this section tell us that one can study rigidity as a graph-theoretic property by focusing solely on the generic frameworks. It does, however, raise the question of whether given frameworks actually admit nontrivial flexes and how to find and study these frameworks. We will now discuss how to determine exactly which bar-joint frameworks do not behave generically, as well as discuss an example that demonstrated that a simple general position assumption is not sufficient to guarantee generic behaviour.

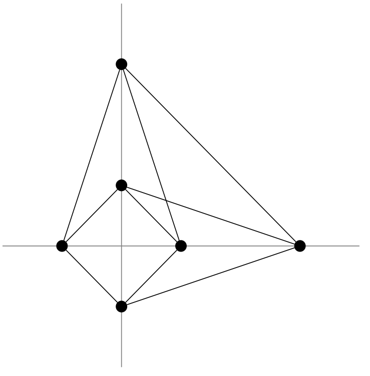

Figure 6: A flexible bar-joint framework of $K_{3,3}$. One can deform the framework by moving the vertices on one line away from the intersection, and the vertices on the other line toward the intersection.

One approach to this problem is to consider when infinitesimal rigidity fails to provide the correct answer, as done by White and Whiteley [86]. Given a minimally rigid graph and a bar-joint framework, one may 'tie down' the framework by adding bars to connect certain vertices to some fixed points in space. Algebraically, this results in adding rows to the rigidity matrix $R(\Gamma, p)$ such that one obtains a square matrix $\tilde{R}$. Then $p = (x_1, \dots, x_{d|V|})$ is infinitesimally flexible if and only if $\det(\tilde{R})(p) = 0$. It turns out that there is a factor C of $\det(\tilde{R})$, called the pure condition, which is independent of the given tie down, such that p is infinitesimally flexible if and only if $C(p) = 0$.

Example 2.16. In 1899, Dixon [23] found two ways that the complete bipartite graph $K_{3,3}$ is flexible in the plane, though $K_{3,3}$ is generically rigid in the plane. This occurs when the vertices of the framework lie on orthogonal lines as in Figure 6, or on a specific conic.

An analysis of the stresses of bipartite graphs [8], shows that $K_{3,3}$ is infinitesimally flexible if and only if the vertices lie on a quadric. This generalises to the vertices of $K_{d+1,d(d+1)/2}$ lying on a quadric. The pure condition gives the same result for $K_{3,3}$.

If one places no restrictions on realisations p of the graph $\Gamma = (V, E)$, one way that a graph can be realised in a flexible way is to proceed as follows. One may take a vertex cut $V' \subseteq V$, and take $p(v) = 0$ for all $v \in V'$ (or, more generally, by placing all vertices in V' on a $(d-2)$-dimensional subspace). By placing the other vertices in a different position, one always obtains a flexible framework. From this argument, it follows in particular that any non-complete graph has a flexible realisation.

Less trivially, one may consider bar-joint frameworks that are edge-injective, meaning that for every edge $e = vw$, one has $p(v) \neq p(w)$. Graseger, Legerský, and Schicho [34] have shown that the existence of a flexible edge-injective bar-joint framework in the plane has an elegant combinatorial description in terms of colourings. Let $\Gamma = (V, E)$ be a graph, and let c be a 2-colouring of the edge set (i.e. a function $c : E \to \{r, b\}$). We say that c is a NAC-colouring if, for every cycle $e_1, \dots, e_m$, either the cycle is monochromatic or there are at least 2 edges of each colour. NAC stands for 'no almost cycle', where a cycle with exactly one edge of a given colour is an almost cycle.

Theorem 2.17. *[34] Let $\Gamma = (V, E)$ be a graph. There exists a flexible edge-injective bar-joint framework in the plane if and only if Γ has a NAC-colouring.*

In dimension ≥ 3, a similar argument with the vertex cuts shows that even for edge-injective graphs, there is always a flexible realisation, or the graph is complete.

2.3 Combinatorial rigidity

2.3.1 Rigidity on the line and in the plane:

Definition 2.18. We say that a graph is locally d-rigid (resp. globally d-rigid) if any generic d-dimensional bar-joint framework (Γ, p) is rigid (resp. globally rigid).

The main problem in rigidity is to find a description of rigidity in purely combinatorial terms. By k-connected, we refer to vertex connectivity. This means that a graph is k-connected if there is no vertex cut of size $< k$. In dimension 1, there is the following folklore result:

Theorem 2.19. *A graph Γ is 1-rigid if and only if Γ is connected. A graph Γ is globally 1-rigid if and only if Γ is 2-connected.*

Being 1-connected can be rephrased in another way, which, in fact, generalises to 2-dimensional rigidity.

Definition 2.20. Let $\Gamma = (V, E)$ be a (multi)-graph, and let $d, \ell \in \mathbb{Z}_{\geq 0}$ with $2d > \ell$. We say that Γ is (d, ℓ)-sparse if for every subset $V' \subseteq V$ with $|V'| \geq 2$, one has

$$|E(V')| \leq d|V'| - \ell.$$

We say that Γ is (d, ℓ)-tight, if Γ is (d, ℓ) sparse and

$$|E| = d|V| - \ell.$$

From Maxwell's rule (Theorem 2.11), it follows that having a $(1, 1)$-tight subgraph is a necessary condition for 1-rigidity. Subgraphs which are $(1, 1)$-tight are precisely trees, so in the 1-dimensional case $(1, 1)$-tightness essentially characterises rigidity. In 2-dimensions, Theorem 2.11 implies that having a $(2, 3)$-tight subgraph is a necessary condition for 2-rigidity. It was proven by Pollaczek-Geiringer, and independently rediscovered by Laman, that this condition is also sufficient for 2-rigidity:

Theorem 2.21. *[51, 64] A graph $\Gamma = (V, E)$ is 2-rigid if and only if Γ has a $(2, 3)$-tight spanning subgraph.*

A high-level view of the proof is as follows. It suffices to show that all $(2, 3)$-tight graphs are 2-rigid. First, one shows that all $(2, 3)$-tight graphs can be built starting from a single edge using simple operations called extensions (also called Henneberg operations). For the first operation, the so-called 0-extension, a vertex v_* is added, and this vertex is connected to two vertices v_1 and v_2. For the second operation, the so-called 1-extension, one removes an edge u_1u_2, adds a vertex v_*, and connects this vertex to u_1, u_2, and an additional vertex v_1. See Figure 7 for an illustration. Then, one shows that these operations preserve rigidity for generic frameworks. It then follows that all graphs that are $(2, 3)-$tight are rigid. See Figure 8 for a construction of $K_{3,3}$ from a single edge using extension moves.

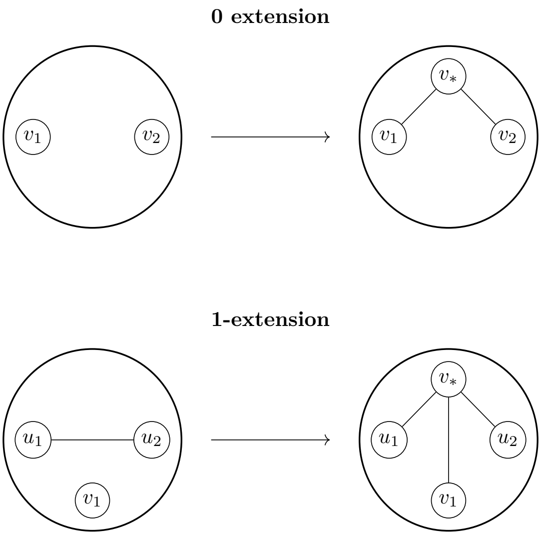

Figure 7: 2-dimensional extensions

There are, however, various other proofs of the Geiringer-Laman theorem. One proof, due to Lovász and Yemini is based on the so-called Dilworth truncation [56]. A proof by Bernstein and Krone uses tropical geometry [7]. A proof by Tay [82] is based on a direct analysis of a generalised rigidity matrix for a very special framework, which then generically has the same rank as the ordinary rigidity matrix.

