Algebras of Bounded Holomorphic Functions

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Abstract

Some problems concerning the algebra of bounded holomorphic functions from bounded domains in $\mathbf{C}^n$ are solved. A bounded domain of holomorphy Ω in $\mathbf{C}^2$ with nonschlicht H^∞- envelope of holomorphy is constructed and it is shown that there is a point in Ω for which Gleason's Problem for $H^\infty(\Omega)$ cannot be solved.

If $A(\Omega)$ is the Banach algebra of functions holomorphic in the bounded domain Ω in $\mathbf{C}^n$ and continuous on the boundary and if p is a point in Ω, then the following problem is known as Gleason's Problem for $A(\Omega)$:

Is the maximal ideal in $A(\Omega)$ consisting of functions vanishing at p generated by $(z_1 - p_1)$, ... , $(z_n - p_n)$?

A sufficient condition for solving Gleason's Problem for $A(\Omega)$ for all points in Ω is given. In particular, this condition is fulfilled by a convex domain Ω with $\mathrm{Lip}_{1+\epsilon}$-boundary $(0 < \epsilon < 1)$ and thus generalizes a theorem of S.L.Leibenzon. One of the ideas in the methods of proof is integration along specific polygonal lines.

If Gleason's Problem can be solved in a point it can be solved also in a neighbourhood of the point. It is shown, that the coefficients in this case depends holomorphically on the points.

Defining a projection from the spectrum of a uniform algebra of holomorphic functions to $\mathbf{C}^n$, one defines the fiber in the spectrum over a point as the elements in the spectrum that projects on that point. Defining a kind of maximum modulus property for domains in $\mathbf{C}^n$, some problems concerning the fibers and the number of elements in the fibers in certain algebras of bounded holomorphic functions are solved. It is, for example, shown that the set of points, over which the fibers contain more than one element is closed. A consequence is also that a starshaped domain with the maximum modulus property has schlicht H^∞-envelope of holomorphy. These kind of problems are also connected with Gleason's problem.

A survey paper on general properties of algebras of bounded holomorphic functions of several variables is included. The paper, in particular, treats aspects connecting H^∞-envelopes of holomorphy and some areas in the theory of uniform algebras.

Key words: holomorphic function, bounded holomorphic function, domain of holomorphy, envelope of holomorphy, Gleason's problem, convex set, uniform algebra, spectrum, fibers, generalized Shilov boundary, analytic structure, plurisubharmonic function

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Papers summarized in this dissertation:

1. A. Fällström, *Algebras of Bounded Holomorphic Functions in Several Variables.*
2. U. Backlund and A. Fällström, *A pseudoconvex domain with nonschlicht H^∞- envelope.* Geometrical and Algebraical Aspects in Several Complex Variables, Cetraro (Italy) June 1989. Seminars and Conferences. Editel 1991.
3. A. Fällström, *On the spectrum of finitely generated algebras of holomorphic functions.*
4. U. Backlund and A. Fällström, *The Gleason Problem for* $A(\Omega)$.
5. A. Fällström, *Plurisubharmonicity and Algebras of Holomorphic Functions.*
6. A. Fällström, *Comparison of some concepts related to bounded holomorphic functions.*

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Summary

This thesis consists of the following six papers:

1. A. Fällström, *Algebras of Bounded Holomorphic Functions in Several Variables.*
2. U. Backlund and A. Fällström, *A pseudoconvex domain with nonschlicht H^∞- envelope.* Geometrical and Algebraical Aspects in Several Complex Variables, Cetraro (Italy) June 1989. Seminars and Conferences. Editel 1991.
3. A. Fällström, *On the spectrum of finitely generated algebras of holomorphic functions.*
4. U. Backlund and A. Fällström, *The Gleason Problem for $A(\Omega)$.*
5. A. Fällström, *Plurisubharmonicity and Algebras of Holomorphic Functions.*
6. A. Fällström, *Comparison of some concepts related to bounded holomorphic functions.*

In paper [2] in this thesis we consider some questions concerning holomorphic continuation of the class of bounded holomorphic functions. A natural question is whether the bounded holomorphic functions on the domain can be holomorphically continued to a strictly larger domain (Riemann domain). The simplest example of such a domain is a punctured disc in the complex plane. With the extra conditions that the interior of the closure of the domain equals the domain itself, the question becomes much harder. N. Sibony [Sib] constructed a domain of that kind in the unit polydisk in two variables. The phenomena can not occur in the complex plane. In [2] (Theorem 3.1) we construct a bounded domain of holomorphy Ω in $\mathbf{C}^2$ with nonschlicht H^∞-envelope of holomorphy. That is, the maximal domain of definition for the bounded holomorphic functions on Ω is a Riemann domain (a covering space) spread over $\mathbf{C}^2$. This also implies that the domain contains a point, for which Gleason's Problem cannot be solved [2] (Proposition 4.1).

In the papers [3],[4] and [5] we consider some questions concerning the spectrum of the Banach Algebras of bounded holomorphic functions on a domain in $\mathbf{C}^n$ ($H^\infty(\Omega)$) and holomorphic functions that are continuous up to the boundary of the domain ($A(\Omega)$). The holomorphic functions on a domain form a Fréchet Algebra and the spectrum of the algebra equals the envelope of holomorphy (the largest domain (Riemann domain), to which the holomorphic functions can be continued holomorphically). In the case of bounded holomorphic functions, the situation is much more complicated.

If Ω is a domain in $\mathbf{C}^n$ and p is a point in Ω, the following problem is known as Gleason's Problem:

> Is the maximal ideal in $\mathcal{B}(\Omega)$ consisting of functions vanishing at p algebraically generated by $(z_1 - p_1), ..., (z_n - p_n)$?

Here $\mathcal{B}(\Omega)$ denotes a uniform algebra of functions on Ω. We also say that Ω has the Gleason $\mathcal{B}$-property in the point.

Using the Gelfand transform of the coordinate functions, one can define a projection from the spectrum of the algebra to $\mathbf{C}^n$. The inverse mapping of this projection, maps a point in $\mathbf{C}^n$ onto what we call the fiber over that point.

In paper [3] we study some results concerning topologically finitely generated algebras and the fibers in such algebras. We show ([3], Theorem 2) that if all ideals in a fiber are algebraically finitely generated, then there are only finitely many elements in the fiber. This also implies that the H^∞-envelope of holomorphy over such a point contains only finitely many sheets.

If a domain has the Gleason $\mathcal{B}$-property in some point, then this is true also in a neighbourhood of the point. In [3] (Theorem 1) it is shown that the generators in that case depends holomorphically on the point.

In paper [4] we give a sufficient condition for solving Gleason's Problem for $A(\Omega)$ for all points in the domain ([4], Theorem 1). In particular ([4], Corollary 1) this condition is fulfilled by a convex domain Ω with $\mathrm{Lip}_{1+\epsilon}$-boundary ($0 < \epsilon < 1$) and thus generalizes a theorem of S.L.Leibenzon.

Basener and Sibony have introduced generalized Shilov boundaries for uniform algebras. These boundaries form a nice setting for finding analytic structure in the spectrum of such algebras.

In paper [5] we use generalized Shilov boundaries and a theorem by Basener [Bas] to study the fibers in certain uniform algebras of holomorphic functions. We show ([5], Theorem 1) that the function

$$\varphi_f(z) = \sup_{m \in I(z)} \log |\hat{f}(m)|$$

where $I(z)$ denotes the fiber over $z \in \mathbf{C}^n$, is plurisubharmonic. This is, for domains that fulfill a certain maximum modulus property, used to show some results concerning the number of elements in the fibers. For example, one proves that the points over which the fibers contain more than one element is closed in the domain of definition of the functions in the algebra. A consequence is also that a starshaped domain with this maximum modulus property has a schlicht H^∞-envelope of holomorphy.

Results connected to Gleason's problem and envelopes of holomorphy are also discussed.

To get a picture of how little we yet know in this area, different tools and concepts in the theory of bounded holomorphic fuctions in several variables, are discussed in paper [6].

For more background on the questions that are discussed in this thesis we refer the reader to the included survey paper [1] on bounded holomorphic functions.

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