Researchers have also provided various equivalent formulations of sparsity. Many of these are based on a theorem independently proved by Nash-Williams [61] and Tutte [84], which has the following corollary, due to Nash-Williams:

Theorem 2.22. *[62] Let $\Gamma = (V, E)$ be a graph. Then Γ is (k, k)-tight if and only if there exist k edge-disjoint spanning trees $T_i = (V, E_i)$ such that $E(\Gamma) = \bigcup_{i=1}^{k} E_i$.*

Theorem 2.23. *The following are equivalent.*

1. *The graph Γ is $(2, 3)$-tight*
2. *The graph Γ can be constructed from a single edge K_2, using 2-dimensional 0 and 1-extensions.*
3. *[56] For any pair uv, the multigraph $\Gamma + uv$ has an edge-disjoint decomposition into 2 spanning trees.*
4. *[18] Γ has a proper $3T2$ decomposition, i.e., the edge set of E has a decomposition into 3 trees T_1, T_2, T_3, such that any vertex is contained in 2 of the trees and for any subset $V' \subseteq V$, $(V', T_i(V'))$ is connected for at most 1 of the trees.*

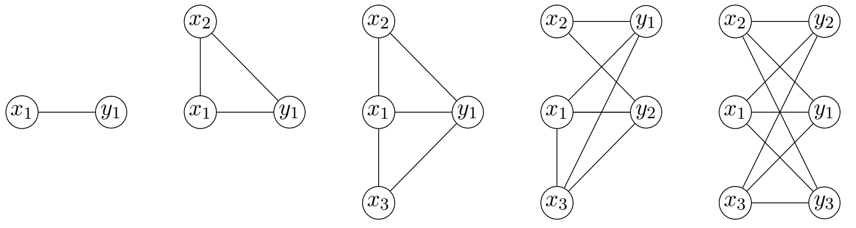

Figure 8: A construction of $K_{3,3}$ by extension moves. The first two steps are 0 extensions. In the third step, the edge x_1x_2 is used for a 1-extension, and y_2 is added. In the fourth step, the edge x_1x_3 is used for a 1-extension, and y_3 is added.

The equivalence of 1 and 2 is the Geiringer Laman theorem. Some of these properties are generalised to arbitrary $(d, \ell)-$sparse graphs by Haas [37]. A series of papers by Szegő, Fekete and Frank extend the characterisation using inductive constructions [25, 29, 79] to other values of d and ℓ.

In the plane, global rigidity is characterised as well. Hendrickson [39] proved two necessary conditions for global rigidity, namely $(d + 1)$-connectivity and redundant rigidity. A graph Γ is said to be redundantly rigid if it is d-rigid, and for any edge $e \in E$, $\Gamma \setminus e$ is also d-rigid. In dimensions ≥ 3, these conditions are not sufficient [16]. On the line, this is equivalent to Γ having a 2-connected subgraph. In the plane, this condition was shown to be sufficient by the stress-condition of Connelly [17], and by the work of Jackson and Jordán [40], who showed that any redundantly rigid 3-connected graph can be constructed from K_4 using 1-extensions and edge additions.

2.3.2 Rigidity and connectivity

In dimensions 3 and higher, the condition from Theorem 2.11 is not sufficient for rigidity, as illustrated by the graph pictured in Figure 9. The flexibility of this graph is easy to see since one has a vertex cut $\{v, w\}$ of two vertices, and one can then rotate one of the components of $\Gamma \setminus \{v, w\}$ around the line spanned by $p(v)$ and $p(w)$ for generic p. One may verify that each subset satisfies

$$|E(V')| \leq 3|V'| - 6.$$

The example points to the following folklore result:

Theorem 2.24. *If a graph Γ has a vertex cut of size $k \leq d-1$, then Γ is d-flexible. Conversely, if Γ with $|V(\Gamma)| \geq d+1$ is d-rigid, then Γ is d-connected.*

Intuitively, one also expects a highly connected graph to be rigid. Lovász and Yemini showed that a 6-connected graph is 2-rigid [56] based on the combinatorial characterisation of rigidity in the plane. They conjectured that similar results should hold in d-dimensions. In a breakthrough result, this conjecture was resolved by Villányi using the probabilistic method.

Theorem 2.25. *[85] Let Γ be a $d(d+1)$ connected graph. Then Γ is globally d-rigid.*

The result is the best possible, in the sense that there exist $d^2 + d - 1$ connected graphs that are not rigid in $\mathbb{R}^d$. These results point towards the fact that rigidity can be seen as some

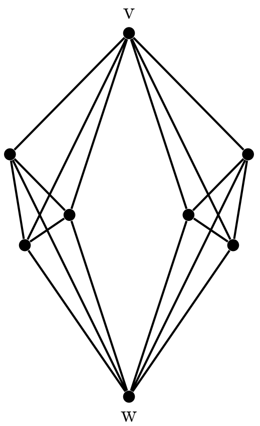

Figure 9: Double banana graph.

rough measure of connectivity. Another way in which this is true is the relation of rigidity to algebraic connectivity. We recall that the Laplacian of a graph is the $|V| \times |V|$ matrix, defined as

$$L(\Gamma)_{u,v} = -1 \text{ if there is an edge } uv \in E(\Gamma)$$
$$L(\Gamma)_{u,v} = 0 \text{ if } uv \notin E(\Gamma)$$
$$L(\Gamma)_{u,u} = \deg(u).$$

The second smallest eigenvalue λ_2 of $L(\Gamma)$ is called the algebraic connectivity of a graph [26], introduced by Fiedler. One has $\lambda_2 > 0$ if and only if Γ is connected, and λ_2 gives a lower bound on the connectivity of Γ.

By analogy, one may define, for any bar-joint framework (Γ, p), the stiffness matrix as the $d|V| \times d|V|$ matrix, defined as $L(\Gamma, p) = R(\Gamma, p)^t R(\Gamma, p)$. The 1-dimensional case is simply the Laplacian for all edge-injective frameworks (up to a scalar). One has that (Γ, p) is rigid if and only if the $\binom{d+1}{2} + 1$-th eigenvalue λ_N is nonzero, and one defines the d-connectivity of Γ as $\sup_{p \in \mathbb{R}^{d|V|}} \lambda_N$, introduced by Jordán and Tanigawa [46]. Jordán and Tanigawa then use this to study the rigidity of random graphs.

2.3.3 Abstract rigidity matroids and conjectures in 3-space

As we have seen, rigidity is not characterised in d-dimensions. There are, however, certain conjectures with regards to 3-rigidity. We shall state one, for which we need to introduce the concept of an abstract rigidity matroid. This also gives us the opportunity to recall some concepts from matroid theory, which will be used in papers II, IV, and V.

Definition 2.26. A matroid is a pair $\mathcal{M} = (E, \mathcal{I})$ consisting of a finite set E, and a collection of subsets $\mathcal{I} \subseteq P(E)$ such that

1. $\emptyset \in \mathcal{I}$

2. If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$
3. If $X, Y \in \mathcal{I}$ with $|Y| < |X|$ then there exists some $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

The sets $\mathcal{I}$ are called the independent sets of the matroid.

Example 2.27. Let $E \subseteq \mathbb{R}^d$ be a finite collection of vectors. We say a subset $F \subseteq E$ is independent if the vectors $v \in F$ are linearly independent. Matroids arising in this way are called linear matroids.

Example 2.28. Let K be a field and let L be a field extension. Suppose that $E = \{\alpha_1, \ldots, \alpha_n\} \subseteq L$. We say $F \subseteq E$ is independent if the elements of F are algebraically independent over K. Such a matroid is called an algebraic matroid.
A broad class of examples arises when L is a function field. Let $\mathfrak{p} \subseteq K[x_1, \ldots, x_n]$ be a prime ideal, and let $L = \mathrm{Frac}\,(K[x_1, \ldots, x_n]/\mathfrak{p})$. Letting $E = \{x_1, \ldots, x_n\}$ we have that a subset $F \subseteq E$ is independent if and only if $\langle F \rangle \cap \mathfrak{p} = \{0\}$.

Example 2.29. Let E be the edge set of a finite graph Γ. We say a subset $F \subseteq E$ is independent if F does not contain a cycle. The resulting matroid is called the graphic matroid.

Example 2.30. Let E be the edge set of a finite graph Γ, and let $d, l \in \mathbb{Z}$ with $2d > l$, $d > 0, l \geq 0$. We say a subset $F \subseteq E$ is independent if

$$|F(V')| \leq d|V'| - l$$

for every subset $V' \subseteq V$. The fact that this defines a matroid was first proved by Lorea [55].

There are various notions associated with matroids, all of which can be used to provide a definition of a matroid. This gives so-called cryptomorphic definitions, meaning the definitions are equivalent but in non-obvious ways. We now introduce some terminology for matroids, most of which is intuitive. For what follows in this section, most important will be the closure operator and rank function.

Definition 2.31. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid.

- We say a subset $F \subseteq E$ is dependent if $F \notin \mathcal{I}$.
- We say a subset $F \subseteq E$ is a circuit if F is a minimally dependent set, meaning F is dependent and for any proper subset $F' \subset F$, F' is independent.
- We define, for any subset $F \subseteq E$, the rank of F to be the maximal size independent set contained in F :
$$\mathrm{Rank}_{\mathcal{M}}(F) = \max_{\substack{I \subseteq F \\ I \in \mathcal{I}}} |I|.$$
- We define, for any subset $F \subseteq E$, the closure of F
$$Cl_{\mathcal{M}}(F) = \{x \in E \mid \mathrm{Rank}_{\mathcal{M}}(F \cup \{x\}) = \mathrm{Rank}_{\mathcal{M}}(F)\}.$$

Example 2.32. Let $\mathcal{M} = (E, \mathcal{I})$ be a linear matroid, as in Example 2.27. The rank of any subset $F \subseteq E$, is the dimension of $\langle v \mid v \in F \rangle$. One also has

$$Cl_{\mathcal{M}}(F) = \{w \in E \mid w \in \langle v \mid v \in F \rangle\}.$$

Rigidity of graphs can be naturally defined as a matroid.

Definition 2.33. Let $\mathcal{R}_d(\Gamma) = (E(\Gamma), \mathcal{I})$, where $F \in \mathcal{I}$ if and only if $\Gamma' = (V, F)$ satisfies

$$\text{Rank}(R(\Gamma', p)) = |F|$$

for almost all $p \in \mathbb{R}^{d|V|}$.

In this definition, one may alternatively take a single generic realisation. One sees that a graph $\Gamma = (V, E)$ is rigid in d-dimensions if and only if the set E contains a basis of $R_d(K_{|V|})$. One can alternatively define the rigidity matroid as an algebraic matroid, namely it is the algebraic matroid of the Cayley-Menger variety [9,65]. For $d = 1$, $\mathcal{R}_1(\Gamma)$ is the graphic matroid, and by the Geiringer-Laman theorem $\mathcal{R}_2$ is the matroid from Example 2.30, with $d = 2$, $l = 3$.

Rigidity matroids have some additional 'gluing' properties, which inspired Graver to define abstract rigidity matroids [35].

Definition 2.34. An abstract d-rigidity matroid is a matroid $\mathcal{M} = (E, \mathcal{I})$ with base set E being the edge set of a complete graph on ground set V, such that

1. If $E_1, E_2 \subseteq E$ are such that $Cl_{\mathcal{M}}(E_i) = E(V_i)$ for some $V_i \subseteq V$, with $|V_1 \cap V_2| \leq d - 1$, then $Cl_{\mathcal{M}}(E_1 \cup E_2) \subseteq E(V_1) \cup E(V_2)$
2. If $E_1, E_2 \subseteq E$ are such that $Cl_{\mathcal{M}}(E_i) = E(V_i)$ for some $V_i \subseteq V$, with $|V_1 \cap V_2| \geq d$, then $Cl_{\mathcal{M}}(E_1 \cup E_2) = E(V_1 \cup V_2)$

Graver proved that the two dimensional rigidity matroid plays a special role. The two dimensional rigidity matroid is, namely, the unique maximal abstract 2-rigidity matroid. Maximal here refers to the weak matroid order: one says that $\mathcal{M}_1 \leq \mathcal{M}_2$ if they have the same ground set, and if any independent set in $\mathcal{M}_1$ is independent in $\mathcal{M}_2$. Graver conjectured that $\mathcal{R}_d(K_n)$ is the unique maximal abstract d-dimensional rigidity matroid. This conjecture is false when $d \geq 4$ [90, Section 11.5]. Whiteley conjectured [90, Conjecture 11.5.1] that another matroid, called a cofactor matroid, is a maximal rigidity matroid. Clinch, Jackson and Tanigawa [13] showed that this conjecture holds true for $d = 3$. When $d = 3$, Graver's conjecture is still open. In [14] it is shown that the relevant cofactor matroid satisfies another conjecture for 3-dimensional rigidity called Dress' conjecture. Dress and others made several related conjectures [24], the strongest of which was shown to be false by Jackson and Jordán [41]. The essential idea behind all of these conjectures is to try to use the $3|V| - 6$ count to define a rank function, subtracting a term whenever one finds hinges, meaning disconnecting sets of size 2, in certain subgraphs.

2.4 Algorithms for generic rigidity

Since one may, for generic points, test rigidity by computing the rank of the rigidity matrix, one obtains a cubic time randomised algorithm to test for the rigidity of a graph by simply taking a randomly chosen realisation and computing its rank. The dimensions of the rigidity matrix are $|E| \times d|V|$, and the complexity of computing the rank of a $n \times m$ matrix is $O(\min(nm^2, mn^2))$. A similar randomised computation can be used to efficiently test for generic rigidity [33]. There are, as we shall shortly see, efficient deterministic algorithms for testing rigidity in the plane. In dimensions 3 and higher, this is not known. One may still test for generic rigidity by computing the dimension of a certain ideal, for example using Gröbner bases, but these methods are not computationally efficient. Using the combinatorial characterisation in the plane yields efficient algorithms. These algorithms are important background for Paper II.

2.4.1 Algorithms for (d, l)-sparsity

By the Geiringer-Laman theorem, it suffices to find the existence of a $(2, 3)-$tight subgraph to test for rigidity. In this section, we shall discuss two similar approaches for testing (d, l)-sparsity: the pebble game and algorithms based on orientations. Since (d, l)-sparse subgraphs form a matroid, one can use a greedy algorithm as soon as one has a way to recognise independent sets. Indeed, in matroids, the procedure given in pseudocode in Algorithm 1 always produces bases.

Algorithm 1 Greedy algorithm for producing bases of matroids

```
Input: A matroid M = (E, I).
Output: A basis of the matroid.
Initialise B = ∅
for x ∈ E do
    if B ∪ {x} ∈ I then
        B ← B ∪ {x}
    end if
end for
return B
```

If one can check whether $A \in \mathcal{I}$ in constant time, the resulting algorithm produces bases in $O(|E|)$-time. In practice, one needs to develop an algorithm to test whether $A \in \mathcal{I}$. Such an algorithm is called an independence oracle. By changing the order in which the elements of E are considered, one can produce any basis of the matroid. In this way, if one is given a cost function on $c : E \to \mathbb{R}$, one may produce minimal value bases by processing the elements $x \in E$ such that $c(x)$ is in increasing order.

The pebble game algorithm provides one such independence oracle. It was first introduced for rigidity by Hendrickson and Jacobs [43]. Lee and Streinu analysed the pebble game algorithm and showed that it can recognise (d, l)-sparse multigraphs [52]. Streinu and Theran [77], and later Lee, Streinu, and Theran [53], showed that pebble game algorithms can be used to recognise more general matroidal structures on hypergraphs.

The pebble game takes a graph $\Gamma = (V, E)$ as its input, and it outputs a maximal sparse subgraph. To do this, one maintains an oriented graph and a number of pebbles on each vertex. We shall denote oriented graphs as being a graph $\Gamma = (V, E)$, together with functions $o, t : E \to V$ such that $o(e)$ is the starting vertex and $t(e)$ is the end vertex. Thus, throughout the pebble-game algorithm one is given:

- An oriented graph $D = (V, E', o, t)$ of accepted edges, where $E' \subseteq E$.
- A number of pebbles $peb(v)$ on each vertex $v \in V$.

One is allowed to perform the following 'moves':

- Accept an edge: If $e = uv \in E \setminus E'$ is not accepted and $peb(v) + peb(u) > l$, one accepts the edge uv, removes a pebble from u (if $peb(u) > 0$), and orients the edge with $o(uv) = u, t(uv) = v$ or removes a pebble from v (if $peb(v) > 0$), and orients the edge by $o(uv) = v, t(uv) = u$.

- Move a pebble: for a directed path $e_1, \dots, e_n$, such that $peb(t(e_n)) > 0$, remove a pebble from $t(e_n)$, add a pebble to $o(e_1)$, and reverse the orientation of all of the edges in the path (i.e. $o(e_i) \leftarrow t(e_i), t(e_i) \leftarrow o(e_i)$).

Given a set of accepted edges $E' \subseteq E$, to see whether an edge e can be added to E', one repeatedly looks for pebbles which can be moved to one of the endpoints until one can accept the edge. The set of accepted edges will always yield a sparse graph, and as such this procedure is an independence oracle to test for sparsity.

Berg and Jordán developed an algorithm able to determine independence in certain matroids on graphs, including the rigidity matroid [5]. In [27, Section 13.5], it is shown to apply to more general matroids called count matroids. We sketch their approach here, which is similar in spirit to the pebble-game algorithm.

Definition 2.35. Suppose we are given a graph $\Gamma = (V, E)$, a non-negative integer $l \in \mathbb{N}$, and a function $m : V \to \mathbb{N}$ such that for every edge uv, one has $m(u) + m(v) > l$. Then, one says that a subset $E' \subseteq E$ is $\mathcal{M}_{m,l}$ independent if and only if for every subset $V' \subseteq V$, one has

$$|E'(V')| \leq \sum_{v \in V'} m(v) - l.$$

For $(2,3)-$sparsity, the algorithm is based on an interpretation of sparsity using orientations. Let us sketch the approach in general. If $\Gamma = (V, E)$ is a graph, and $o, t : E \to V$ is an orientation, then one defines $\deg_+(v) = |\{e \in E \mid o(e) = v\}|$ and $\deg_-(v) = |\{e \in E \mid t(e) = v\}|$. Given a function $a : V \to \mathbb{N}$, one says an orientation is an a-orientation if $\deg_-(v) \leq a(v)$. The following result due to Hakimi [38] and Frank and Gyárfás [28] indicates the relation to count matroids, and the proof includes an algorithm for checking the existence of such an orientation. We sketch this proof, as it provides the algorithm.

Lemma 2.36. *Let $\Gamma = (V, E)$ be a graph and let $a : V \to \mathbb{N}$ be a function. The graph Γ has an a-orientation if and only if for every $V' \subseteq V$, one has*

$$|E(V')| \leq \sum_{v \in V'} a(v).$$

Proof. The proof of 'only if' is fairly straightforward. To prove 'if' one follows the following iterative procedure. One starts with an orientation of Γ, given by $o, t : E \to V$, and one considers the vertices $B = \{v \in V \mid \deg_-(v) > a(v)\}$. For any vertex $v \in B$, if there exists a path from some $w \in V \setminus B$ with $\deg_-(w) < a(w)$ to v, then changing the orientation of this path decreases $\deg_-(v) - a(v)$, and does not create any new vertices with $\deg_-(v) > a(v)$. One repeats this process until one cannot decrease $\sum_{v \in B} \deg_-(v) - a(v)$.
Assume for a contradiction that $B \neq \emptyset$ after this process, and let Z be the set of vertices x such that there exists a path from x to a vertex of B. For all $v \in Z$ one has that $\deg_-(v) \geq a(v)$, and by the definition of Z, there do not exist edges with $t(e) \in Z, o(e) \notin Z$. One thus has

$$\sum_{v \in Z} a(v) \leq \sum_{v \in Z} \deg_-(v) = |E(Z)| \leq \sum_{v \in Z} a(v),$$

and thus $a(v) = \deg_-(v)$ for all $v \in Z$, a contradiction. Thus $B = \emptyset$. □

Suppose that we are given a count matroid on a graph $\Gamma = (V, E)$ defined by $m : V \to \mathbb{N}$ and $l \in \mathbb{N}$. Suppose that one is given an independent set I in the count matroid, and one wants to consider whether $F = I \cup \{e\}$ is independent. If the endpoints of e are u, w, since $m(u) + m(w) > l$ we can find $a_u, a_w \in \mathbb{N}$ such that $a_u + a_w = l$, and $a_u \leq m(u), a_w \leq m(w)$. One can then define a as

$$\begin{aligned} a(u) &= m(u) - a_u \\ a(w) &= m(w) - a_w \\ a(v) &= m(v) \text{ for all } v \in V \setminus \{u, w\} \end{aligned}$$

One can then pick an arbitrary orientation of (V, F) and use the procedure in the proof of Lemma 2.36 to check whether the resulting graph has an a-orientation. By Lemma 2.36, if $(V, I \cup \{e\})$ has an a-orientation, then $|F(X)| \leq \sum m(v) - l$ for any subset $X \subseteq V$ with $\{u, w\} \subseteq X$. For other subsets, this is automatically satisfied since $F \setminus \{e\}$ was assumed to be independent. These observations taken together give an algorithm for testing independence.

2.5 Variants of rigidity (i): Symmetry-forced rigidity

In this section, we shall discuss an important first variant of rigidity: symmetric rigidity. This is a broad topic, an overview of which can be found in [72]. We will only cover local rigidity in this section. We shall outline the construction of the orbit rigidity matrix, introduced by Schulze and Whiteley [69]. In Paper IV, we will extensively use terminology and techniques introduced in this section.

Definition 2.37. Let Γ be a graph with a group S acting freely on V, given by $\theta : S \to \mathrm{Aut}(\Gamma)$, and such that S is a discrete subgroup of $E(d)$, made explicit by giving an injective group homomorphism $\alpha : S \to E(d)$. We say that an S-symmetric framework is a bar-joint framework such that for all vertices $v \in V$, and all $s \in S$, one has

$$p(\theta(s)(v)) = \alpha(s)(p(v)).$$

See Figure 10 for an example of a framework which is symmetric with respect to a reflectional symmetry. Note that a symmetry of a framework consists of a group together with a (geometric) action of the group on $\mathbb{R}^d$, and not simply the underlying group. For instance the group $\mathbb{Z}/2$ may act on $\mathbb{R}^2$ by a rotation or as a reflection.

Since one requires $p(\theta(g) \cdot v) = \alpha(g)p(v)$, one has, for any edge $uv \in E$, that

$$\|p(\theta(g)(u)) - p(\theta(g)(v))\| = \|\alpha(g)(p(u)) - \alpha(g)p(v)\| = \|p(u) - p(v)\|.$$

When one enforces symmetry, it thus suffices to consider distance constraints for one representative of each edge-orbit. Reasoning in this way, one obtains only one equation that needs to hold for each different edge orbit. To represent this data efficiently, a natural idea is to take the quotient graph Γ/S of the graph by the group S. This alone is not sufficient to recover the graph Γ; one needs to label the edges to do this, leading to the concept of a gain graph. See [91] for a general reference on gain graphs. Gain graphs were first used in [45] in rigidity theory. As for oriented graphs in Section 2.4, we describe oriented multigraphs by giving two functions $o, t : E \to V$ that provide the endpoints of each edge.

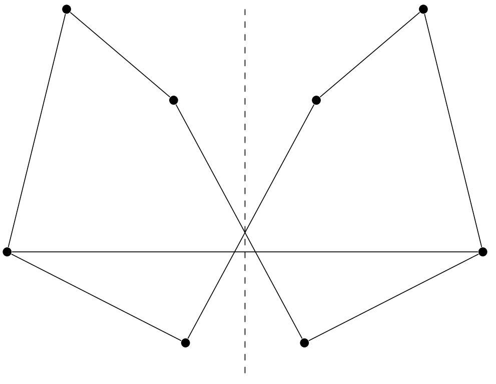

Figure 10: A $\mathbb{Z}/2$-symmetric framework with reflection symmetry.

Definition 2.38. We define an S-gain graph to be a pair consisting of an oriented (multi)-graph $\Gamma = (V, E, o, t)$, together with a function $\psi : E \to S$, such that the following conditions hold:

(i) for all loops $e \in E(\Gamma)$, $\psi(e) \neq \mathrm{id}$;

(ii) for all parallel edges $e, f \in E(\Gamma)$ with the same orientation, $\psi(e) \neq \psi(f)$; and

(iii) for all parallel edges $e, f \in E(\Gamma)$ with the opposite orientation, $\psi(e) \neq \psi(f)^{-1}$.

We denote an edge with $o(e) = v_1, t(e) = v_2$ and $\psi(e) = g$ by $v_1 \xrightarrow{g} v_2$.

One can construct a gain graph (Γ, ψ) from an S-symmetric graph $\Gamma = (V, E)$ as follows. We shall assume that Γ is such that V/S and E/S are finite. Pick some orientation on Γ and pick a representative v_i from each orbit $S \cdot v \in V/S$, yielding $\tilde{V} = \{v_1, \dots, v_n\}$. Then, we pick a representative e from each orbit $S \cdot e \in E/S$, such that $o(e) \in \tilde{V}$ or $t(e) \in \tilde{V}$, which clearly exists by construction of $\tilde{V}$. This yields a set $\tilde{E} = \{e_1, \dots, e_m\}$, and for every $e_i \in E$, we have $e_i = \{v_j, g_e v_k\}$. We define $o, t : \tilde{E} \to \tilde{V}$ such that $\{o(e_i), t(e_i)\} = \{v_j, v_k\}$ and we set $\psi(e_i) = g_e$ if $o(e_i) = v_j$ and $\psi(e_i) = g_e^{-1}$ if $o(e_i) = v_k$.

This procedure clearly produces a gain graph from a symmetric graph, however the resulting gain graph is not unique, as one gets different gains $\psi(e)$ from different choices of representatives, and one may also change the orientation of edges. The different choices lead to gain graphs that are called *equivalent*. From a gain graph, one can construct a symmetric graph, and two equivalent gain graphs yield the same symmetric graph. The construction of a gain graph is illustrated in Figure 11.

To obtain a S-symmetric realisation, it suffices to only provide coordinates for one of the representatives of each orbit, i.e. to the vertices of the gain graph Γ. This is called a joint-configuration in [45].

Definition 2.39. A joint configuration is a pair of an S-gain graph (Γ, ψ) and a function $p : V(\Gamma) \to \mathbb{R}^d$. One defines the rigidity map as

$$f_{\Gamma,\psi} : \mathbb{R}^{d|V(\Gamma)|} \to \mathbb{R}^{|E(\Gamma)|} : (x_v)_{v \in V} \mapsto \left(\|x_v - \alpha(g)(x_w))\|^2 \right)_{v \xrightarrow{g} w \in E}.$$

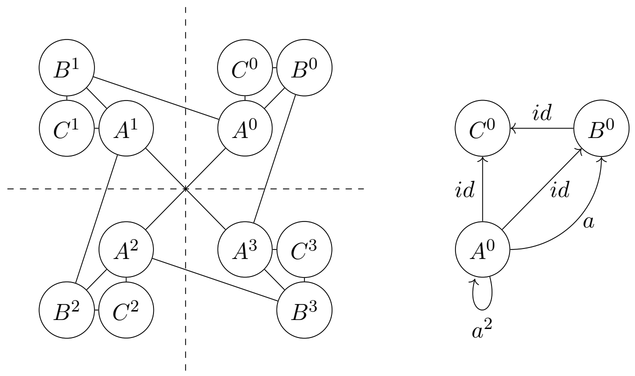

Figure 11: A $\mathbb{Z}/4$-symmetric graph and the gain graph encoding the action of $\mathbb{Z}/4$. The element a is a generator of $\mathbb{Z}/4$, which acts on Γ by $\theta(a)(A^i) = A^{i+1}$, $\theta(a)(B^i) = B^{i+1}$ and $\theta(a)(C^i) = C^{i+1}$, where $i+1$ is interpreted modulo 4. The representatives chosen in the construction of the gain graph are A^0, B^0, C^0.

One defines the orbit rigidity matrix by $O(\Gamma, \psi, p) := \frac{1}{2} d(f_{\Gamma,\psi})_p$.

Similarly to Section 2.1 and Section 2.2, one can define rigidity, as well as infinitesimal rigidity. However, there is one issue that needs to be resolved: namely, that of determining the trivial infinitesimal motions. For example, for reflectionally symmetric frameworks as in Figure 10, translations in the direction of the axis of reflection always define motions of the framework. For rotationally symmetric frameworks, rotations around the centre of rotation always define motions of the framework. These motions are the suitable trivial infinitesimal motions in this context.

Moreover, suppose we are given a joint configuration in $\mathbb{R}^2$ such that all edges have gain $g_e = \mathrm{id}$. Then, the symmetry imposes no additional conditions on the graph. In the plane, such graphs can thus have orbit rigidity matrices with rank at most $2|V(\Gamma)| - 3$, since one always has a 3-dimensional space of trivial motions. The following definition is useful for capturing this phenomenon:

Definition 2.40. Let S be a group and let (Γ, ψ) be a S-gain graph. Let $W = e_1, e_2, \ldots, e_n$ be a walk in Γ, where e_i has end-vertices $v_i, v_{i+1} \in V(\Gamma)$ for all $1 \leq i \leq n$. We define the *gain* of W as

$$\psi(W) = \prod_{i=1}^{n} \psi(e_i)^{\mathrm{sign}(e_i)}, \tag{3}$$

where $\mathrm{sign}(e_i) = 1$ if $e_i = v_i \xrightarrow{\psi(e_i)} v_{i+1}$ and $\mathrm{sign}(e_i) = -1$ if $e_i = v_{i+1} \xrightarrow{\psi(e_i)} v_i$. Suppose Γ is connected. We define the *gain of* Γ with *base vertex* $v \in V(\Gamma)$ and *gain map* ψ to be

$$\langle \Gamma \rangle_{\psi,v} = \langle \psi(W) \in S \mid W \text{ is a closed walk in } \Gamma \text{ starting at } v \rangle.$$

We say that (Γ, ψ) is cyclic if $\langle \Gamma \rangle_{\psi,v}$ is cyclic for some $v \in V$, and we say that (Γ, ψ) is balanced if $\langle \Gamma \rangle_{\psi,v} = \{\mathrm{id}\}$.

This leads to the following adapted definitions of sparsity.

Definition 2.41. Let S be a cyclic group. We define a S-gain graph (Γ, ψ) to be $(2,3,1)$-*gain sparse* if the following hold:

(i) $|E(\Gamma')| \leq 2|V(\Gamma')| - 1$ for all subgraphs Γ' of Γ with $|E(\Gamma')| \geq 1$;

(ii) $|E(\Gamma')| \leq 2|V(\Gamma')| - 3$ for all balanced subgraphs Γ' of Γ with $|V(\Gamma')| \geq 2$.

We say (Γ, ψ) is $(2,3,1)$-*gain tight* if it is $(2,3,1)$-*gain sparse* and, in addition,

$$|E(\Gamma)| = 2|V(\Gamma)| - 1.$$

Malestein and Theran showed that $(2,3,1)-$gain sparsity characterises symmetry-forced rigidity with respect to finite cyclic groups [59], which was demonstrated using different methods in [45].

Definition 2.42. Let S be a dihedral group. We define a S-gain graph (Γ, ψ) to be $(2,3,1,0)$-*gain sparse* if the following hold:

(i) $|E(\Gamma')| \leq 2|V(\Gamma')|$ for all subgraphs Γ' of Γ;

(ii) $|E(\Gamma')| \leq 2|V(\Gamma')| - 1$ for all cyclic subgraphs Γ' of Γ with $|V(\Gamma')| \geq 1$; and

(iii) $|E(\Gamma')| \leq 2|V(\Gamma')| - 3$ for all balanced subgraphs Γ' of Γ with $|V(\Gamma')| \geq 2$.

We say (Γ, ψ) is $(2,3,1,0)$-*gain tight* if it is $(2,3,1,0)$-*gain sparse* and, in addition,

$$|E(\Gamma)| = 2|V(\Gamma)|.$$

In [45], $(2,3,1,0)$ sparsity was shown to be both necessary and sufficient for D_{2n}-symmetric graphs for odd n, where D_{2n} is a dihedral group of order $2n$.

2.5.1 Graphs on surfaces viewed intrinsically

Whiteley [88] studied rigidity of graphs on surfaces by considering the edges between vertices to be modelled as geodesics. The allowed motions of such graphs are the deformations such that the length of each geodesic segment is constant. Many surfaces arise as quotients of discrete groups of isometries: a cone is a quotient of $\mathbb{R}^2$ by a cyclic group, a cylinder is a quotient of $\mathbb{R}^2$ by the group $\mathbb{Z}$, and a torus is a quotient of $\mathbb{R}^2$ by the group $\mathbb{Z}^2$.

In this way, one may think of a graph on surfaces $\mathbb{R}^2/S$ as an S-symmetric graph Γ. For infinite groups S, such as $\mathbb{Z}^2$, the resulting symmetric graph is infinite. The resulting frameworks are called periodic frameworks, the mathematics of which was formalised by Borcea and Streinu [10], who also pointed out applications to material science.

The rigidity of $\mathbb{Z}^2$-periodic frameworks, i.e. rigidity on the torus, was characterised by Ross [66]. Malestein and Theran also characterise $\mathbb{Z}^2$-symmetric rigidity [57, 58], and they also study more general crystallographic groups, as well as allow the representation of $\mathbb{Z}^2$ on $\mathbb{R}^2$ to vary.

In addition to surfaces which are quotients of $\mathbb{R}^2$, two other well-understood cases are rigidity on the sphere and on the hyperbolic plane. These surfaces also have 3-dimensional Lie groups of isometries, and the rigidity of graphs on the sphere or on the hyperbolic plane is characterised by $(2,3)$-tightness, similarly to the plane. One way in which this can be deduced is by using a technique due to Whiteley [87] called coning. For any graph $\Gamma = (V, E)$, one defines the cone graph $\Gamma * v_* = (V \cup \{v_*\}, E_c)$, where $E_c = E \cup \{\{v_*, v\} \subseteq V \cup \{v_*\} \mid v \in V\}$.

Theorem 2.43. *[87] Let* (Γ, p) *be a* 2*-dimensional bar-joint framework, and let* $(\Gamma * v_*, p')$ *be the* 3*-dimensional framework defined by*

$$\begin{aligned} p'(v) &= (p(v), 1) \text{ for } v \in V \\ p'(v_*) &= (x, y, z), \end{aligned}$$

where $z \neq 1$*. Then* $S(\Gamma, p) \cong S(\Gamma * v_*, p')$*. Additionally* (Γ, p) *is rigid if and only if* $(\Gamma * v_*, p')$ *is.*

Saliola and Whiteley [67] used this construction to transfer infinitesimal rigidity and stresses between frameworks on the plane to frameworks on a hemisphere or to frameworks on the hyperbolic plane. We describe the construction only for the hemisphere and the plane. Let (Γ, p) be a bar-joint framework in the plane. Consider the coned framework $(\Gamma * v_*, p')$ as in Theorem 2.43, where one chooses $p'(v_*) = (0, 0, 0)$. Consider then for each edge of the form $v_* v$ the intersection point x_v of the line through $p'(v)$, and $p'(v_*)$ with the unit sphere S^2. Then let $q : V(\Gamma) \to S^2 : v \mapsto x_v$. In terms of infinitesimal rigidity, the plane framework (Γ, p) has the same infinitesimal rigidity properties as (Γ, q). More intrinsically, the resulting map between the sphere and the plane is the so-called gnomonic projection. In other words, given a framework on a hemi-sphere, its gnomonic projection is a framework which has the same properties concerning infinitesimal rigidity.

The transfer between the hyperbolic plane and the Euclidean plane is similar, viewing the hyperbolic plane as a hyperboloid in a pseudo-Euclidean space, and by adapting the coning result. These results were extended by Schulze and Whiteley to show that rigidity-theoretic properties of symmetric frameworks with respect to a dihedral symmetry group transfer between these distinct geometries as well [70].

2.6 Variants of rigidity (ii): other variants

In this section, we will introduce more variants of rigidity. Two of these variants, namely parallel redrawings and linearly constrained frameworks will be discussed in Paper I. In Paper III, a parallel redrawings problem will also be discussed. The other variants which we will discuss in this section are added because of historical interest and importance.

Some other variants of rigidity, which each represent active areas of research, are volume rigidity [11], rigidity in normed spaces [50], uniqueness of low-rank matrix completion [74], and projective rigidity [6]. For each of the variants below, we will only discuss local rigidity.

2.6.1 Linearly constrained frameworks and graphs on surfaces viewed extrinsically

In Section 2.5.1, an approach was described to model graphs on surfaces using symmetric graphs. Another approach, initiated by Nixon, Owen, and Power [63], is as follows. One is given a graph Γ, and one considers maps $p : V \to X$, where X is a surface embedded in $\mathbb{R}^3$. More generally, one may consider graphs such that the vertices lie on an embedded submanifold of $\mathbb{R}^d$. One then wants to consider only the flexes of the framework such that the vertices remain on X; however, the requirement for a flex is still that the Euclidean lengths are preserved. In this way, infinitesimal motions $(W_v)_{v \in V}$ need to satisfy $(W_v)_{v \in V} \in \ker(R(\Gamma, p))$, but they additionally need to satisfy

$$W_v \in T_{p(v)} X$$

for all $v \in V$.

The same type of condition arises in the study of linearly constrained frameworks. Similarly to the graphs on surfaces viewed extrinsically, one can define a linearly constrained framework to be a bar-joint framework where the vertices are constrained to lie on a linear space. First, the rigidity of linearly constrained frameworks was studied in the plane by Streinu and Theran [78], where some of the vertices are constrained to lie on a line. It was then studied for graphs in $\mathbb{R}^d$ by Cruickshank, Guler, Jackson, and Nixon [19]. They characterised rigidity when all vertices are required to lie in t-dimensional affine spaces for $t \in \{1, 2\}$ and $d \geq 3$, and also in the case where $d \geq \max(2t, t(t-1))$. This latter bound was improved by Jackson, Nixon and Tanigawa to hold whenever $d \geq 2t$ [42].

2.6.2 Body-bar and panel-hinge structures

While rigidity is not characterised in dimensions greater than 2, there are many similarly defined structures which are characterised in arbitrary dimensions. The definitions of such structures use the Grassmann-Cayley algebra, which geometrically describes the join and the intersections of affine subspaces of $\mathbb{R}^d$. This algebra can also be used to describe the kinematics of geometric structures, which is how it is used in this context. Since it would take us too far, we will not state precise definitions in this section.

Tay [80] characterised the rigidity of so-called body-bar frameworks. These consist of rigid bodies in $\mathbb{R}^d$ that are linked by rigid bars. The bars are attached to the rigid bodies by joints. This is naturally modelled by a multi-graph: each rigid body is represented by a vertex, and each bar is represented by an edge. A generic body-bar framework in dimension d with an underlying multigraph $\Gamma = (V, E)$ is shown to be rigid if and only if it has a (n_d, n_d)-tight spanning subgraph $\Gamma' = (V, F)$, where $n_d = \binom{d+1}{2}$ is the dimension of $SE(d)$.

A similar structure that was characterised by Tay [81] and Whiteley [88] is that of a body-hinge framework. These consist of rigid bodies that are joined at hinges (which are $(d-2)$-dimensional subspaces), around which each of the bodies may rotate. If each hinge is attached to exactly 2 bodies, then one can create a graph, representing each body by a vertex and each hinge by an edge. If one creates the multigraph B where one creates $n_d - 1$ copies of each edge, one has that the body-hinge framework is rigid if and only if B contains a spanning subgraph which is $(n_d, n_d)-$tight.

Tay and Whiteley conjectured [83] that one could replace the bodies by $(d-1)$-dimensional subspaces, which was known as the molecular conjecture. The resulting structures are called panel-hinge frameworks. In a celebrated result by Katoh and Tanigawa, this conjecture was proved [48].

In [81], Tay also characterises a related structure: namely rod-bar frameworks. These consist of $(d-2)-$dimensional vertices, which are joined by rigid bars, i.e., 1-dimensional line segments.

2.6.3 Parallel redrawings

One may consider the following variant of the rigidity of graphs. Given a graph $\Gamma = (V, E)$ and a realisation $p : V \to \mathbb{R}^d$, one may move the vertices of the graph as long as the edges remain parallel. This problem was characterised by Whiteley [90, Theorem 8.2.2], and an alternative proof was given by Develin, Martin, and Reiner [22].

This structure naturally generalises to hypergraphs. Recall that a hypergraph is a pair (V, E), where V is a set and E is a set of nonempty subsets of V. Elements of E are called hyperedges. A hypergraph is called r-uniform if every hyperedge e contains exactly r elements.

One considers realisations of hypergraphs $\mathcal{H} = (V, E)$ where each vertex v is realised as a point $p(v)$ in $\mathbb{R}^d$, and each $e \in E$ is realised as a hyperplane H_e with a prescribed normal α_e. If such realisations exist, in two distinct realisations H_e and H'_e will be parallel for every edge. In this context, rigidity means that the only redrawings are the trivial redrawings: i.e. those coming from dilations and translations. The rigidity of parallel redrawings was characterised using a sparsity condition by Whiteley [89].

A problem which is dual to the parallel redrawings problem is that of determining the scenes that give rise to a given picture. We only describe the 3-dimensional case, other cases are analogous. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$. One is given a picture of a hypergraph, where a picture is a map $v \mapsto p(v) \in \mathbb{R}^2$, such that each hyperedge forms a convex polygon. One wants to find the dimension of the space of maps $q : V \to \mathbb{R}^3$ such that $\pi \circ q = p$ and for each edge e, the set $\{q(v) \mid v \in e\}$ is contained in a plane. Such maps are called liftings or scenes. Determining when a generic picture lifts to a nontrivial scene, was determined by Whiteley [89].

2.6.4 Rigidity of polytopes

In this thesis, we are only concerned with realisations of graphs and hypergraphs. However, the rigidity of polytopes is, in fact, a much older topic. We only state the most central results.

It was first proven by Cauchy [12] that given two convex polytopes P and Q in $\mathbb{R}^d$, such that there is a continuous bijection $\varphi : P \to Q$ where each face F of P maps isometrically onto a face F' of Q, then φ is an isometry from P to Q. From this, it follows that the graph of any convex simplicial polytope in $\mathbb{R}^d$ is rigid. Gluck proved that almost all triangulations of a sphere are rigid [31]. It was long believed that all non-intersecting polyhedra are rigid, though a counterexample was found by Connelly [15].

The rigidity of polytopes has some direct applications in polytopal combinatorics. As observed by Kalai [47], the lower bound theorem for simplicial polytopes due to Barnette [4] can be deduced from the inequality which holds for rigid graphs:

$$|E| \geq d|V| - \binom{d+1}{2}.$$

2.7 General rigidity models

Since some of the research in this thesis will concern a general rigidity model, we will provide an overview of other rigidity models.

One way to give a general model of rigidity is to consider the embeddings of graphs in a space X to be given by providing the locations of the vertices $p : V \to X$, and to have a function $g : X^2 \to Y$ which gives some type of measurement. One may then define the rigidity map

$$f_\Gamma : X^V \to Y^E : (x_v)_{v \in V} \mapsto g(x_v, x_w)_{vw \in E},$$

and define rigidity along the lines of Section 2.1 and Section 2.2. This perspective, for functions defined on $\mathbb{R}^d$, is developed by Cruickshank, Mohammadi, Nixon and Tanigawa [21], and they define the notion of g-rigidity. Similar definitions, where the map f_Γ was not informed by some combinatorial structure, were given by Stacey, Mahony, and Trumpf [76] and expanded upon by Stacey and Mahony [75].

Another line of work models rigidity using group actions. In this setting, a graph is realised in a space X, and admissible deformations are prescribed by a group G acting on X. Classical Euclidean rigidity arises when G is the Euclidean group, leading to the preservation of edge lengths. Such a perspective was taken by Gortler, Gotsman, Liu, and Thurston [32]. In fact, they allow a monoid M to act on $\mathbb{R}^d$, which they take to be the monoid of affine self-maps, and they use this to study what they call affine rigidity. However, they soon reformulate and study their problem in terms of so called affinity matrices.

Another use of groups in a rigidity-like context is due to Scherk and Mathis [68]. They consider the various symmetry groups associated with different CAD constraints in Euclidean space, which they used to develop a decomposition algorithm. In their model, they associate to each geometric constraint the group of transformations that preserve this constraint.

The model that we define in Paper I follows this group-theoretic approach. In contrast to the previous approaches based on group theory, we develop the model in full generality, and we consistently only use group-theoretic tools in our study of the model, providing unified results for graph-of-groups realisations.

3 Summary of papers

3.1 Paper I: Structural rigidity and flexibility using graphs of groups

Klara Stokes, Joannes Vermant, Structural rigidity and flexibility using graphs of groups
Applicable Algebra in Engineering, Communication and Computing 2026.

This paper develops the definition of a graph-of-groups realisation, providing a general model for rigidity theoretic problems. The graph-of-groups approach is shown to lead to a natural algebraic structure on the set of motions, namely that of a groupoid.

In the most general setting, we prove that there is a one-to-one correspondence between motions and sections of a certain graph homomorphism. This construction can be seen as similar in spirit to the construction of the universal covering in Bass-Serre theory, which is where graph-of-groups originate.

Much of the paper focuses on Lie-graph-of-groups realisations, where one assigns Lie subgroups to each vertex and edge. In this setting, local and infinitesimal rigidity are defined. We generalise the Maxwell bound and that rigidity implies a degree of vertex connectivity in this setting. One usually obtains sparsity conditions for hypergraphs $\mathcal{H} = (V, E)$ such that for all subsets $V' \subseteq V$ and $E' \subseteq E$, one has

$$\lambda|I'| \leq k_1|V'| + k_2|E'| - \dim(G), \tag{4}$$

for some integer parameters k_1, k_2 and λ, which are defined using the dimensions of the subgroups at the vertices and the edges, and where $I' = \{(v, e) \in V' \times E' \mid v \in e\}$.

The applicability of the approach is illustrated through examples. The first discussed example is Euclidean rigidity, where we show all definitions correspond to the classical definitions. We also recover some known results on centres of rotation of infinitesimal motions of graphs in the plane. We then discuss the example of scenes and parallel redrawings, where a correspondence between them is proved by presenting an isomorphism of the groupoids and projective rigidity is treated in a similar manner. Finally, we show that by adapting the model, one can consider linearly constrained frameworks, and we also examine the unique colourability of graphs as a rigidity problem, where the graph homomorphism interpretation of motions becomes particularly simple.

Contribution: I contributed substantially to all parts of the article. This includes the development of the ideas, the proofs, the examples, the literature study, the writing of the article, and the creation of figures.

3.2 Paper II: Sparsity greedoids and pebble game algorithms for posets

Signe Lundqvist, Klara Stokes, Tovohery Randrianarisoa, Joannes Vermant, Sparsity greedoids and pebble game algorithms for posets *arXiv preprint,* `arXiv:2306.05050v3` 2026.

In the context of this thesis, the motivation behind this paper was to understand the computational aspects of checking the sparsity conditions (4) that arise in Paper I, as well as to find potential sparsity conditions that could be useful when trying to generalise the definitions to higher rank structures such as polytopes and graded posets.

The sparsity condition from Paper I can be checked using the algorithms mentioned in Section 2.4, though we generalised a result of Birgitte Servatius [73] and compared this notion of sparsity to other notions of sparsity for hypergraphs. We define a general sparsity condition that arises when one runs the pebble game algorithm on certain graphs, which come, for example, from graded posets. Our definition is as follows:

Definition 3.1. Let $V = V_1 \cup \cdots \cup V_n$ be a partition of the vertex set, with edges allowed only between V_i and V_{i+1} for the different values of i. Let $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $L = (l_{1,2}, \ldots, l_{n-1,n}) \in \mathbb{Z}^{n-1}$. An edge set $E' \subseteq E$ is called (K, L)*–sparse* if every $V' \subseteq V$ satisfies

$$|E'(V')| \;\leq\; \sum_{i=1}^{n} k_i \, |V' \cap V_i| \;-\; l(V'),$$

with

$$l(V') = \sup\{\, l_{i,i+1} : V' \cap V_i \neq \emptyset,\;\; V' \cap V_{i+1} \neq \emptyset \,\},$$

and (K, L)*-tight* if equality holds for $V' = V$.

While this structure does not define a matroid, we show that it defines a greedoid. More precisely, the definitions are as follows.

Let $K = (k_1, ..., k_n)$ and $L = (l_{1,2}, ..., l_{n-1,n})$ be sequences of integers and let $\mathcal{L}_{(K,L)}$ be the set of strings on E that satisfy the following three properties:

1. For any $\alpha = e_1 \ldots e_m$, the set $\{e_1, \ldots, e_m\}$ is (K, L)-sparse.
2. For any $\alpha = e_1 \ldots e_m$ and $1 \leq i < j \leq n-1$, if $e_{k_1} \in E(V_i \cup V_{i+1})$ and $e_{k_2} \in E(V_j \cup V_{j+1})$, then $k_1 < k_2$.
3. For any $\alpha = e_1 \ldots e_m$ and $i \in \{1, \ldots n-1\}$ such that there is an e_j with $e_j \in E(V_i \cup V_{i+1})$, the edge set $\{e_1, \ldots, e_m\} \cap E(V_1 \cup \cdots \cup V_i)$ is maximally (K, L)-sparse on $V_1 \cup \cdots \cup V_i$.

Theorem 3.2. *Suppose that the parameters*

$$K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{>0}, L = (l_{1,2}, \ldots, l_{n-1,n}) \in \mathbb{Z}^{n-1}_{\geq 0},$$

satisfy the following inequalities:

$$k_i + k_{i+1} > l_{i,i+1},$$
$$k_{i+1} + l_{i-1,i} > l_{i,i+1},$$
$$l_{n-1,n} \geq \ldots \geq l_{1,2}.$$

Let $G = (V, E)$ be a graph such that the vertex set has a partition $V = V_1 \cup \cdots \cup V_n$ such that all edges $e \in E$ have endpoints in $V_i \cup V_{i+1}$ for some $1 \leq i \leq n-1$. Then $\mathcal{L}_{(K,L)}$ is a greedoid language.

We also show that the pebble game algorithm recognises the maximal feasible sets of this greedoid language.

Contribution: I contributed substantially to all parts of the article. This includes the development of the ideas, the proofs, the examples, the literature study, the writing of the article, the creation of figures and implementing algorithms for computational experiments.

3.3 Paper III: Homological methods in rigidity theory using graphs of groups

Joannes Vermant, Homological methods in rigidity theory using graphs of groups, *arXiv preprint* `arXiv:2603.05435v2` 2026.

The main question addressed in this article is when the necessary conditions for rigidity are also sufficient in Lie-graph-of-groups realisations. The main theorem of the article states that when the realisations are defined using 1-dimensional stabilisers, then for any graph Γ, almost all realisations are (infinitesimally) rigid if and only if the graph has the relevant sparsity property.

A tool that is used to prove this theorem is that of *cellular sheaves*. In a sense, these sheaves are combinatorial analogues of the sheaves on topological spaces often used in geometry and topology. Sheaves can be seen as tools allowing one to address local-to-global questions. Within rigidity theory, one may think of infinitesimal motions as being defined by essentially local conditions: each edge gives a restriction on how the endpoints may move. A motion of all of the vertices respecting this condition is then a global solution.

Concretely, one describes the infinitesimal motions IM_ρ as the zeroth cohomology group of a cellular sheaf. This description also provides a natural analogue of the stresses of a Lie graph-of-groups realisation, namely the first cohomology group of this sheaf.

The sheaves are more naturally thought of as belonging to a more general family of 'motion sheaves' which are parametrised by (real) Grassmannians. For motion sheaves, we prove some results that imply the necessary condition for rigidity from Paper I. We show that within the Grassmannians, the dimension of the cohomology groups is a lower semicontinuous function. It follows that the infinitesimal rigidity is a generic property within certain classes of graph-of-groups realisations.

To prove the main result, one uses a homological approach to show that inductive constructions preserve the relevant notion of independence. This then gives a characterisation of independence for certain motion sheaves. By analysing the set of motion sheaves that come from graph-of-groups realisations, one obtains a characterisation for the motion sheaves. Finally, we analyse in some detail the set of motion sheaves that arise from parallel-redrawing type problems, since these are particularly well-behaved.

Contribution: This is a single-authored paper.

3.4 Paper IV: Hyperbolic symmetric rigidity and intrinsic surface rigidity

Sean Dewar, Alison La Porta, Rebecca Monks, Anthony Nixon, Klara Stokes and Joannes Vermant, Hyperbolic symmetric rigidity and intrinsic surface rigidity *Preprint, not submitted.*

In this paper, we want to characterise the rigidity of graphs which are embedded on compact orientable surfaces of genus $g \geq 2$. This is a problem of studying rigidity in a locally homogeneous space, and, as for the torus, one can study graphs on the surface by viewing them as symmetric frameworks on its covering. In this case, since surfaces of genus $g \geq 2$ are quotients of the hyperbolic plane, we study forced symmetric rigidity with respect to Fuchsian groups, i.e. discrete subgroups of isometries of the hyperbolic plane. We need one definition to state our main result.

The following theorem is the main result of the paper, characterising rigidity on genus $g \geq 2$ surfaces. The notion of $(2, 3, 1, 0)$-tightness is the same as the one introduced in Definition 2.42, replacing the dihedral group with a surface group.

Theorem 3.3. *Let* (Γ, ψ) *be a* S*-gain graph, where* S *is a discrete subgroup of* $\mathrm{PSL}(2, \mathbb{R})$ *that is isomorphic to a surface group. Then the following are equivalent:*

1. (Γ, ψ) *is minimally infinitesimally rigid in the hyperbolic plane* $\mathbb{H}$*; and*
2. (Γ, ψ) *is* $(2, 3, 1, 0)$*-gain tight.*

The proof is an inductive proof based on the characterisation of $(2, 3, 1, 0)$-tight graphs given in [45]. The main difficulty in the paper is showing that the inductive constructions preserve independence and that the base graphs for the induction are independent. A technical innovation that is used is that the orbit rigidity matrix is described using the Riemannian metric and the geodesics between points in the framework, which makes it possible to use arguments based on Riemannian geometry.

Contribution: I contributed substantially to all parts of the article. This includes the development of the ideas, the proofs, the examples, the literature study, the writing of the article, and the creation of figures.

3.5 Paper V: The rigidity of graphs in positive characteristic fields

Sean Dewar, Stefano Lia, Anthony Nixon, Klara Stokes, and Joannes Vermant: The rigidity of graphs in positive characteristic fields *Preprint, not submitted.*

In this paper, we consider rigidity in quadratic spaces: vector spaces with a non-degenerate quadratic form. The goal was to initiate the study of rigidity and rigidity matroids over general fields of characteristic different from 2. We use three distinct approaches: one using framework rigidity, one using algebraic matroids, and one using graph-of-groups.

We first investigate framework rigidity, where we establish some basic notions associated with well-behaved frameworks. When the field is algebraically closed, we provide a novel definition of local rigidity, and we show that this is equivalent to infinitesimal rigidity for non-singular points in a certain algebraic variety. We show that some of the concepts from framework rigidity can be reformulated in terms of graph-of-groups realisation.

The algebraic matroid is defined essentially as in the Euclidean case. We show that if the characteristic is sufficiently large, then the algebraic matroid is isomorphic to the linear matroid that one obtains when computing over a polynomial ring. A bound on the field size is also provided for the existence of frameworks p such that the generic linear rigidity matroid is isomorphic to the linear matroid obtained from p.

Contribution: I contributed substantially to all parts of the article. This includes the development of the ideas, the proofs, the examples, the literature study and the writing of the article.

4 Concluding remarks

In this thesis, we have studied rigidity from a group-theoretic perspective, and we have developed the theory of the rigidity of graphs in homogeneous and locally homogeneous spaces. Graph-of-groups realisations can be naturally thought of as embedding graphs or hypergraphs in homogeneous spaces. The work on hyperbolic symmetric rigidity can be seen as the study of graphs in locally symmetric spaces, albeit a special case.

Necessary conditions for infinitesimal rigidity were proved in quite general settings, and they are shown to be sufficient in the case where the rigidity problem is defined by a 1-dimensional stabiliser group in a Lie group G. However, many open questions remain, and there seem to be natural ways to extend these ideas to other contexts.

First, some technical improvements to the results seem possible. In Paper IV, we have restricted our attention to surface groups. Most arguments do not use this hypothesis, and it is possible that the results extend to all Fuchsian groups, or at least to a subclass of Fuchsian groups.

There are several more ambitious generalisations of the work in this thesis that could be made. One natural direction that can be studied is that of symmetric graph-of-groups realisations, which would give a more complete definition of the rigidity of graphs in locally symmetric spaces.

Another substantial research direction is to extend the research of Paper III to characterise motion sheaves for other values of s and n, as well as to provide an interpretation for the associated sheaves. On a related note, the theory of stresses for graph-of-groups realisations as defined in Paper III is mostly unexplored. Providing a suitable geometric theory of these objects is an open problem.

Another research project could be to extend the ideas in this thesis to the study of the rigidity of other combinatorial structures, such as polytopes, as introduced in Section 2.6.4. More generally, one could consider realisations of higher-rank incidence geometries. Preliminary work shows that such structures can also be studied in a group-theoretic way, similar to how graphs were studied in this thesis. The use of homological methods, as used for graphs in Paper III, seems natural in this context, in which case higher cohomology groups H^k would also play an important role.

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